On the other hand, the y's with $Ay \geq 0$ the the feasible directions for x^*, the points of the red region.

Since x^* is optimal, there is no direction both feasible and increasing. In other words, $c^T y \geq 0$ for all y such that $Ay \geq 0$, or $\max\{c^T y : Ay \leq 0\} = 0$. By the strong duality, there is a dual solution $\lambda \in \mathbb{R}^2$: $c = A^T \lambda = \lambda_1 A_{1\cdot}^T + \lambda_2 A_{2\cdot}^T$, $\lambda_1 \geq 0$, $\lambda_2 \geq 0$.

It means c, in the 2-dim example, is between two vectors $A_{1\cdot}^T$ and $A_{2\cdot}^T$ so that the blue and red regions have no intersection:

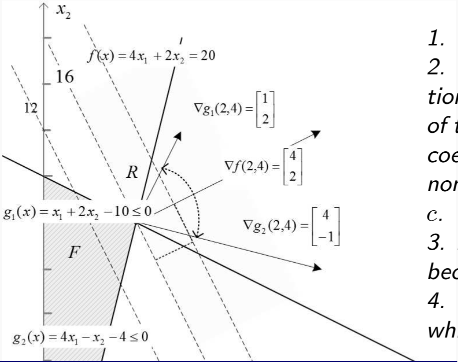
Exercise 4.4

The argument applies to a general case $\min\{c^T x : Ax \geq 0\}$ *for optimal solution* $\bar{x}$*, and the constraints* $\bar{x}$ *satisfies with equality (called active constraints).*

Exercise 4.5

Consider the linear program

$$
\begin{array}{rrrrrr}
\max & f(x) = & 4x_1 & +2x_2 & & \\
\textit{sub. to} & g_1(x) = & x_1 & +2x_2 & -10 & \leq 0 \\
& g_2(x) = & 4x_1 & -x_2 & -4 & \leq 0 \\
& g_3(x) = & -x_1 & & & \leq 0 \\
& g_4(x) = & & -x_2 & & \leq 0
\end{array}
$$

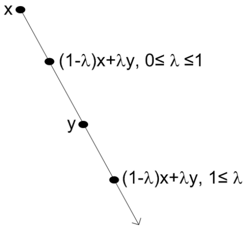


1. *Write the dual.*
2. *From the graph, $(2,4)$ is an optimal solution. Repeat the same observation in terms of the active constraints and c. Compute the coefficients λ_i's of the vectors $A_{i.}$ in their nonnegative linear combination representing c.*
3. *It we set $\lambda_i = 0$ for i not active at $(2,4)$, λ becomes an optimal solution of dual problem.*
4. *Compute the objective coefficient d for which $(0,5)$ is optimal.*

The *computational efforts* required to solve (or *computational complexity* of) an optimization problem depend on the characteristics of the functions of the objective and constraints. In particular, if both objective function and the feasible region are convex, an optimization problem is solvable efficiently.

A straight line passing through two points x, $y \in \mathbb{R}^n$ can be parameterized in λ as follows.

$$x + \lambda(y - x) = (1 - \lambda)x + \lambda y. \tag{5.11}$$

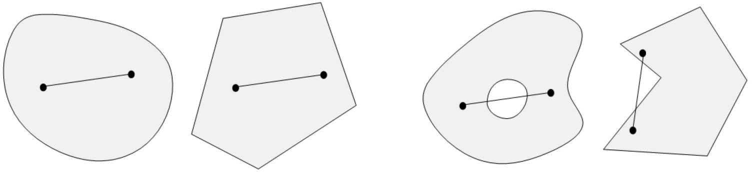


If λ is allowed to be any real number, (5.11) is called a *affine combination* of x and y. For $0 \leq \lambda \leq 1$, (5.11) is called a *convex combination* of x and y.

Definition 5.1

A set $S \subseteq \mathbb{R}^n$ is convex if it contains every line segment connecting its two points: $(1-\lambda)x + \lambda y \in S,\ \forall\ x, y \in S,\ \forall\ 0 \le \lambda \le 1$.

The figure shows convex and nonconvex sets.



Proposition 5.2

If S_1 and S_2 are convex, then their intersection $S_1 \cap S_2$ is convex.

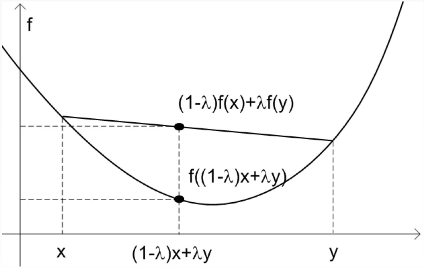
Proof: Take any two points x_1 and x_2 from $S_1 \cap S_2$ and any λ with $0 \le \lambda \le 1$. From the convexity of S_1, $(1-\lambda)x_1 + \lambda x_2 \in S_1$. Similarly, it is also contained in S_2. Hence $(1-\lambda)x_1 + \lambda x_2 \in S_1 \cap S_2$. $\square$

Definition 5.3

For a set $S \subseteq \mathbb{R}^n$, a function $f : S \to \mathbb{R}$ is convex if S is a convex set and

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y), \ \forall x, y \in S, \ \forall \lambda : \ 0 \leq \lambda \leq 1.$$

f is concave if $-f$ is convex. f is affine if f is both convex and concave.



Example 5.4

$f(x) = x^2$ is a convex function. For if $0 \leq \lambda \leq 1$, we have $(1-\lambda)f(x) + \lambda f(y) - f((1-\lambda)x + \lambda y) = (1-\lambda)\lambda(x-y)^2 \geq 0$. Hence $f(x) = -x^2$ is concave.

Proposition 5.5

Suppose f and g are convex. Then for any $\alpha \geq 0$, $f+g$, αf is convex.

Exercise 5.6

A linear functional $f(x) = c^T x$ is both convex and concave, and hence affine.

Exercise 5.7

Is $f(x, y) = 2x^2 - xy + y^2 + 2x - 3y$ convex?

A convex combination can be extended in terms of a finite number of vectors.

Definition 5.8

Convex combination For an integer $k \geq 2$ and $\lambda \in \mathbb{R}^k$ such that $\lambda_1 + \cdots + \lambda_k = 1$ and $\lambda_1 \geq 0, \ldots, \lambda_k \geq 0$, the following vector v is called a convex combination of $v_1, v_2, \ldots, v_k$.

$$v = \lambda_1 v_1 + \cdots + \lambda_k v_k \tag{5.12}$$

Suppose v is a convex combination $k+1$ vectors, $v = \lambda_1 v_1 + \cdots + \lambda_k v_k + \lambda_{k+1} v_{k+1}$. $0 < \lambda_1 < 1$. If we let $\mu = \lambda_2 + \cdots + \lambda_{k+1}$, then $0 < \mu < 1$, and v can be rewritten as follows.

$$v = (1-\mu)v_1 + \mu\left(\frac{\lambda_2 v_2 + \cdots + \lambda_{k+1} v_{k+1}}{\lambda_2 + \cdots + \lambda_{k+1}}\right).$$

Every convex combination can be represented by a repeated convex combination of two vectors.

Definition 5.9

Convex hull For a set $S \subseteq \mathbb{R}^n$, the *convex hull*, conv.S, is the smallest convex set that includes S.

$\mathrm{conv}.S$ is the intersection of convex sets including S. Why? Let the latter be S'. Then S' is included in the convex set $\mathrm{conv}.S$. On the other hand, since S' is convex (why?) and includes S, it also includes the convex hull of S. Hence $\mathrm{conv}.S = S'$.

Theorem 5.10

$\mathrm{conv}.S$ is the set of convex combinations from $S \subseteq \mathbb{R}^n$.

Proof: Let T be the set of convex combinations from $S \subseteq \mathbb{R}^n$.

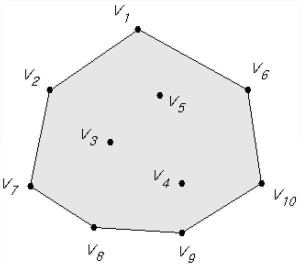
'$T \subseteq \mathrm{conv}.S$' Since any element of T is a convex combination of vectors of S, it is contained in any convex set including S. Hence it is contained in $\mathrm{conv}.S$.

'$T \supseteq \mathrm{conv}.S$' Note that a convex combination of convex combinations of vectors of S is a convex combination of vectors of S (Check it!). Therefore T is a convex set including S and thus includes $\mathrm{conv}.S$. $\square$

Corrolary 5.11

The convex hull of $S = \{v_1, v_2, \ldots, v_k\} \subseteq \mathbb{R}^n$ is $\{\lambda_1 v_1 + \cdots + \lambda_k v_k |\ \lambda_1 + \cdots + \lambda_k = 1,\ \lambda_1 \geq 0,\ \ldots,\ \lambda_k \geq 0\}$.

When $k = 2$, the convex hull is the line segment connecting two points. When $k = 3$, it is triangular with the three vectors as its vertices. In general, the convex hull of a finite set S of vectors is the convex region defined by a polygon whose vertices are from S.



Definition 5.12

(볼록다각형, polytope) We call the convex hull of a finite set of vectors a *polytope*.

Proposition 5.13

For a finite set $S = \{ v_1, v_2, \dots, v_k \}$, *consider the optimization* $\min\{c^T x | x \in \text{conv}.S\}$. *Then an optimum is attained in* S: *there is* v_i *which is an optimal solution the problem.*

증명 Suppose v_1 makes smallest the objective value $c^T x$ over S. Then for any convex combination of S, $c^T(\lambda_1 v_1 + \cdots + \lambda_k v_k) = c^T\Big((1 - \lambda_2 - \cdots - \lambda_k)v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k\Big) = c^T v_1 + \lambda_2(c^T v_2 - c^T v_1) + \cdots + \lambda_k(c^T v_k - c^T v_1) \geq c^T v_1$. □
Thus, the optimizations on S and its convex hull are essentially the same thing.

Remark 5.14

It can be more convenient to deal with the convex hull of a set of vectors than the set itself. For instance, the convex hull can be represented by a compact linear system whereas the elements of S are too large.

Definition 6.1

If a set $L \subseteq \mathbb{R}^n$ contains every straight line through any two points of itself, L is said to be an *affine set* (평면 집합). In other words, an affine set is a set closed under affine combination:

$$(1-\lambda)x + \lambda y \in L,\ \forall x, y \in L,\ \forall \lambda \in \mathbb{R}. \tag{6.13}$$

A subspace is an affine set. Affine set is a convex set.

Definition 6.2

Affine combination also can be extended in terms of a finite number of vectors $S = \{v_1, v_2, \ldots v_k\}$.

$$\lambda_1 v_1 + \cdots + \lambda_k v_k,\ \lambda_1 + \cdots + \lambda_k = 1. \tag{6.14}$$

Similarly, any affine combination can be expressed by repeated affine combinations of two vectors. (Check it). Therefor a set is affine set if and only if it is closed under affine combination of any finite set of its elements.

Definition 6.3

Let $S \subseteq \mathbb{R}^n$. By the *affine hull*, $\mathrm{aff}.S$, of S, we mean the smallest affine set that includes S.

Exercise 6.4

$\mathrm{aff}.S$ *is the intersection of affine sets including* S*.*

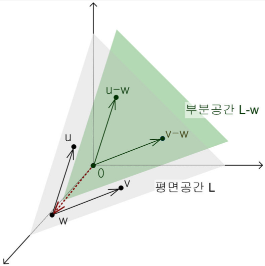
Exercise 6.5

$\mathrm{aff}.S$ *is the same as the set of affine combinations of vectors of* S*.*

Proposition 6.6

If we translate an affine set $L \subseteq \mathbb{R}^n$ by a vector w so that $L - w$ contains the origin, then the set $L - w \equiv \{v - w : v \in L\}$ is a subspace of $\mathbb{R}^n$. (For $L - w$ to contain the origin, necessarily $w \in L$.)

Proof: We show that $L - w$ is closed under addition and scalar multiplication: $\forall\, u, v \in L, \forall\, \alpha \in \mathbb{R}$, $(u-w) + \alpha(v-w) \in L-w$. But, $(u-w) + \alpha(v-w) = (u + \alpha v - \alpha w) - w$ and the parenthesized term $\in L$ (why?). Hence the proposition follows. $\square$



Note that for every $w, w' \in L$, $L - w = L - w'$. For $l \in L$, $l - w = l - w + w' - w' \in L - w'$ and similarly $l - w' \in L - w$. Thus the subspace is uniquely determined independently on the choice of w.

Conversely,

Exercise 6.7

A subspace S translated by any vector w, $S + w$, is an affine set.

We have seen that affine sets are exactly the translated subspaces.

Exercise 6.8

Let S be a subspace. Then $w \notin S$ iff S and $S + w$ do not intersect.

Proposition 6.9

There is a unique subspace whose translation yields the same affine subspace.

Proof: Suppose, two subspaces S and S' are translated by w and w', respectively, to be the same affine set: $S + w = S' + w'$ or $S = S' + (w' - w)$. This implies $w' - w$ and hence $-(w' - w) \in S$. Therefore, $S' = S - (w' - w) = S$. $\square$

An affine space has a very special structure.

Proposition 6.10

Every affine space L is the set of solutions of a linear equality system $Ax = b$.

Proof: Take any $w \in L$. Since $S = L - w \subseteq \mathbb{R}^n$ is a subspace, it has a basis of k vectors for some $k \leq n$. Then by applying G-J method to the matrix having the k vectors as rows, we can construct a matrix A whose null space is S. (Provide the details!) Let $b = Aw$. Then we can easily check, i) $Ax = b$ defines an affine set, and ii) $Ax = b$ has the solution set of $Ax = 0$ translated by w, namely, L. $\square$

Exercise 6.11

Let the affine hull of vectors $x_1 = [1, 2, 3, 6]^T$, $x_2 = [-1, 4, 1, 2]^T$ be L.

(1) *Represent L as $L = S + w$ for some subspace S and a vector $w \in L$.*

(2) *Compute A and b so that $L = \{x \in \mathbb{R}^4 : Ax = b\}$.*

Definition 6.12

The vectors $v_1, v_2, \ldots, v_k$ are affinely independent if $v_2 - v_1, \ldots, v_k - v_1$ is linearly independent; affinely dependent, otherwise.

We first check the subtracting vector is irrelevant in the definition: $v_2 - v_1$, $v_3 - v_1, \ldots, v_k - v_1$ are linearly independent $\Leftrightarrow v_1 - v_2, v_3 - v_2, \ldots, v_k - v_2$ are linearly independent $\Leftrightarrow \cdots$. (Check it.) It is also easy to see that affine independence is invariant with a translation.

Proposition 6.13

The vectors $v_1, v_2, \ldots, v_k$ are affinely independent only $\lambda = 0$ satisfies the following. The converse is also true.

$$\begin{gathered} \lambda_1 v_1 + \cdots + \lambda_k v_k = 0 \\ \lambda_1 + \cdots + \lambda_k = 0. \end{gathered} \tag{6.15}$$

(6.15) is equivalent to that the following vectors are linearly independent.

$$\begin{bmatrix} v_1 \\ 1 \end{bmatrix}, \begin{bmatrix} v_2 \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} v_k \\ 1 \end{bmatrix}.$$

Exercise 6.14

Linearly independent vectors are also affinely independent.

If we translate, by $w \notin S$, a basis of a subspace S, and add w to it, then the resulting set is a set of affinely independent vectors. Therefore, the maximum number of affinely independent vectors from $S + w$ is $\geq \dim(S) + 1$. But it can not exceed $\dim(S) + 1$ (why?).

Proposition 6.15

The maximum number of affinely independent vectors in $S + w$ is $\dim S + 1$.