Definition 2.16

If a face F of P has no face among its proper subsets, it is called a *minimal face* (극소면).

Proposition 2.17

A minimal face F of $P = \{x : Ax \geq b\}$ has a face subsystem whose rank is equal to $r(A)$: $\exists$ a subsystem $A^{\circ}x \geq b^{\circ}$ of $Ax \geq b$ such that $F = \{x \in P : A^{\circ}x = b^{\circ}\}$ and $r(A^{\circ}) = r(A)$.

Proof: It suffices to show that the maximum face subsystem $A^{\circ}x \geq b^{\circ}$ of F has rank $r(A^{\circ}) = r(A)$. Suppose on the contrary $r(A^{\circ}) < r(A)$. Then there is a constraint ${a_{i'}}^T x \geq b_{i'}$ not in $A^{\circ}x \geq b^{\circ}$ such that $r\begin{pmatrix} A^{\circ} \\ a_{i'}^T \end{pmatrix} > r(A^{\circ})$. Since $A^{\circ}x = b^{\circ}$ has a solution, the augmented system $A^{\circ}x = b^{\circ}$, $a_{i'}^T x = b_{i'}$ also has a solution. Let x' be any solution of the augmented system.

Proof(*cont'd*):
Case 1 $x' \in P$: Then $F' = \{x \in P : A^\circ x = b^\circ,\ a_{i'}^T x = b_{i'}\}$ is nonempty and hence a face. Since $A^\circ x \geq b^\circ$ is the maximum subsystem of F, F also has a point satisfying ${a_{i'}}^T x > b_{i'}$. Thus $F' \subsetneq F$. A contradiction.

Case 2 $x' \notin P$: Let y be any point of P. Then the line segment from y to x', there is the last point $z \in P$. Since the whole line is contained affine set $A^\circ x = b^\circ$, there is a blocking constraint $a_{i''}^T x \geq b_{i''}$ not from $A^\circ x \geq b^\circ$ such that $a_{i''}^T z = b_{i''}$. Since $a_{i''}^T y > b_{i''}$, similarly with Case 1, $F' = \{x \in P : A^\circ x = b^\circ,$ $a_{i''}^T x = b_{i''}\}$ is a face of P such that $F' \subsetneq F$. A contradiction. □

In the proof, if the maximum subsystem $A^\circ x = b^\circ$ of a face F has at the same time the points of P and not of P, then F has a face F' which is a proper subset of F. Thus if F is a minimal face, other constraints are all valid inequalities for $\{x : A^\circ x = b^\circ\}$. Therefore, we have

$$F = \big\{x \in P : A^\circ x = b^\circ\big\} = \big\{x : A^\circ x = b^\circ\big\}. \tag{2.1}$$

In the proof of Proposition 2.17, we assumed $A^\circ x \geq b^\circ$ is the maximum face subsystem of F. But from (2.1), for any subsystem of $A^\circ x \geq b^\circ$ with the same rank has the corresponding affine set equal to F.

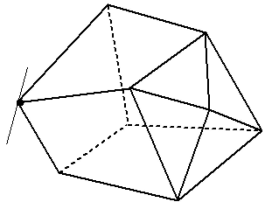
Corrolary 2.18

Every minimal face is an affine set with the dimension $n - r(A)$.

Exercise 2.19

If a minimal face of P is a point, then every minimal face is also a point which is a unique solution of $A^\circ x = b^\circ$ where $A^\circ x \geq b^\circ$ is a subsystem with $r(A^\circ) = n$.

The figure shows an extreme point or vertex of a 3-dimensional polyhedron. It has the following features.



First, it is a geometrically protruded point; it can not be in the middle of any two distinct points of P.

Second, it is a face whose dimension is 0. More specifically, from Exercise 2.19, it is a unique point satisfying with equality a subsystem $A^{\circ}x \geq b^{\circ}$ such that $r(A^{\circ}) = n$. In the above example, the vertex is a unique point at which a full rank subsystem of three inequalities is satisfied with equality.

The two are actually equivalent.

Definition 2.20

A point x° of $P = \{x : Ax \geq b\}$ ($A \in \mathbb{R}^{m\times n}$) is called a *vertex* of P if it is not strictly between two end points of a line segment included in P, namely, if $x^\circ = (1-\lambda)y + \lambda z$ for some $0 < \lambda < 1$ and y, $z \in P$, then $y = z$ $(= x)$.

Proposition 2.21

A point x° of $P = \{x : Ax \geq b\}$ is a vertex if and only if it is a 0-dimensional (minimal) face of P. In other words, x° is a vertex of P if and only if it is the solution of the equality system of some subsystem $A^\circ x \geq b^\circ$ with $r(A^\circ) = n$.

Proof : ($\Rightarrow$) Let x° be a vertex of P and let $A^\circ x \geq b^\circ$ be the subsystem of consraints satisfied with equality by x°. Then since x° satisfies other constraints with $>$, there is $\epsilon > 0$ such that every point of $B_\epsilon(x^\circ)$ also satisfies them.
Why $r(A^\circ) = n$? If $r(A^\circ) < n$, $A^\circ x = b^\circ$ has a point y° distinct from x° and the line passing through the two points lie in the affine set $A^\circ x = b^\circ$. Therefore, the intersection of the line and $B_\epsilon(x^\circ)$ is a line segment contained in P. Let the two end points be u and v. Then $u \neq v$, u, $v \in P$, and $x^\circ = \frac{1}{2}u + \frac{1}{2}v$, a contradiction to the assumption that x° is a vertex.

Proof(*cont'd*):($\Leftarrow$) Let x° be a 0-dimensional face with the face subsystem $A^\circ x \geq b^\circ$ with $r(A^\circ) = n$. Then x° is a unique solution of $A^\circ x = b^\circ$. Therefore, every point of P distinct from x° satisfies at least one inequality from $A^\circ x \geq b^\circ$ with $>$. And so does any strict convex combination having at least one endpoint not equal to x°. Thus if $x^\circ = (1-\lambda)y + \lambda z$ for y, $z \in P$ and $0 < \lambda < 1$, then $y = z = x^\circ$. $\square$

Lemma 2.22

A bounded polyhedron has a vertex.

Proof: Consider any point $x^1 \in P$. Let $A^1 x \geq b^1 x^1$ be the subsystem of constraints that are satisfied by x^1 with equality. If the $r(A^1) = n$, we are done.

Otherwise, the system $A^1 x = b^1$ has a solution y^1 distinct from x^1. Since P is bounded, for any line from x^1 to y^1, there is a blocking constraint, say, $a_{i'}^T x \geq b_{i'}$. Let z^1 be the last point of P on the line so that $a_{i'}^T x \geq b_{i'}$. Note that $a_{i'}$ increases the rank of A^1. (For otherwise the solution set of the augmented system is the same as that of $A^1 x = b^1$ or empty, neither of which is the case.)
If $r(A^2) = n$, z^1 is a vertex. Otherwise, setting $x^2 = z^1$ we repeat to compute y^2 and z^2 and so on. Since the rank of the system strictly increases each time, x^k will be a vertex for some $k \leq n$. $\square$

Proposition 2.23

A bounded polyhedron is the convex hull of its vertices.

Proof: Let $x^1, \ldots, x^K$ be the vertices of P and $Q = \text{conv.}\{x^1, \ldots, x^K\}$.

$(P \supseteq Q)$ Clear since P is convex.

$(P \subseteq Q)$ Let x^1 be any element of P and $A^1x \geq b^1$ the active constraints. If $r(A^1) = n$, x^1 is a vertex and we are done. Otherwise as in Lemma 2.22, we consider a half line in $\{x | \; A^1x \geq b^1$ and passing through x^1. Let the last point of P on the line be z^1. Similarly, let the last point of P in the half line from x in the opposite direction be w^1. Then x is a convex combination of z^1 and w^1. Hence if z^1 and w^1 are both vertices of P, we are done.

Otherwise, we repeat the procedure after setting $x^2 = z^1$ or $x^2 = w^1$ (or both). Then every branch of the procedure will terminate with w or z that is a vertex. Since a convex combination of convex combinations of vertices is a convex combination of vertices, we have the proposition. $\square$

Exercise 2.24

Consider the following dual problem.

$$\begin{array}{lrrrrcl} \max & -4y_1 & & +2y_3 & -2y_4 & & \\ s.t. & -2y_1 & & +2y_3 & & = & 5 \\ & 4y_1 & +y_2 & +y_3 & -y_4 & = & 6 \\ & y_1, & y_2, & y_3, & y_4 & \geq & 0 \end{array}$$

$y^* = [0, -\frac{7}{2}, \frac{5}{2}, 0]^T$ *is an optimal solution with objective value* 5*.*

(1) *Let* $P = \{x : Ax \geq b\}$ *be the polyhedron of the primal problem. Show* $F = \{x \in P : 5x_1 + 6x_2 = 5\}$ *is a face* P*.*

(2) *Find a face subsystem* $A^\circ x \geq b^\circ$ *of* F*.*

Exercise 2.25

Find a vertex of the following polyhedron starting with $x^1 = [2\ 2\ 2]^T$.

$$\begin{array}{llll} x_1 & +2x_2 & +x_3 & \geq 3 \\ x_1 & & -x_3 & \leq 1 \\ & & x_3 & \leq 4 \\ x_1 & +x_2 & +x_3 & \geq 6 \\ x_1 & & & \geq 0 \\ & x_2 & & \geq 0 \\ & & x_3 & \geq 0 \end{array}$$

Exercise 2.26

Find a vertex of the polyhedron starting with $x^1 = [\frac{1}{2}\ \frac{1}{2}\ \frac{9}{2}\ 6]^T$.

$$\begin{array}{ccccc} 3x_1 & +2x_2 & -x_3 & +x_4 & = 4 \\ 2x_1 & -x_2 & +x_3 & & = 5 \\ x_1 \geq 0 & x_2 \geq 0 & x_3 \geq 0 & x_4 \geq 0 & \end{array}$$

Exercise 2.27

Suppose LP minimizing $c^T x$ *over*$P = \{Ax \geq b\}$ *has an optimal solution. Discuss how to find an optimal solution which is a vertex.*

Proposition 2.23 implies that if the feasible solution set is bounded, it suffices to an optimal vertex. In fact it also applies to an LP with a polyhedron not necessarily bounded.

Exercise 2.28

If LP $\min\{c^T x | Ax \geq b\}$ *has an optimal set and a vertex of its polyhedron, an optimal solution is attained at a vertex, i.e. there is a vertex which is an optimal solution.*

Hint: Consider the face of optimal solutions. Recall that a minimal face is affine.

Exercise 2.29

Prove or disprove by a counterexample.

(1) *If* $P = \{x : Ax \geq b\}$ $(A \in \mathbb{R}^{m \times n})$ *has a vertex,* $R(A)$ *is* $\mathbb{R}^n$.

(2) *A bounded polyhedron has a facet.*

(3) *A minimal representation of a polyhedron* P *is unique up to a positive multiplication.*

(4) *If a polyhedron does not have a vertex, it has no facet.*

(5) *If a polyhedron is not full-dimensional, it is a face of itself.*

(6) *If a polyhedron has no facet, it is not full-dimensional.*

(7) *The converse of (6).*

Exercise 2.30

Consider the polyhedron whose linear system description is as follows:

$$\begin{array}{rrrl}
x_1 & +2x_2 & +x_3 & \geq 3 \\
x_1 & & -x_3 & \leq 1 \\
 & & x_3 & \leq 4 \\
x_1 & +x_2 & +x_3 & \geq 6 \\
x_1 & & & \geq 0 \\
 & x_2 & & \geq 0 \\
 & & x_3 & \geq 0
\end{array}$$

(1) *Show* $x^\circ = (1,5,0)^T$ *is a vertex of the polyhedron.*

(2) *How many facets does* P *have and why.*

Recall the sum of two sets S, $T \subseteq \mathbb{R}^n$ is defined as $S + T = \{s + t : s \in S,\ t \in T\}$. Then a half-line $L^+ = \{x + \lambda y : \lambda \geq 0\}$ is the sum of $\{x\}$ and the set $\{\lambda y : \lambda \geq 0\}$.

The set $\{\lambda y : \lambda \geq 0\}$ is closed in nonnegative multiplication.

Definition 3.1

We call a set cone (뿔) if it closed in nonnegative multiplication.

By a cone, we normally mean a cone which is convex as well. And it is not difficult to prove the following proposition.

Proposition 3.2

A set $K \subseteq \mathbb{R}^n$ is a convex cone if it is closed in nonnegative linear combination or conic combination: $\forall\ x$, $y \in K$ *and* $\forall \lambda$, $\mu \geq 0$, $\lambda x + \mu y \in K$.

The conic combination also can be extended to a finite number of vectors. Similarly we can define conic hull of a set S to be the smallest cone including S as a subset. Also, then we can show the conic hull of S is the set of conic combinations of vectors from S.

Definition 3.3

If a convex cone K is a conic hull of a finite set of vectors $\{y^1, \ldots, y^k\}$

$$K = \text{cone}\{y^1, \ldots, y^k\} = \{\lambda_1 y^1 + \cdots + \lambda_k y^k : \lambda_1 \geq 0, \ldots, \lambda_k \geq 0\},$$

then K is said to be a *finitely generated cone.*

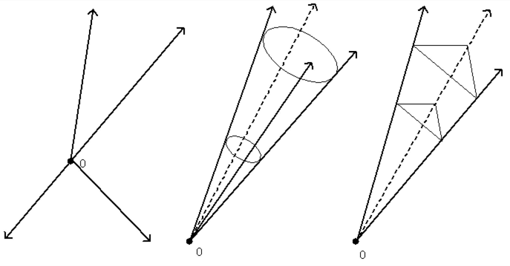


Figure: Cone, convex cone and finitely generated cone.

Definition 3.4

For any matrix $A \in \mathbb{R}^{m \times n}$, the polyhedron $K = \{y : Ay \geq 0\}$ is a convex cone by Proposition 3.2. We call K a polyhedral cone.

Proposition 3.5

Minkowski's theorem: *Every polyhedral cone is finitely generated.*

Proof: Any vector y of $K = \{y : Ay \geq 0\}$ can be scaled down to be contained in a unit hypercube centered at the origin. Hence every $y \in K$ is a positive multiplication of a vector from $\bar{K} = \{y : Ay \geq 0, \; -e \leq y \leq e\}$. Since $\bar{K}$ is bounded, it is the convex hull of its finite number of vertices. It implies $K = \{y : Ay \geq 0\}$ is the conic hull of the vertices of $\bar{K}$, and hence finitely generated. $\square$

If a polyhedron $P = \{x : Ax \geq b\}$ includes a half line $x + \lambda y$, then we should have $Ay \geq 0$. Conversely, if $Ay \geq 0$, then for any $x \in P$, the half line $x + \lambda y$ is included in P. It suggests the following proposition.

Proposition 3.6

Every polyhedron $P = \{x : Ax \geq b\}$ is the sum of a bounded polyhedron and a polyhedral cone. In other words, there is a finite set of vectors $\{x^1$, x^2, ..., $x^p\}$ and $\{y^1$, y^2, ..., $y^q\}$ such that for any $x \in P$, there are λ and μ such that

$$\begin{gathered} x = \lambda_1 x^1 + \lambda_2 x^2 + \cdots + \lambda_p x^p + \mu_1 y^1 + \mu_2 y^2 + \cdots + \mu_q y^q, \\ \lambda_1 + \lambda_2 + \cdots + \lambda_p = 1,\ \lambda_1 \geq 0,\ \lambda_2 \geq 0,\ \cdots, \lambda_p \geq 0, \\ \mu_1 \geq 0,\ \mu_2 \geq 0,\ \cdots, \mu_q \geq 0. \end{gathered} \tag{3.2}$$