Optimal Design of Energy Systems (M2794.003400)

Chapter 11. Geometric Programming

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Chapter 11. Geometric Programming

11.1 Introduction

- Geometric Programming : Adaptable to problems where **the objective function** and **the constraints equations** are sum of polynomials of variables
- The first stage of the solution is to find the optimum value of the function.
- In this chapter, **degree of difficulty** (presented at 11.3) is **0** to be suited to geometric programming.

Chapter 11. Geometric Programming

11.2 Form of The objective Function and Constraint

- It can treat both of a constrained and unconstrained objective functions. For example;
  - Unconstrained

$$\text{Minimize} \quad y = 5x + \frac{10}{\sqrt{x}}$$

  - Constrained

$$\text{Minimize} \quad y = 5x_1\sqrt{x_2} + 2x_1^2 + x_2^{3/2}$$

$$\text{Where} \qquad x_1x_2 = 50$$

Chapter 11. Geometric Programming

11.3 Degree of Difficulty(DOD)

- Definition;

$$\mathrm{DoD} \equiv T - (N + 1)$$

Where T : Number of terms in objective function and constrains

N : Number of variables

- Example;

$$y = 6 - 3x + 10x^{0.5} \quad \rightarrow \quad \begin{aligned} &T = 2\ (-3x\,, 10x^{0.5}) \\ &N = 1\,(x) \\ &\therefore \mathrm{DoD} = 0 \end{aligned}$$

$$\begin{aligned} &y = 5x_1x_2^{0.5} + 2x_1^2 + x_1^{0.5} \\ &x_1x_2 = 50 \end{aligned} \quad \rightarrow \quad \begin{aligned} &T = 4\ \left(5x_1x_2^{0.5}, 2x_1^2, x_1^{0.5}, x_1x_2\right) \\ &N = 2\,(x_1, x_2) \\ &\therefore \mathrm{DoD} = 1 \end{aligned}$$

In this chapter, however, DOD=0

Chapter 11. Geometric Programming

11.4 Mechanics of Solution for One Independent Variable, Unconstrained

- A basic mechanics of geometric programming is suggested.
- For the optimal value y^*

$$y = c_1 x^{a_1} + c_2 x^{a_2} = u_1 + u_2$$

- Using geometric programming, y^* can be represented in product form. Then,

$$y^* = g^* = \left(\frac{c_1 x^{a_1}}{w_1}\right)^{w_1} \left(\frac{c_2 x^{a_2}}{w_2}\right)^{w_2}$$

$$y^* = c_1 (x^*)^{a_1} + c_2 (x^*)^{a_2} = u_1^* + u_2^* \quad \text{Where } u_1^* = c_1 (x^*)^{a_1},\ u_2^* = c_2 (x^*)^{a_2}$$

- Provided that $w_1 + w_2 = 1$ (Optimized)

  $a_1 w_1 + a_2 w_2 = 0$ (Optimized)

11.4 Mechanics of Solution for One Independent Variable, Unconstrained

- At the optimum value,

$$y^* = g^* = \left(\frac{c_1 x^{a_1}}{w_1}\right)^{w_1} \left(\frac{c_2 x^{a_2}}{w_2}\right)^{w_2}$$

- Provided that $w_1 + w_2 = 1$, and $a_1 w_1 + a_2 w_2 = 0$;

$$x^{a_1 w_1 + a_2 w_2} = 1$$

- So, from a given equation

$$\therefore \; w_1 = \frac{u_1^*}{y^*} = \frac{u_1^*}{u_1^* + u_2^*} \qquad w_2 = \frac{u_2^*}{y^*} = \frac{u_2^*}{u_1^* + u_2^*}$$

Example 11.1

- Determine **the optimum pipe diameter** which results in minimum first plus operating cost for 100 m of pipe.

(Given)

- The objective function, the cost y ($), in terms of pipe diameter, D (mm) is then

$$y = 160D + \frac{32 \times 10^{12}}{D^5}$$

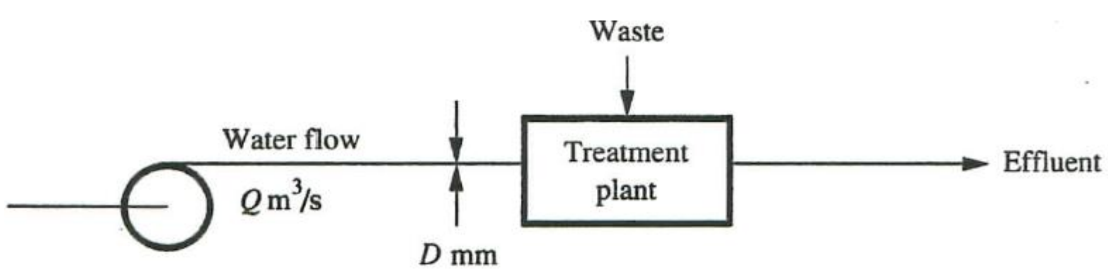

Fig. Waste-treatment system in Example 11.1 and 11.4

Chapter 11. Geometric Programming

Example 11.1

(Solution)

- For the optimum value y^*

$$y^* = g^* = 160D^* + \frac{32 \times 10^{12}}{(D^*)^5}$$

$$y^* = g^* = \left(\frac{160D}{w_1}\right)^{w_1} \left(\frac{32 \times 10^{12}}{D^5 w_2}\right)^{w_2}$$

- Provided that $w_1 + w_2 = 1$

$$a_1 w_1 + a_2 w_2 = w_1 - 5w_2 = 0$$

- Solving gives $w_1 = \frac{5}{6}$, $w_2 = \frac{1}{6}$

Chapter 11. Geometric Programming

Example 11.1

(Solution)

- By substituting w_1 and w_2 to the above optimizing equation,

$$y^* = g^* = \left(\frac{160}{5/6}\right)^{5/6} \left(\frac{32 \times 10^{12}}{1/6}\right)^{1/6} = \$\ 19{,}200$$

$$w_1 = \frac{5}{6} = \frac{u_1^*}{y^*} = \frac{160D^*}{19{,}200} \qquad \rightarrow \qquad \therefore D^* = 100\ mm$$

(Answer)

- The optimum diameter : 100 mm (at minimum cost, $19,200)

Example 11.2

- Determine the maximum power of which this engine is capable and the rotative speed at which the maximum occurs.

(Given)

- The torque T (N·m) is represented by

$$T = 23.6\omega^{0.7} - 3.17\omega$$

Where ω : the rotative speed (rad/s)

Example 11.2

(Solution)

- The power P (Watt) is the product of the torque and the rotative speed.

$$P = T\omega = 23.6\omega^{1.7} - 3.17\omega^2$$

$$y^* = g^* = \left(\frac{23.6}{w_1}\right)^{w_1} \left(\frac{-3.17}{w_2}\right)^{w_2}$$

- Provided that $w_1 + w_2 = 1$, and $a_1 w_1 + a_2 w_2 = 1.7w_1 + 2w_2 = 0$

- Solving gives $w_1 = 6.667$, $w_2 = -5.667$

Example 11.2

(Solution)

$$y^* = g^* = \left(\frac{23.6}{6.667}\right)^{6.667} \left(\frac{-3.17}{-5.667}\right)^{-5.667} = 122{,}970\ \mathrm{W}$$

$$w_1 = 6.667 = \frac{u_1^*}{y^*} = \frac{23.6(\omega^*)^{1.7}}{122{,}970} \quad \rightarrow \quad \omega^* = 469\ \mathrm{rad/s}$$

- If we focus on w_2 instead of w_1, then the result is also

$$w_2 = -5.667 = \frac{u_2^*}{y^*} = \frac{-3.17(\omega^*)^2}{122{,}970} \quad \rightarrow \quad \omega^* = 469\ \mathrm{rad/s}$$

(Answer)

- The maximum power : 122,970 W (at the rotative speed, 469 rad/s)

11.5 Why Geometric Programming Works; One Independent Variable

- Substantiation :

$$y = c_1 x^{a_1} + c_2 x^{a_2} = u_1 + u_2 \qquad \text{Where } u_1 = c_1 x^{a_1}, u_2 = c_2 x^{a_2}$$

$$g = \left(\frac{u_1}{w_1}\right)^{w_1}\left(\frac{u_2}{w_2}\right)^{w_2} = \left(\frac{c_1 x^{a_1}}{w_1}\right)^{w_1}\left(\frac{c_2 x^{a_2}}{w_2}\right)^{w_2} \qquad \text{Where } w_1 + w_2 = 1$$

- Using logarithm, it can be represented as

$$\ln g = w_1(\ln u_1 - \ln w_1) + w_2(\ln u_2 - \ln w_2)$$

Subject to $\emptyset = w_1 + w_2 - 1 = 0$

11.5 Why Geometric Programming Works; One Independent Variable

- To find a combination w_1, w_2 at which function g has the maximum value, Lagrange multiplier is used

$$\nabla(\ln g) - \lambda \nabla \emptyset = 0$$

$$\emptyset = 0$$

- which provide the three equations:

w_1: $$\ln u_1 - 1 - \ln w_1 - \lambda = 0$$

w_2: $$\ln u_2 - 1 - \ln w_2 - \lambda = 0$$

$$w_1 + w_2 - 1 = 0$$

- The unknowns are w_1, w_2, λ, and the solutions for w_1 and w_2 are

$$w_1 = \frac{u_1}{u_1 + u_2} \qquad w_2 = \frac{u_2}{u_1 + u_2}$$

$$\therefore \boldsymbol{g} = \left(\frac{\boldsymbol{u_1}}{\boldsymbol{w_1}}\right)^{\boldsymbol{w_1}} \left(\frac{\boldsymbol{u_2}}{\boldsymbol{w_2}}\right)^{\boldsymbol{w_2}} = \left(\frac{u_1}{u_1/(u_1+u_2)}\right)^{\frac{u_1}{(u_1+u_2)}} \left(\frac{u_2}{u_2/(u_1+u_2)}\right)^{\frac{u_2}{(u_1+u_2)}} = \boldsymbol{u_1} + \boldsymbol{u_2} = \boldsymbol{y}$$

Chapter 11. Geometric Programming

11.5 Why Geometric Programming Works; One Independent Variable

- To get optimum value x^*, y^*, g^*, the derivative of y is used.

$$y' = a_1 c_1 x^{(a_1 - 1)} + a_2 c_2 x^{(a_2 - 1)} = 0$$

- Multiplying by x and so,

$$xy' = a_1 c_1 x^{a_1} + a_2 c_2 x^{a_2} = 0$$

$$a_1 u_1^* + a_2 u_2^* = 0$$

- Where u_1^* and u_2^* are the values of u_1 and u_1 at the optimum value of y

$$u_1^* = -\frac{a_2 u_2^*}{a_1}$$

11.5 Why Geometric Programming Works; One Independent Variable

- When substituting u_1^* and u_2^* into w_1 and w_2,

$$w_1 = \frac{u_1^*}{u_1^* + u_2^*} = \frac{-\frac{a_2 u_2^*}{a_1}}{-\frac{a_2 u_2^*}{a_1} + u_2^*} = \frac{-a_2}{a_1 + a_2} \qquad w_2 = \frac{u_2^*}{u_1^* + u_2^*} = \frac{u_2^*}{-\frac{a_2 u_2^*}{a_1} + u_2^*} = \frac{a_1}{a_1 + a_2}$$

- When these values of w_1 and w_2 are substituted into the solution for g,

$$g^* = \left(\frac{c_1 x^{a_1}}{w_1}\right)^{\frac{-a_2}{a_1+a_2}} \left(\frac{c_2 x^{a_2}}{w_2}\right)^{\frac{a_1}{a_1+a_2}} = \left(\frac{c_1}{w_1}\right)^{\frac{-a_2}{a_1+a_2}} \left(\frac{c_2}{w_2}\right)^{\frac{a_1}{a_1+a_2}} \qquad \because\ x^{\frac{-a_2 a_1}{a_1+a_2} + \frac{a_2 a_1}{a_1+a_2}} = 1$$

$$\therefore\ \boldsymbol{g^* = \left(\frac{c_1}{w_1}\right)^{w_1} \left(\frac{c_2}{w_2}\right)^{w_2} = y^*}$$

Chapter 11. Geometric Programming

11.7 Unconstrained, Multivariable Optimization

- The solving procedure for one variable extend to multivariable optimizations.
- If **DoD=0**, a two variable problem have an objective function containing 3 terms.
- An example 11.4 represents a detailed solution for two variable problem.

Example 11.4

- Use geometric programming to optimize the total system.

(Given)

- The objective function, the cost y (\$), in terms of pipe diameter, D (mm) and flow rate, Q (m^3/s) is then

$$y = 160D + \frac{32 \times 10^{15} Q^2}{D^5} + \frac{150}{Q}$$

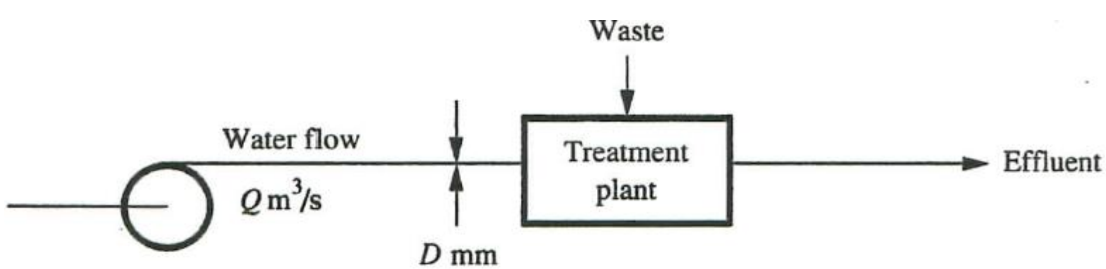

Fig. Waste-treatment system in Example 11.1 and 11.4

Example 11.4

(Solution)

$$y = 160D + \frac{32 \times 10^{15} Q^2}{D^5} + \frac{150}{Q}$$

$$y^* = g^* = \left(\frac{160}{w_1}\right)^{w_1} \left(\frac{220 \times 10^{15}}{w_2}\right)^{w_2} \left(\frac{150}{w_3}\right)^{w_3}$$

- Provided that $w_1 + w_2 + w_3 = 1$

  D: $w_1 - 5w_2 = 0$

  Q: $2w_2 - w_3 = 0$

- Solving gives $w_1 = \frac{5}{8}$, $w_2 = \frac{5}{8}$, $w_3 = \frac{2}{8}$

Example 11.4

(Solution)

$$\left(\frac{160}{5/8}\right)^{5/8}\left(\frac{220\times 10^{15}}{1/8}\right)^{1/8}\left(\frac{150}{1/4}\right)^{1/4} = \$30{,}224$$

$$u_1^* = 160D = \frac{5}{8}(30{,}224) \quad \rightarrow \quad D^* = 118\ mm$$

$$u_3^* = \frac{150}{Q^*} = \frac{2}{8}(30{,}224) \quad \rightarrow \quad Q^* = 0.0198\ m^3/s$$

(Answer)

- The optimum diameter : 118 mm
- The optimum flow rate : 0.0198 ㎥/s
- The minimum cost : $30,224

11.8 Constrained Optimization with Zero Degree of Difficulty

- Suppose that the objective function to be minimized and its constraint is

$$y = u_1 + u_2 + u_3$$

$$u_4 + u_5 = 1 \qquad \text{Where } u_i = f_i(x_1, x_2, x_3, x_4)$$

- The objective function can be rewritten

$$y = g = \left(\frac{u_1}{w_1}\right)^{w_1} \left(\frac{u_2}{w_2}\right)^{w_2} \left(\frac{u_3}{w_3}\right)^{w_3}$$

- Provided that

$$w_1 + w_2 + w_3 = 0$$

$$w_i = \frac{u_i}{u_1 + u_2 + u_3}$$

11.8 Constrained Optimization with Zero Degree of Difficulty

- The constraint equation can also be rewritten as

$$u_4 + u_5 = 1 = \left(\frac{u_4}{w_4}\right)^{w_4} \left(\frac{u_5}{w_5}\right)^{w_5}$$

- Provided that

$$w_4 + w_5 = 1$$

$$w_4 = \frac{u_4}{u_4 + u_5} = u_4 \qquad w_5 = \frac{u_5}{u_4 + u_5} = u_5 \qquad \because u_4 + u_5 = 1$$

- The above equation can be raised to the M th power

$$1 = \left(\frac{u_4}{w_4}\right)^{Mw_4} \left(\frac{u_5}{w_5}\right)^{Mw_5}$$ M is an arbitrary constant

- When multiplying the transformed objective function by the transformed constraint equation

$$\boldsymbol{y = g = \left(\frac{u_1}{w_1}\right)^{w_1} \left(\frac{u_2}{w_2}\right)^{w_2} \left(\frac{u_3}{w_3}\right)^{w_3} \left(\frac{u_4}{w_4}\right)^{Mw_4} \left(\frac{u_5}{w_5}\right)^{Mw_5}}$$

11.8 Constrained Optimization with Zero Degree of Difficulty

- Return to the original equation and solve by Lagrange multipliers

$$\nabla(u_1 + u_2 + u_3) - \lambda[\nabla(u_4 + u_5)] = 0$$

$$u_4 + u_5 = 1$$

- For u_i 's variable x_1, x_2, x_3, x_4, the number of possible equations is 4

$$\begin{cases} a_{11}u_1^* + a_{21}u_2^* + a_{31}u_3^* - \lambda a_{41}u_4^* - \lambda a_{51}u_5^* = 0 \\ a_{12}u_1^* + a_{22}u_2^* + a_{32}u_3^* - \lambda a_{42}u_4^* - \lambda a_{52}u_5^* = 0 \\ a_{13}u_1^* + a_{23}u_2^* + a_{33}u_3^* - \lambda a_{43}u_4^* - \lambda a_{53}u_5^* = 0 \\ a_{14}u_1^* + a_{24}u_2^* + a_{34}u_3^* - \lambda a_{44}u_4^* - \lambda a_{54}u_5^* = 0 \end{cases}$$

Chapter 11. Geometric Programming

11.8 Constrained Optimization with Zero Degree of Difficulty

- Dividing the combinations of 4 equations by y^*, $\boldsymbol{\frac{u_i^*}{y^*} \rightarrow w_i}$

- Since M was arbitrary, let it equal; $\boldsymbol{M = \frac{-\lambda}{y^*}}$

$$\begin{cases} a_{11}w_1 + a_{21}w_2 + a_{31}w_3 - Ma_{41}w_4 - Ma_{51}w_5 = 0 \\ a_{12}w_1 + a_{22}w_2 + a_{32}w_3 - Ma_{42}w_4 - Ma_{52}w_5 = 0 \\ a_{13}w_1 + a_{23}w_2 + a_{33}w_3 - Ma_{43}w_4 - Ma_{53}w_5 = 0 \\ a_{14}w_1 + a_{24}w_2 + a_{34}w_3 - Ma_{44}w_4 - Ma_{54}w_5 = 0 \end{cases}$$

$$w_1 + w_2 + w_3 = 1$$

$$Mw_4 + Mw_5 = M$$

- These 6 equations can be solved for the 6 unknowns $w_1, w_2, w_3, Mw_4, Mw_5, M$

Chapter 11. Geometric Programming

11.8 Constrained Optimization with Zero Degree of Difficulty

- Because u_i is a polynomials of x_i, all of the x terms in u can be cancelled
- So the y equation can be simplified as

$$\therefore \boldsymbol{y} = \boldsymbol{g} = \left(\frac{\boldsymbol{c_1}}{\boldsymbol{w_1}}\right)^{\boldsymbol{w_1}} \left(\frac{\boldsymbol{c_2}}{\boldsymbol{w_2}}\right)^{\boldsymbol{w_2}} \left(\frac{\boldsymbol{c_3}}{\boldsymbol{w_3}}\right)^{\boldsymbol{w_3}} \left(\frac{\boldsymbol{c_4}}{\boldsymbol{w_4}}\right)^{\boldsymbol{Mw_4}} \left(\frac{\boldsymbol{c_5}}{\boldsymbol{w_5}}\right)^{\boldsymbol{Mw_5}}$$

Example 11.5

- A water pipeline extends 30 km. Select the number of pumps and the pipe diameter that results in the minimum total cost for the system.

(Given)

Cost of each pump = $2500 + 0.00032\Delta p^{1.2}(\$)$

Cost of 30 km of pipe = $2{,}560{,}000D^{1.5}(\$)$

Δp: pressure drop in each pipe, Pa

D : diameter of pipe, m

- A friction factor : 0.02
- A flow rate : 0.16 ㎥/s

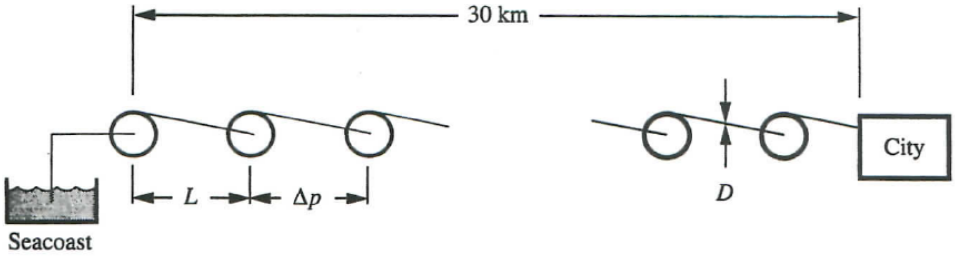

Fig. Water pipeline in Example 11.5

Example 11.5

(Solution)

$$y = n(2500 + 0.00032\Delta p^{1.2}) + 2{,}560{,}000D^{1.5} \qquad n = \frac{30{,}000\ m}{L}$$

- Where L is the length of each pipe section (m)

$$\Delta p = f\frac{L}{D}\frac{V^2}{2}\rho = (0.02)\frac{L}{D}\left(\frac{0.16}{\frac{\pi D^2}{4}}\right)^2\frac{1}{2}(1000\ kg/m^3) \qquad \rightarrow \qquad \frac{\Delta p D^5}{L} = 0.4150$$

- The statement of the problem is

$$y = \frac{75{,}000{,}000}{L} + \frac{9.6\Delta p^{1.2}}{L} + 2{,}560{,}000D^{1.5}$$

$$\frac{2.410\Delta p D^5}{L} = 1$$

Example 11.5

(Solution)

$$y^* = \left(\frac{75{,}000{,}000}{w_1}\right)^{w_1} \left(\frac{9.6}{w_2}\right)^{w_2} \left(\frac{2{,}560{,}000}{w_3}\right)^{w_3} \left(\frac{2.41}{w_4}\right)^{w_4}$$

- Provided that

$$\begin{aligned}
L:&\quad -w_1 - w_2 \qquad\qquad - Mw_4 = 0\\
\Delta p:&\quad 1.2w_2 \qquad\qquad + Mw_4 = 0\\
D:&\quad 1.5w_3 + 5Mw_4 = 0
\end{aligned}$$

$$\begin{aligned}
w_1 + w_2 + w_3 &= 1\\
Mw_4 &= M
\end{aligned}$$

$$\therefore w_1 = 0.0385,\ w_2 = 0.1923,\ w_3 = 0.7692,\ M = -0.2308,\ Mw_4 = -0.2308$$

Example 11.5

(Answer)

$$y^* = \left(\frac{75{,}000{,}000}{0.0385}\right)^{0.0385} \left(\frac{9.6}{0.1923}\right)^{0.1923} \left(\frac{2{,}560{,}000}{0.7692}\right)^{0.7692} \left(\frac{2.41}{1}\right)^{-0.2308} = \$410{,}150$$

$$u_1^* = (410{,}150)(0.0385) = \frac{75{,}000{,}000}{L^*} \quad \rightarrow \quad L^* = 4750\ m$$

$$u_3^* = (410{,}150)(0.769) = 2{,}560{,}000 D^{1.5} \quad \rightarrow \quad D^* = 0.246\ m$$

$$\Delta p^* = \frac{L^*}{2.410 D^{*5}} = 2{,}188{,}000\ Pa = 2188\ kPa$$