Optimal Design of Energy Systems (M2794.003400)

Chapter 12. Linear Programming

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Chapter 12. LINEAR PROGRAMMING

12.1 The origins of Linear Programming

objective function
constraints
linear combinations
equality or inequality

1930s, economic models

1947 USAF simplex method

(United States Air Force)

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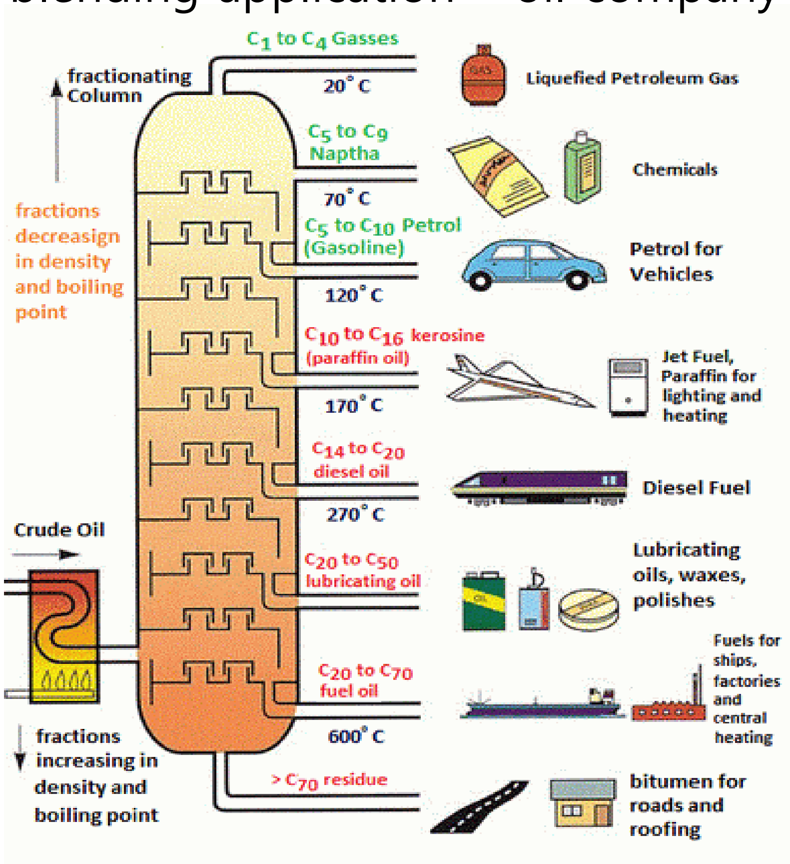
12.2 Some examples

(1) blending application – oil company (2) machine allocation – manufacturing plant

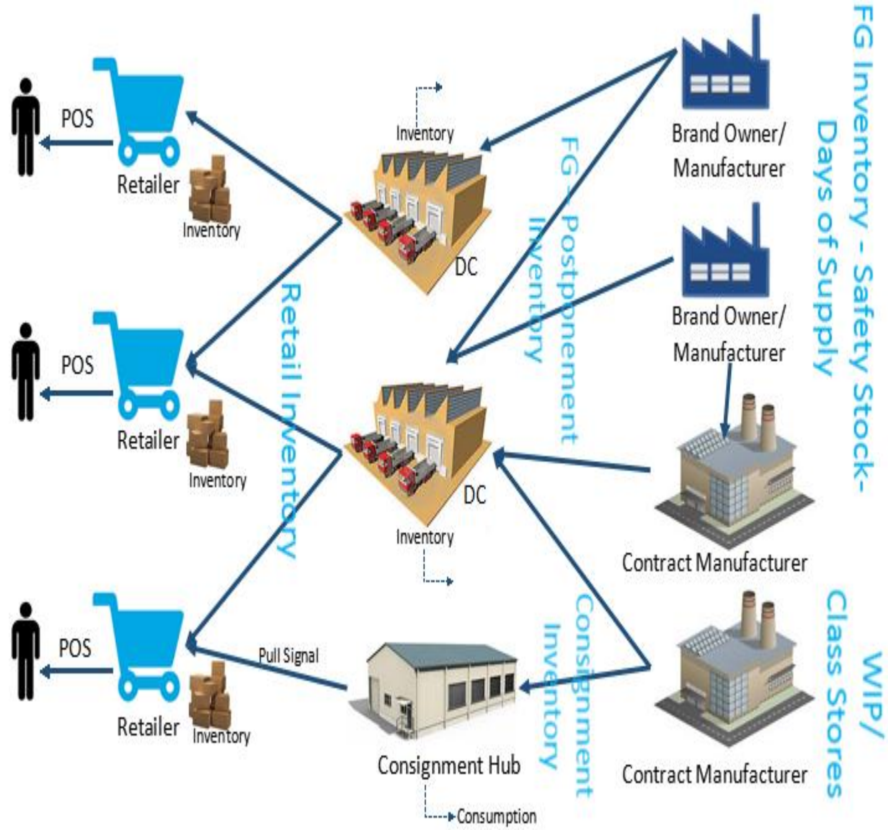


Worldofchemicals.com

Hyundai Motors, in Beiging

12.2 Some examples

(3) inventory and production planning



(4) transportation

Solvanni.com

Formacionindustrial.es

12.3 Mathematical statement

objective function

$$y = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

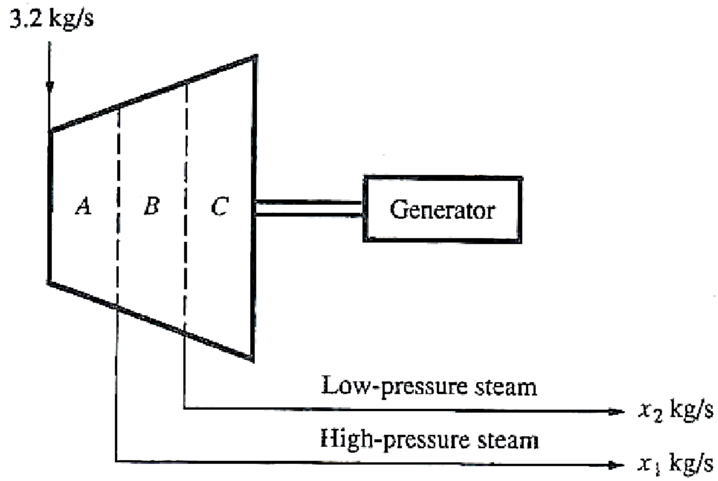
Constraints

$$\left.\begin{aligned} \emptyset_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq r_1 \\ &\vdots \\ \emptyset_m &= a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq r_m \end{aligned}\right\} \text{\underline{inequality} constraints}$$

cf) <u>Lagrange method</u> is applicable for <u>equality</u> constraints

12.4 Developing the mathematical statement

Example 12.1 : Objective function of revenue and constraint equations



Revenue rates

Electricity : 0.03 $/kWh

Low-pressure steam : 1.10 $/Mg

High-pressure steam : 1.65 $/Mg

Flow rates w_A, w_B, w_C

$P_A, \mathrm{kW} = 48w_A$

$P_B, \mathrm{kW} = 56w_A$

$P_C, \mathrm{kW} = 80w_A$ (w : kg/s)

No less than 0.6kg/s must always flow through section C

Permissible combination of extraction rates : If x_1 =0 → x_2 ≤1.8 kg/s

For each x_1 kg extracted 0.25 kg less can be extracted of x_2

The customer of the process steam is primarily interested in total energy and will purchase no more than $4x_1 + 3x_2 \leq 9.6$

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12.4 Developing the mathematical statement

(Solution)

$$\text{Revenue} = \frac{1.65}{1000}(3600x_1) + \frac{1.10}{1000}(3600x_2) + 0.03(48w_A + 56w_B + 80w_C)$$

$$(w_A = 3.2kg/s\ ,\ w_B = 3.2 - x_1, w_C = 3.2 - x_1 - x_2)$$

$$= 17.66 + 1.86x_1 + 1.56x_2$$

Maximize $y = 1.86x_1 + 1.56x_2$

Constraints $x_1 + x_2 \leq 2.6$

$x_1 + 4x_2 \leq 7.2$

$4x_1 + 3x_2 \leq 9.6$

12.5 Geometric Visualization of the Linear-Programming Problem

(Solution)

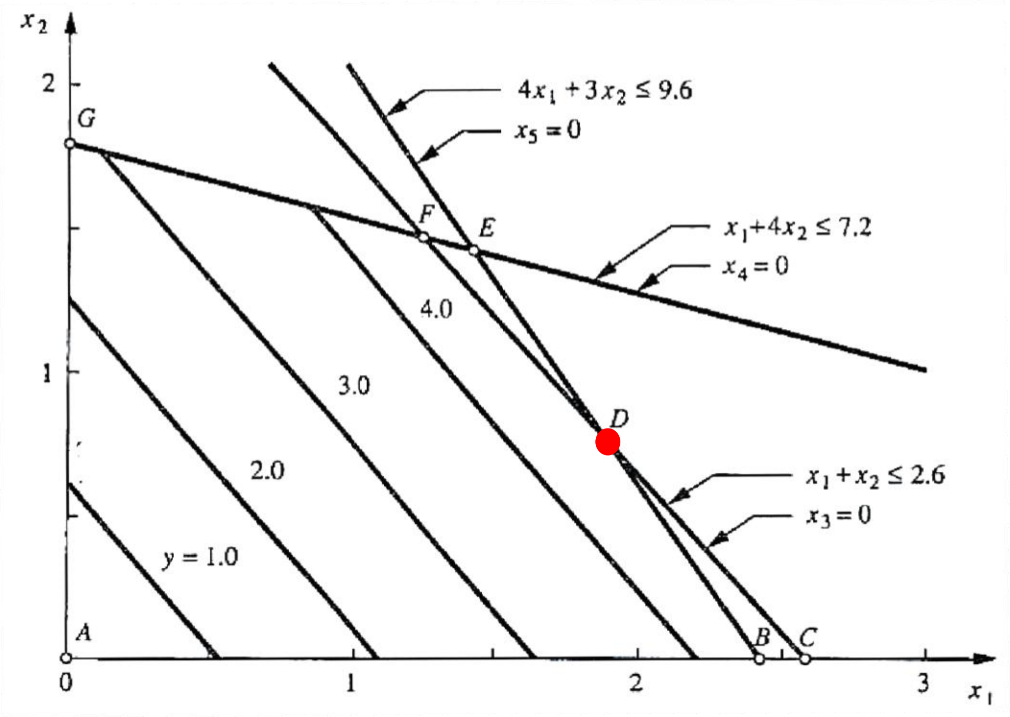

Fig. Constraints and lines of constant profit

- Permitted region : ABDFG
- Optimal point : D
- Optimum solution lies at a corner

12.6 Introduction of Slack Variables

(Solution)

From Ex 12.1 inequalities can be converted into equalities by introduction of another variable in each equation.

$$x_1 + x_2 + x_3 = 2.6 \qquad x_3 \geq 0$$

$$x_1 + 4x_2 + x_4 = 7.2 \qquad x_4 \geq 0$$

$$4x_1 + 3x_2 + x_5 = 9.6 \qquad x_5 \geq 0$$

Slack variables : $x_3,\ x_4,\ x_5$

12.7~8 Preparation for the simplex algorithm, Including the objective function in the tableau

(Solution)

objective function : $y - 1.86x_1 - 1.56x_2 = 0$

	X_1	x_2	x_3	x_4	x_5	
	1	1	1			2.6
	1	4		1		7.2
	4	3			1	9.6
	-1.86	-1.56				0

current value of objective function

12.9 Starting at the origin

(Solution)

Move from one corner to the next corner starting point $x_1=0$, $x_2=0$

	$x_1=0$	$x_2=0$	x_3	x_4	x_5	
	1	1	1			2.6
	1	4		1		7.2
	4	3			1	9.6
	-1.86	-1.56				0

12.10 The simplex algorithm

(Solution)

1. Decide the variable

   Maximization – largest negative difference coefficient
   Minimization – largest positive difference coefficient

2. Determine the controlling constraint

3. Transfer of the controlling constraint

4. For all other boxes

New value = $v - wz$

old value

value in the preview controlling equation

coefficient of the variable being programmed

12.11 Solution of Example 12.1

Table 1	x_1=0	x_2=0	x_3	x_4	x_5	
2.6/1=2.6	1	1	1			2.6
7.2/1=7.2	1	4		1		7.2
9.6/4=2.4	4	3			1	9.6
	-1.86	-1.56				0

controlling constraint (smallest)

Step 1 : largest negative

x_1 should be programmed (increased from zero)

Step 2 : How much x_1 can be increased?

$$x_1 = 0 \rightarrow x_1 \neq 0$$

$$x_2 = 0 \rightarrow x_2 = 0$$

$$x_5 \neq 0 \rightarrow x_1 = 0$$

x_1 *increases until* x_5 *becomes zero.*

$(In\ Fig. 12-2, A \rightarrow B)$

12.11 Solution of Example 12.1

Step 3 :

	x_1	$x_2=0$	x_3	x_4	$x_5=0$	
	1-(1)(1)	1-(1)(0.75)	1-(1)(0)	0-(1)(1)	0-(1)(0.25)	2.6-(1)(2.4)
	1-(1)(1)	4-(1)(0.75)	0-(1)(0)	1-(1)(0)	0-(1)(0.25)	7.2-(1)(2.4)
÷ 4	1	0.75	0	0	0.25	2.4
	-1.86-(-1.86)(1)	-1.56-(-1.86)(0.75)	0-(-1.86)(0)	0-(-1.86)(0)	0-(-1.86)(0.25)	0-(-1.86)(2.4)

Step 4 :

12.11 Solution of Example 12.1

Table 2	x_1	$x_2=0$	x_3	x_4	$x_5=0$	
0.2/0.25=0.8	0	0.25	1	0	-0.25	0.20
4.8/3.25=1.48	0	3.25	0	1	-0.25	4.8
2.4/0.75=3.2	1	0.75	0	0	0.25	2.4
controlling constraint (smallest)	0	-0.165	0	0	0.465	4.464

Step 1 : largest negative (x_2 is programmed next)

$$\rightarrow\ x_3 = 0.2,\quad x_4 = 4.8,\quad x_1 = 2.4, y = 4.4464$$

12.11 Solution of Example 12.1

Step 2 : x_2 increases to its limit until x_3 becomes zero

Step 3 :

	x_1	x_2	$x_3=0$	x_4	$x_5=0$	
÷0.25	0	1	4	0	-1	0.8
	0-(3.25)(0)	3.25-(3.25)(1)	0-(3.25)(4)	1-(3.25)(0)	-0.25-(3.25)(-1)	4.8-(3.25)(0.8)
	1-(0.75)(0)	0.75-(0.75)(1)	0-(0.75)(4)	0-(0.75)(0)	0.25-(0.75)(-1)	2.4-(0.75)(0.8)
	0-(-0.165)(0)	-0.165-(-0.165)(1)	0-(-0.165)(4)	0-(-0.165)(0)	0.465-(-0.165)(-1)	4.464-(-0.165)(0.8)

Step 4 :

12.11 Solution of Example 12.1

Table 3	x_1	x_2	$x_3=0$	x_4	$x_5=0$	
	0	1	4	0	-1	0.80
	0	0	-13	1	3	2.2
	1	0	-3	0	1	1.8
	0	0	0.66	0	0.3	4.596

$$\therefore\ x_2 = 0.8,\quad x_1 = 1.8,\quad x_4 = 2.2, y = 4.596$$

→ no negative coefficients

→ no further improvement is possible (second constraint has no influence)

12.12 Another Geometric Interpretation of Table Transformation

└ by changing the coordinates so that the current point is always at origin

$$y = 1.86x_1 + 1.56x_2$$

$$\begin{cases} x_1 + x_2 + x_3 = 2.6 \\ x_1 + 4x_2 + x_4 = 7.2 \\ 4x_1 + 3x_2 + x_5 = 9.6 \end{cases}$$

Table 1	$x_1,\ x_2$	- physical variables	$x_1 = 0,\ x_2 = 0$	origin
Table 2	$x_2,\ x_5$		$x_2 = 0,\ x_5 = 0$	
Table 3	$x_3,\ x_5$		$x_3 = 0,\ x_5 = 0$	

12.12 Another Geometric Interpretation of Table Transformation

	Table 1		Table 2
3rd constraint	$4x_1 + 3x_2 + x_5 = 9.6$	$\rightarrow$	$x_1 = -0.75x_2 - 0.25x_5 + 2.4$
1st constraint	$x_1 + x_2 + x_3 = 2.6$	$\rightarrow$	$0.25x_2 + x_3 - 0.25x_5 = 0.2$
2nd constraint	$x_1 + 4x_2 + x_4 = 7.2$	$\rightarrow$	$3.25x_2 + x_4 - 0.25x_5 = 4.8$
Objective function	$y - 1.86x_1 + 1.56x_2 = 0$	$\rightarrow$	$y - 0.165x_2 + 0.465x_5 = 4.464$

12.12 Another Geometric Interpretation of Table Transformation

	Table 2		Table 3
1st constraint	$0.25x_2 + x_3 - 0.25x_5 = 0.2$	$\rightarrow$	$x_2 = -4x_3 + x_5 + 0.8$
2nd constraint	$3.25x_2 + x_4 - 0.25x_5 = 4.8$	$\rightarrow$	$-13x_3 + x_4 + 3x_5 = 2.2$
3rd constraint	$x_1 = -0.75x_2 - 0.25x_5 + 2.4$	$\rightarrow$	$x_1 - 3x_3 + x_5 = 1.8$
Objective function	$y - 0.165x_2 + 0.465x_5 = 4.464$	$\rightarrow$	$y + 0.66x_3 + 0.3x_5 = 4.596$

12.12 Another Geometric Interpretation of Table Transformation

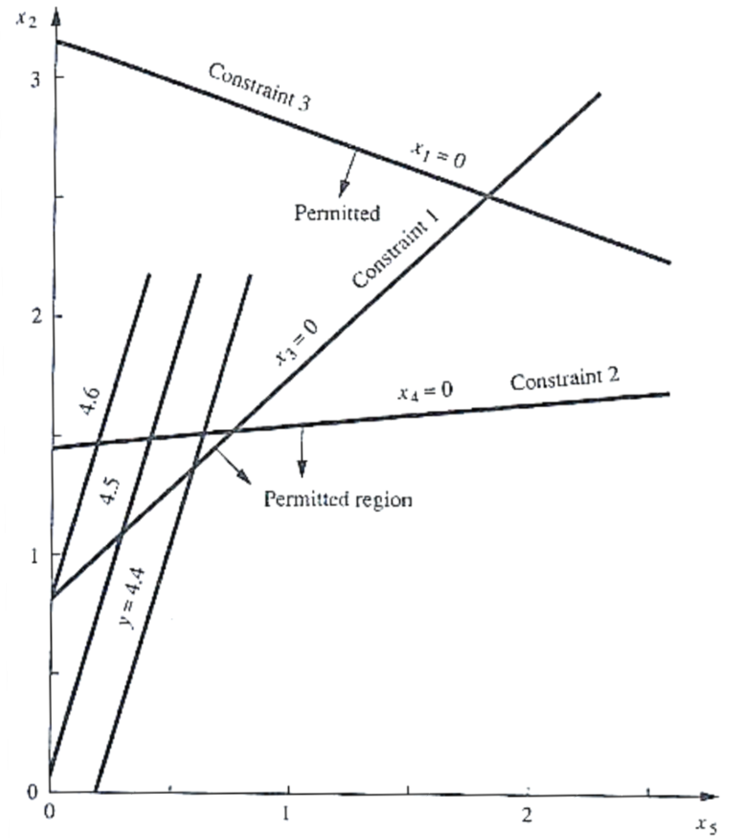

FIGURE 12-3
Tableau 2 expressed on x_5x_2 coordinates.

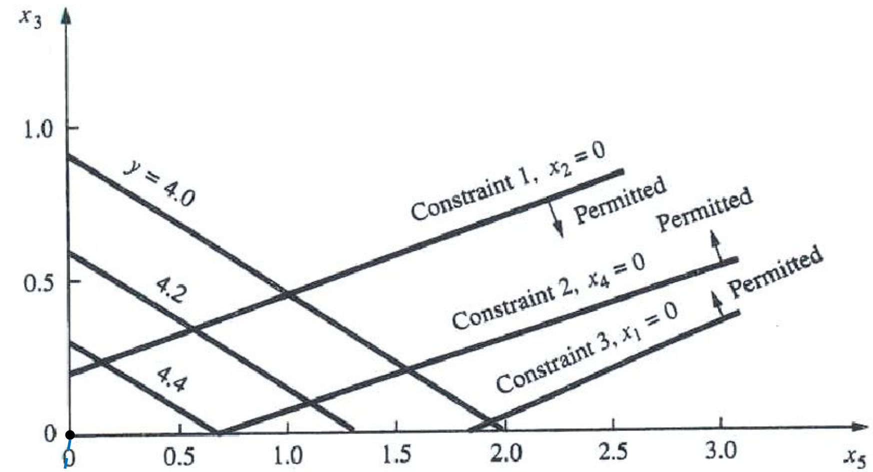

FIGURE 12-4
Tableau 3 expressed on x_5x_3 coordinates.

$$y = 4.596 \quad at \quad x_3 = 0, x_5 = 0$$

12.14 # of variables and # of constraints

n (m = # of slack variables)

At optimum, n variables are zero (corner)

m>n m-n constraints play no role

m<n n-m variables are zero

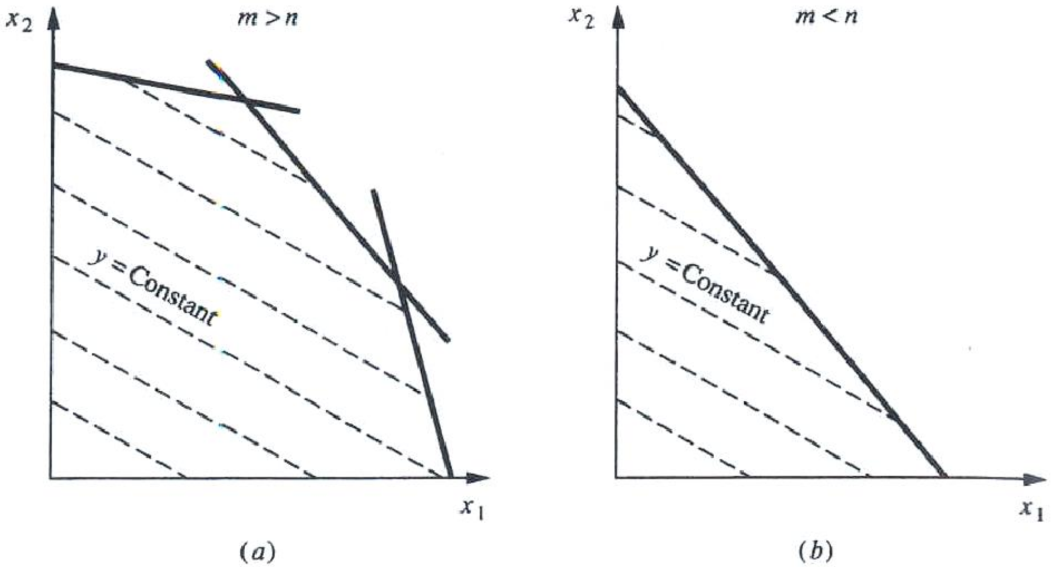

FIGURE 12-5
Relation of number of physical and slack variables.

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12.15 Minimization with greater than constraints

- ✓ Maximization with less than constraints
  - Moving from one corner to another adjacent corner
    (start from the origin)
- ✓ Minimization with greater than constraints
  - Locating the first feasible point - difficult
    → introduction of artificial variable (12.16)

12.16 Artificial variables

$$3x_1 + 4x_2 \geq 12$$

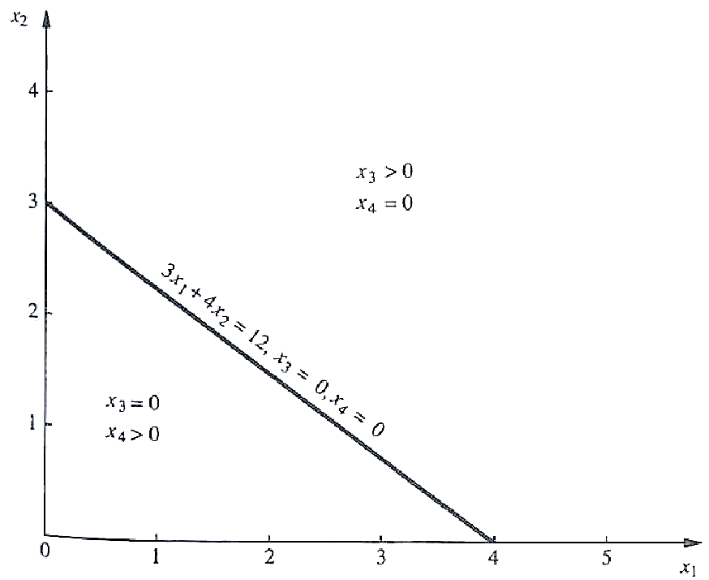
$$3x_1 + 4x_2 - x_3 = 12 \qquad\qquad x_3 \geq 0$$

If $x_1 = x_2 = 0$ (origin), it is not realistic

$$3x_1 + 4x_2 - x_3 + x_4 = 12$$

artificial variable

slack variable



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12.17 Simplex algorithm to minimization problem

Example 12.2 : Minimum value y, magnitudes $\boldsymbol{x_1, x_2}$
(at the minimum : $\boldsymbol{y = 6_{x1} + 3_{x2}}$)

Subject to the constraints

$$\boldsymbol{5_{x1} + x_2 \geq 10, \quad 9_{x1} + 13_{x2} \geq 74, \quad x_1 + 3_{x2} \geq 9}$$

(Solution)

$$\begin{cases} 5x_1 + x_2 - x_3 + x_6 = 10 \\ 9x_1 + 13x_2 - x_4 + x_7 = 74 \\ x_1 + 3x_2 - x_5 + x_8 = 9 \end{cases}$$

$$y = 6x_1 + 3x_2 + Px_6 + Px_7 + Px_8$$

P → A numerical value which is extremely large

$x_3,\ x_4,\ x_5$ → slack variables

$x_6,\ x_7,\ x_8$ → artificial variables

12.17 Simplex algorithm to minimization problem

(Solution)

Starting point-origin with all slack variable=0

artificial variable>0

$x_6 = 10 - 5x_1 - x_2 + x_3$

$x_7 = 74 - 9x_1 - 13x_2 + x_4$

$x_8 = 9 - x_1 - 3x_2 + x_5$

$y = (6 - 15P)x_1 + (3 - 17P)x_2 + Px_3 + Px_4 + Px_5 + 93P$

12.17 Simplex algorithm to minimization problem

(Solution)

Table 1		x_1	x_2	x_3	x_4	x_5	
10	x_6	5	1	-1	0	0	10
74/13	x_7	9	13	0	-1	0	74
3	x_8	1	3	0	0	-1	9
		15P-6	17P-3	-P	-P	-P	93P

controlling constraint
(smallest)

largest positive
difference coefficient

12.17 Simplex algorithm to minimization problem

(Solution)

Table 1 **Table 2**

$$x_6 = 10 - 5x_1 - x_2 + x_3 \rightarrow x_6 = 7 - \left(\frac{14}{3}\right)x_1 + x_3 - \left(\frac{1}{3}\right)x_5 + \left(\frac{1}{3}\right)x_8$$

$$x_7 = 74 - 9x_1 - 13x_2 + x_4 \rightarrow x_7 = 35 - \left(\frac{14}{3}\right)x_1 + x_4 - \left(\frac{13}{3}\right)x_5 + \left(\frac{13}{3}\right)x_8$$

$$x_8 = 9 - x_1 - 3x_2 + x_5 \rightarrow x_2 = 3 - \left(\frac{1}{3}\right)x_1 + \left(\frac{1}{3}\right)x_5 - \left(\frac{1}{3}\right)x_8$$

$$y = (6 - 15P)x_1 + (3 - 17P)x_2 + Px_3 + Px_4 + Px_5 + 93P$$

$$\rightarrow y = \frac{15 - 28P}{3}x_1 + Px_3 + Px_4 + \frac{3 - 14P}{3}x_5 + \frac{-3 + 17P}{3}x_8 + 42P + 9$$

12.17 Simplex algorithm to minimization problem

(Solution)

Table 2		x_1	x_3	x_4	x_5	x_8	
3/2	x_6	14/3	-1	0	1/3	-1/3	7
105/14	x_7	14/3	0	-1	13/3	-13/3	35
9	x_2	1/3	0	0	-1/3	1/3	3
		(28P-15)/3	-P	-P	(14P-3)/3	(-17P+3)/3	42P+9

controlling constraint
(smallest)

largest positive
difference coefficient

12.17 Simplex algorithm to minimization problem

(Solution)

Table 2 **Table 3**

$$x_6 = 7 - \left(\frac{14}{3}\right)x_1 + x_3 - \left(\frac{1}{3}\right)x_5 + \left(\frac{1}{3}\right)x_8 \rightarrow x_1 = \frac{3}{2} + \left(\frac{3}{14}\right)x_3 - \left(\frac{1}{14}\right)x_5 - \left(\frac{3}{14}\right)x_6 + \left(\frac{1}{14}\right)x_8$$

$$x_7 = 35 - \left(\frac{14}{3}\right)x_1 + x_4 - \left(\frac{13}{3}\right)x_5 + \left(\frac{13}{3}\right)x_8 \rightarrow x_7 = 28 - x_3 + x_4 - 4x_5 + x_6 + 4x_8$$

$$x_2 = 3 - \left(\frac{1}{3}\right)x_1 + \left(\frac{1}{3}\right)x_5 - \left(\frac{1}{3}\right)x_8 \rightarrow x_2 = \frac{5}{2} - \left(\frac{1}{14}\right)x_3 + \left(\frac{5}{14}\right)x_5 + \left(\frac{1}{14}\right)x_6 - \left(\frac{5}{14}\right)x_8$$

$$y = \frac{15 - 28P}{3}x_1 + Px_3 + Px_4 + \frac{3 - 14P}{3}x_5 + \frac{-3 + 17P}{3}x_5 + 42P + 9$$

$$\rightarrow y = \frac{15 - 14P}{14}x_3 + Px_4 + \frac{9 - 56P}{14}x_5 + \frac{-15 + 28P}{14}x_6 + \frac{-9 + 70P}{14}x_8 + \frac{56P + 33}{2}$$

12.17 Simplex algorithm to minimization problem

(Solution)

Table 3		x_3	x_4	x_5	x_6	x_8	
21	x_1	-3/14	0	1/14	3/14	-1/14	3/2
7	x_7	1	-1	4	-1	-4	28
-7	x_2	1/14	0	-5/14	-1/14	5/14	5/2
		(14P-15)/14	-P	(56P-9)/14	(-28P+15)/14	(-70P+9)/14	(56P+33)/2

controlling constraint (smallest)

largest positive difference coefficient

12.17 Simplex algorithm to minimization problem

(Solution)

Table 3 **Table 4**

$$x_1 = \frac{3}{2} + \left(\frac{3}{14}\right)x_3 - \left(\frac{1}{14}\right)x_5 - \left(\frac{3}{14}\right)x_6 + \left(\frac{1}{14}\right)x_8 \rightarrow x_1 = 1 + \left(\frac{13}{56}\right)x_3 - \left(\frac{1}{56}\right)x_4 - \left(\frac{13}{56}\right)x_6 + \left(\frac{1}{56}\right)x_7$$

$$x_7 = 28 - x_3 + x_4 - 4x_5 + x_6 + 4x_8 \rightarrow x_5 = 7 - \left(\frac{1}{4}\right)x_3 + \left(\frac{1}{4}\right)x_4 + \left(\frac{1}{4}\right)x_6 - \left(\frac{1}{4}\right)x_7 + x_8$$

$$x_2 = \frac{5}{2} - \left(\frac{1}{14}\right)x_3 + \left(\frac{5}{14}\right)x_5 + \left(\frac{1}{14}\right)x_6 - \left(\frac{5}{14}\right)x_8 \rightarrow x_2 = 5 - \left(\frac{9}{56}\right)x_3 + \left(\frac{5}{56}\right)x_4 + \left(\frac{9}{56}\right)x_6 - \left(\frac{5}{56}\right)x_7$$

$$y = \frac{15 - 14P}{14}x_3 + Px_4 + \frac{9 - 56P}{14}x_5 + \frac{-15 + 28P}{14}x_6 + \frac{-9 + 70P}{14}x_8 + \frac{56P + 33}{2}$$

$$\rightarrow \mathrm{y} = \frac{51}{56}x_3 + \frac{9}{56}x_4 + \left(P - \frac{15}{56}\right)x_6 + \left(P - \frac{9}{56}\right)x_7 + Px_8 + 21$$

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12.17 Simplex algorithm to minimization problem

(Solution)

Table 4	x_3	x_4	x_6	x_7	x_8	
x_1	-13/56	1/56	13/56	-1/56	0	1
x_5	1/4	-1/4	-1/4	¼	-1	7
x_2	9/56	-5/56	-9/56	5/56	0	5
	-51/56	-9/56	-P+51/56	-P+9/56	-P	21

$$\therefore x_1 = 1, \quad x_5 = 7, \quad x_2 = 5, \quad y = 21$$

12.17 Simplex algorithm to minimization problem

(Solution)

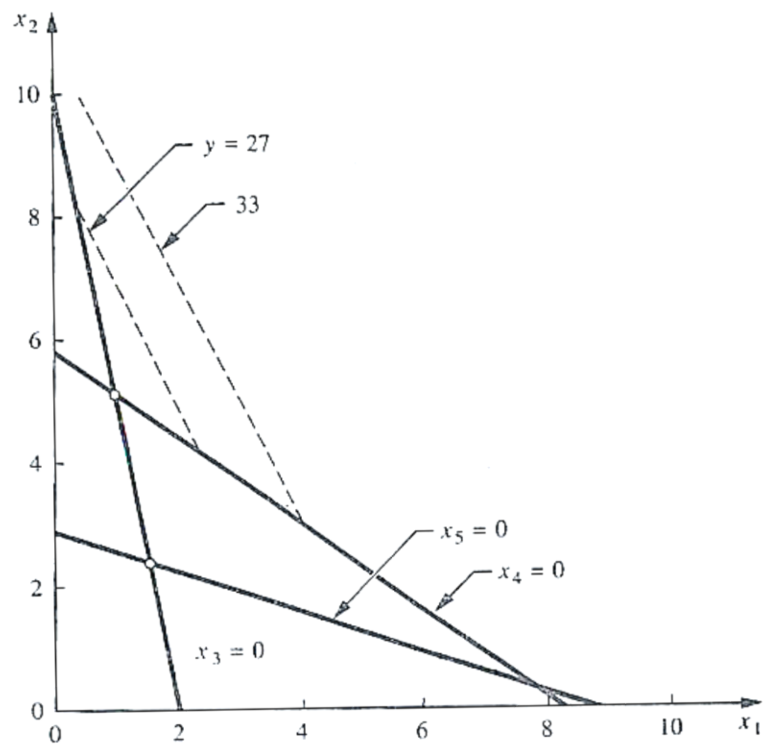

Fig. Minimization

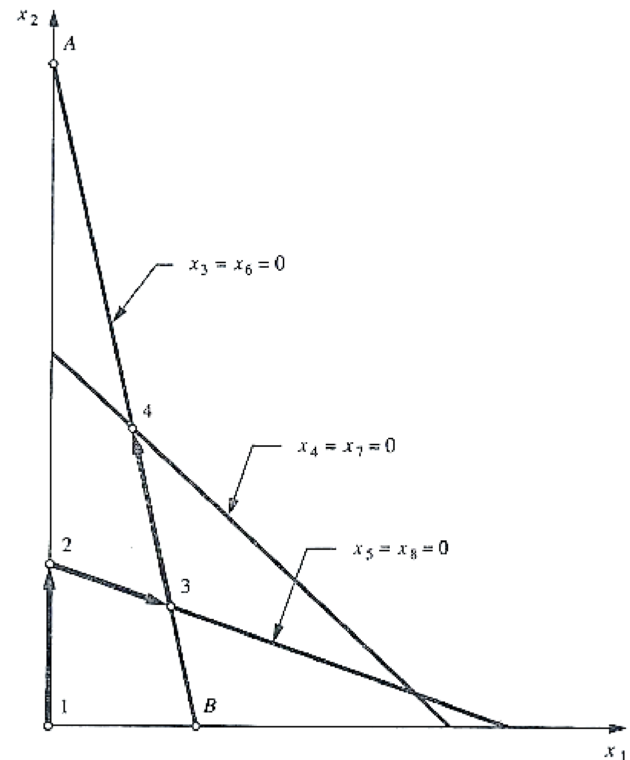

Fig. Points represented by successive tableaux