Chapter 18. Calculus of Variation and Dynamic Programming

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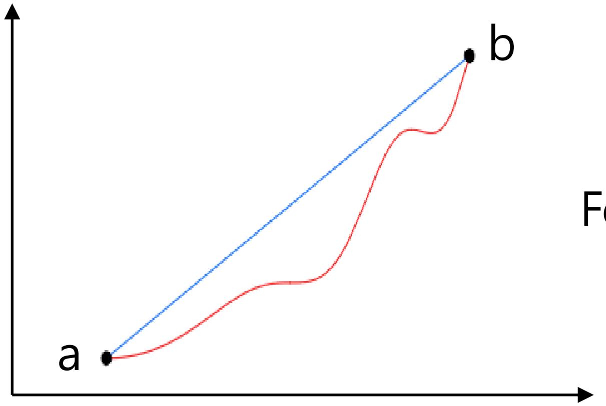
Chapter 18. Calculus of variation and dynamic programming

18.1 The relationship between calculus of variations and dynamic programming

- Dynamic programming method applies to problems seeking an **optimal function**, rather than an optimal point.
- Calculus of variation (COV) and dynamic programming (DP) are companion methods since both are techniques to determine the optimal function y(x).
- A difference between COV and DP is that DP breaks the function into discrete stages, while COV treats it as a continuous function.



For minimum $\int_a^b F(x)dx$

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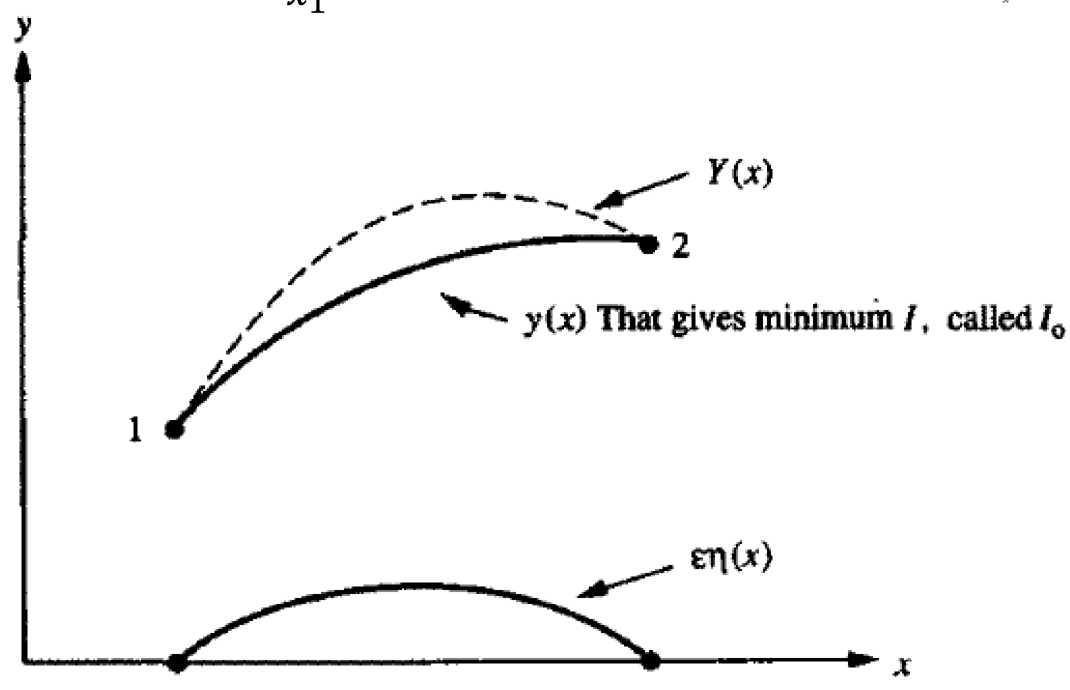
18.2 The Euler-Lagrange equation

- The Euler-Lagrange (E-L) equation is a basic tool in the calculus of variation (COV)

$$I = \int_{x_1}^{x_2} F(x, y, y')dx \qquad\qquad y = y(x)$$



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18.2 The Euler-Lagrange equation

Suppose

1. $y(x)$ is a solution

2. $\eta(x)$ is a test function with $\eta(x_1) = \eta(x_2) = 0$, ε is a numerical value

$$Y(x) = y(x) + \varepsilon\eta(x)$$

3. $\eta(x)$ is a test function, ε is a numerical value

$$I(\varepsilon) = \int_{x_1}^{x_2} F(x, y + \varepsilon\eta, y' + \varepsilon\eta')dx = \int_{x_1}^{x_2} F(x, Y, Y')dx$$

4. $I(\varepsilon)$ can be expressed as a Talyer series expansion of ε

$$I(\varepsilon) = I_o + \left[\left.\frac{\partial I(\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0}\right]\varepsilon + \frac{1}{2}\left[\left.\frac{\partial^2 I(\varepsilon)}{\partial \varepsilon^2}\right|_{\varepsilon=0}\right]\varepsilon^2 + \cdots \qquad I_o\text{: optimal of } I$$

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18.2 The Euler-Lagrange equation

5. Verify the existence of an optimal point,

$$\left.\frac{\partial I(\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} = \left[\frac{\partial}{\partial \varepsilon}\int_{x_1}^{x_2} F(x,Y,Y')dx\right]_{\varepsilon=0} = 0$$

6. Applied the chain rule,

$$\frac{\partial x}{\partial \varepsilon} = 0\ ,\ \ \frac{\partial Y}{\partial \varepsilon} = \eta\ ,\ \ \frac{\partial Y'}{\partial \varepsilon} = \eta'$$

$$\left[\int_{x_1}^{x_2}\left(\frac{\partial F}{\partial x}\frac{\partial x}{\partial \varepsilon} + \frac{\partial F}{\partial Y}\frac{\partial Y}{\partial \varepsilon} + \frac{\partial F}{\partial Y'}\frac{\partial Y'}{\partial \varepsilon}\right)dx\right]_{\varepsilon=0} = \left[\int_{x_1}^{x_2}\left(\frac{\partial F}{\partial Y}\eta + \frac{\partial F}{\partial Y'}\eta'\right)\right]_{\varepsilon=0} dx$$

$$= \int_{x_1}^{x_2}\frac{\partial F}{\partial y}\eta dx + \underline{\int_{x_1}^{x_2}\frac{\partial F}{\partial y'}\eta' dx} = 0$$

$$\int_{x_1}^{x_2}\frac{\partial F}{\partial y'}\eta' dx = \frac{\partial F}{\partial y}[\overset{0}{\cancel{\eta(x_2)}} - \overset{0}{\cancel{\eta(x_1)}}] - \int_{x_1}^{x_2}\frac{d(\partial F/\partial y')}{dx}\eta(x)dx$$

$$= \int_{x_1}^{x_2}\left[\frac{\partial F}{\partial y} - \frac{d(\partial F/\partial y')}{dx}\right]\eta(x)dx = 0$$

18.2 The Euler-Lagrange equation

7. $\eta(x)$ can be arbitrary, so the integral is always zero

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

Euler-Lagrange equation

8. The E-L equation general form,

$$\frac{\partial F}{\partial y} - \frac{\partial(\partial F/\partial y')}{\partial x}\frac{dx}{dx} - \frac{\partial(\partial F/\partial y')}{\partial y'}\frac{dy'}{dx} - \frac{\partial(\partial F/\partial y')}{\partial y}\frac{dy}{dx} = 0$$

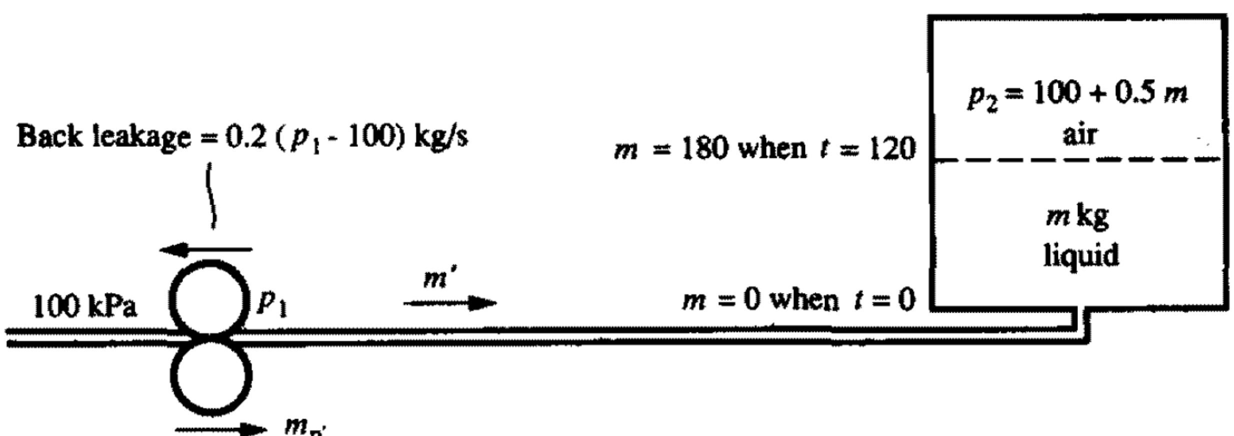
$$F_y - F_{xy} - F_{y'y'}y'' - F_{yy'}y' = 0$$

Example 18.1

- Determine the equation for m' as a function of time t such that **minimum pumping energy** is required during the process

(Given)

- time, t : 0 → 120 s, vessel pressure p : 100 → 190 kPa, liquid mass m : 0 → 180 kg
- Laminar flow, $p_1 - p_2 = 15m'$, vessel pressure $p_2 = 100 + 0.5m$
- Pumping power $m_p'(p_1 - 100)$, back leakage $m' = m_p' - 0.2(p_1 - 100)$



m : liquid mass in the vessel [kg]
m' : flow rate [kg/s]
m_p' : ideal pump flow rate [kg/s]

Fig. Pump delivering hydraulic fluid through a long pipe to build up pressure in a vessel.

Example 18.1

(Solution)

- The integral to minimize

$$I = \int_0^{120} m_p'\underline{(p_1 - 100)}dt = \int_0^{120} [m' + 0.2(p_1 - 100)](p_1 - 100)dt$$

$$p_1 - 100 = 15m' + p_2 - 100 = 15m' + 0.5m$$

$$= \int_0^{120} [3.5mm' + 60(m')^2 + 0.05m^2]dt$$

- Subjecting F to the E-L equation of the form in Eq. (18.14)

$$m'' - 0.0008333m = 0 \qquad F_y - F_{xy} - F_{y'y'}y'' - F_{yy'}y' = 0$$

$$m = C_1 \sinh\sqrt{0.0008333}\ t + C_2 \cosh 0.02887t$$

Example 18.1

(Solution)

- The boundary condition, $m(0) = 0$ required that $C_2 = 0$, so,

$$m = C_1 \sinh \sqrt{0.02887}\ t$$

- The second boundary condition, $m(120) = 180 \rightarrow\ C_1 = 180/\sinh 3.464\ = 11.28$

$$m = 11.28 \sinh(0.02887t)$$
$$m' = 0.3257 \cosh(0.02887t)$$

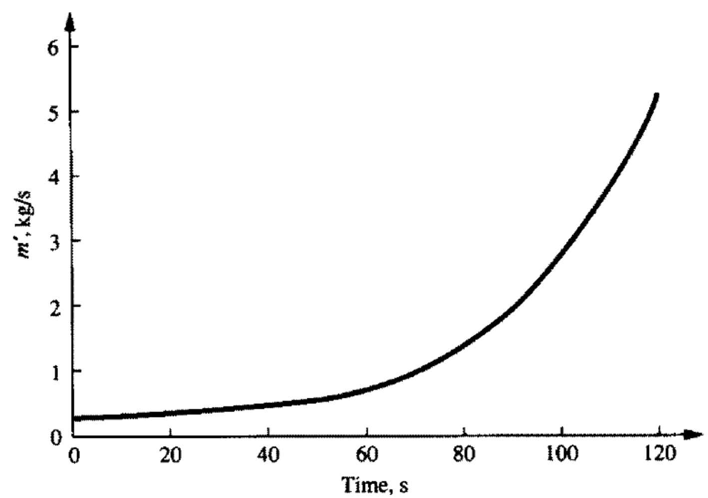


Fig. Optimal plan for flow rate to build up pressure

18.3 The Euler-Lagrange equation to optimize a constrained function

- The constrained problem is find y that minimizes I and J

$$I = \int_{x_1}^{x_2} F(x, y, y')dx\,, \quad J = \int_{x_1}^{x_2} G(x, y, y')dx$$

(J is a constant)

- The E-L equation for the constrained case will be shown to be

$$\frac{\partial(F - \lambda G)}{\partial x} - \frac{d}{dx}\left[\frac{\partial(F - \lambda G)}{\partial y'}\right] = 0$$

- Test function $y(x)$ as the one that minimizes I and J

$$Y(x) = y(x) + \alpha\eta(x) + \beta\gamma(x)$$

(α, β are constants)

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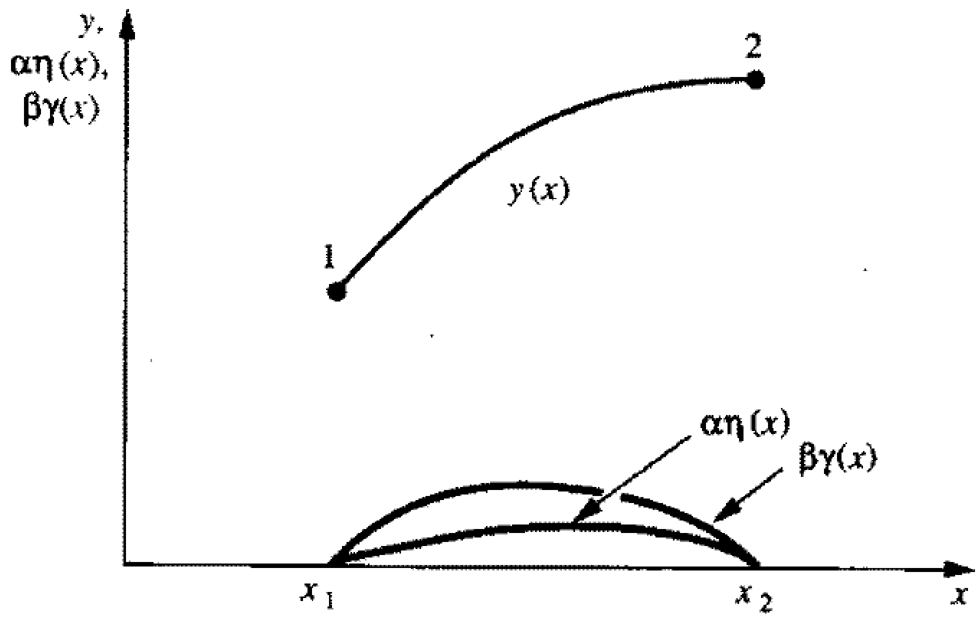
18.3 The Euler-Lagrange equation to optimize a constrained function

- Replace y and y' to the revised function Y and Y'

$$I = \int_{x_1}^{x_2} F(x, Y, Y')dx\,, \quad K = \int_{x_1}^{x_2} G(x, Y, Y')dx$$



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18.3 The Euler-Lagrange equation to optimize a constrained function

- Express I and K in Talyor series of α and β

$$I(\alpha,\beta) = I(0{,}0) + \alpha \left.\frac{\partial I}{\partial \alpha}\right|_{\alpha,\beta=0} + \beta \left.\frac{\partial I}{\partial \beta}\right|_{\alpha,\beta=0} + \cdots$$

$$K(\alpha,\beta) = K(0{,}0) + \alpha \left.\frac{\partial K}{\partial \alpha}\right|_{\alpha,\beta=0} + \beta \left.\frac{\partial K}{\partial \beta}\right|_{\alpha,\beta=0} + \cdots$$

- $I(0{,}0) = \int_{x_1}^{x_2} y(x)dx$, $K(0{,}0) = J$

- In order for $I(\alpha,\beta)$ and $K(\alpha,\beta)$ to be a minimum the first-degree terms

$$\left.\frac{\partial I}{\partial \alpha}\right|_{\alpha,\beta=0} \alpha + \left.\frac{\partial I}{\partial \beta}\right|_{\alpha,\beta=0} \beta = 0 \qquad \left.\frac{\partial K}{\partial \alpha}\right|_{\alpha,\beta=0} \alpha + \left.\frac{\partial K}{\partial \beta}\right|_{\alpha,\beta=0} \beta = 0$$

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18.3 The Euler-Lagrange equation to optimize a constrained function

- Combining the equations

$$\left.\begin{aligned} \left.\frac{\partial I}{\partial \alpha}\right|_{\alpha,\beta=0} \alpha + \left.\frac{\partial I}{\partial \beta}\right|_{\alpha,\beta=0} \beta = 0 \\ \left.\frac{\partial K}{\partial \alpha}\right|_{\alpha,\beta=0} \alpha + \left.\frac{\partial K}{\partial \beta}\right|_{\alpha,\beta=0} \beta = 0 \end{aligned}\right\} \quad -\frac{\beta}{\alpha} = \frac{\partial I/\partial \alpha}{\partial I/\partial \beta} = \frac{\partial K/\partial \alpha}{\partial K/\partial \beta} \Rightarrow \frac{\partial I/\partial \alpha}{\partial K/\partial \alpha} = \lambda = \frac{\partial I/\partial \beta}{\partial K/\partial \beta}$$

- Rewritten the equations

$$\frac{\partial}{\partial \alpha}[I - \lambda K] = \frac{\partial}{\partial \alpha}\left[\int_{x_1}^{x_2} F(x, Y, Y')dx - \lambda \int_{x_1}^{x_2} G(x, Y, Y')dx\right] = 0$$

$$\frac{\partial}{\partial \beta}[I - \lambda K] = \frac{\partial}{\partial \beta}\left[\int_{x_1}^{x_2} F(x, Y, Y')dx - \lambda \int_{x_1}^{x_2} G(x, Y, Y')dx\right] = 0$$

- The E-L equation for a constrained function

$$\frac{\partial(F - \lambda G)}{\partial x} - \frac{d}{dx}\left[\frac{\partial(F - \lambda G)}{\partial y'}\right] = 0$$

Example 18.2

- Calculate the pump power by solving the constrained problem

(Given)

- 40 m^3 tank is to be completely pumped in 1200 second
- The pressure difference : $\Delta p = (360 + 6t)V'$
- The mean pressure difference : $\dfrac{1}{1200}\displaystyle\int_0^{1200} \Delta p \, dt = 210$ [kPa]

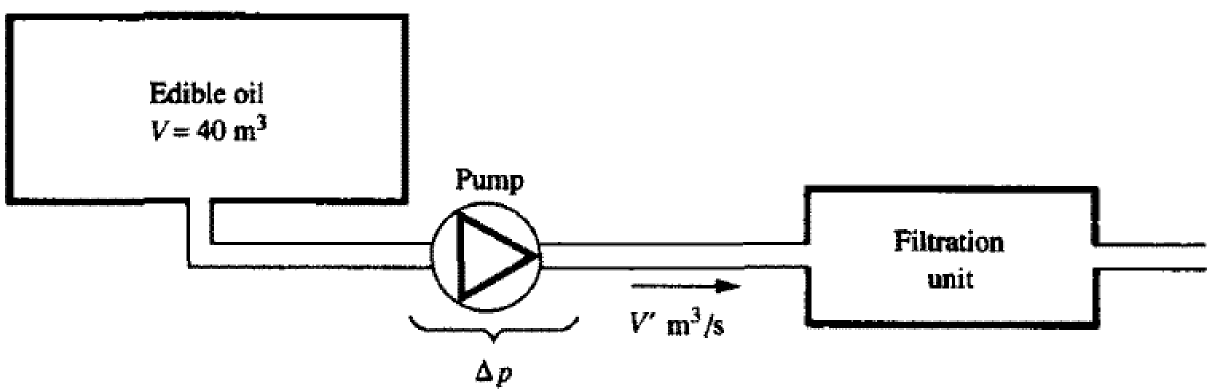

Fig. Pumping edible oil from a tank through a filter

Example 18.2

(Solution)

- Pump power $\Delta p\, V'$ [kW] is to be minimized

$$\int_0^{1200} (3600+6t)(V')^2 dt = 252{,}000 \qquad \begin{cases} F = (3600+6t)(V')^2 \\ G = (3600+6t)V' \end{cases}$$

- Solve for the minimum energy function **without** the constraint

$$\begin{cases} V = 36.41\ln(3600+6t) - 298.15 & [\mathrm{m}^3] \\ V' = 218.46/(3600+6t) & [\mathrm{m}^3/\mathrm{s}] \\ \Delta p = 218.46 & [\mathrm{kPa}] \end{cases}$$

$$\therefore\ \Delta p\, V' = 8738 \quad [\mathrm{kJ}]$$

Example 18.2

(Solution)

- Solve for the minimum energy function **with** the constraint

$$F - \lambda G = (3600 + 6t)[(V')^2 - \lambda V'] \ , \quad \frac{\partial(F - \lambda G)}{\partial V'} = (3600 + 6t)[2V' - \lambda] = constant$$

$$\frac{dV}{dt} = \frac{C_1}{3600 + 6t} + \frac{\lambda}{2} \xrightarrow{\text{integrating}} V - C_2 \ln(3600 + 6t) + \frac{\lambda}{2} t + C_3 = 0$$

- Boundary condition; $(V, t)_{\text{initial}} = (0,0)$, $(V, t)_{\text{final}} = (40,120)$

$$\therefore \begin{cases} C_2 = 36.41 - 546.1\lambda \\ C_3 = 4472\lambda - 298.2 \\ \lambda = -0.02616 \end{cases} \qquad \begin{cases} V = 50.7 \ln(3600 + 6t) - 0.01308t - 415.2 \\ V' = 304.2 \ln(3600 + 6t) - 0.01308 \\ \Delta p = 257.1 - 0.07848t \end{cases}$$

$$\therefore \Delta p \, V' = 8872 \quad [\text{kJ}]$$

Example 18.2

(Solution)

- Value of λ represents the sensitivity coefficient

$$\lambda = \frac{(p_{ave} - 218.46)}{323.4}$$

$$\lambda_{p_{ave}=210} = -0.02616$$

$$\lambda_{p_{ave}=214.23} = -0.01308$$

$$\lambda_{p_{ave}=218.46} = 0$$

- The mean value of λ by the difference in p_{ave},

$$(\lambda_{p_{ave}=214.23}) \times (218.46 - 210) = 133$$

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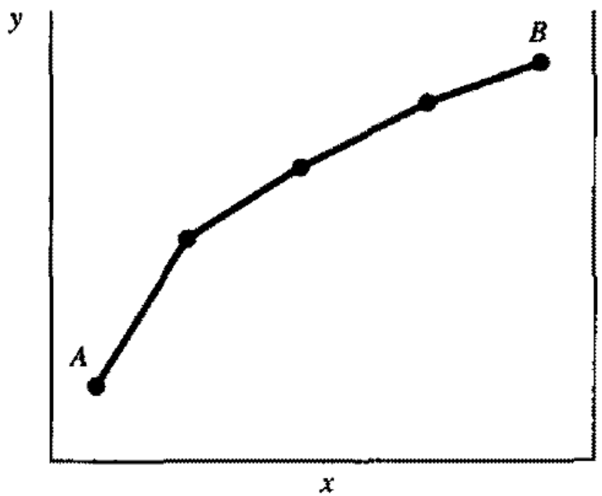
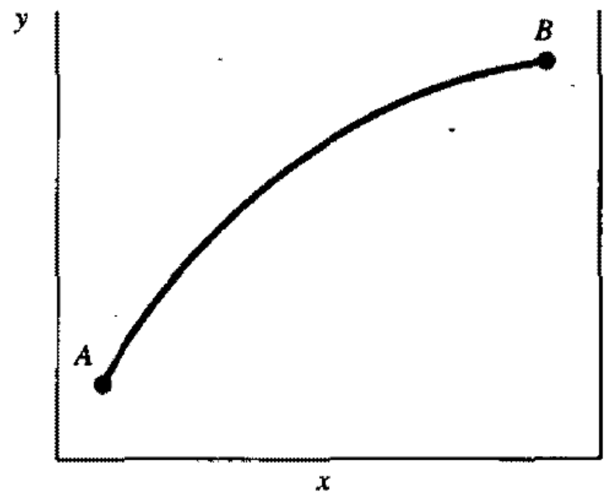
18.4 Dynamic programming and calculus of variations

- Calculus of variation(COV) develops an equation for the function that is continuous with continuous derivatives. COV is more representative of the physical problem.
- If the physical situation is one of discrete stages, dynamic programming(DP) is more accurate and COV is an approximation.



Optimize $\sum_{i=1}^{4} F_i(y, y', x)$



Optimize $\int_A^B F_i(y, y', x)dx$

Fig. (a) Dynamic programming (b) calculus of variation

18.5 Stage operation with recycle

- The objective is to maximize the summation

$$\sum_{n=1}^{N} r_n$$

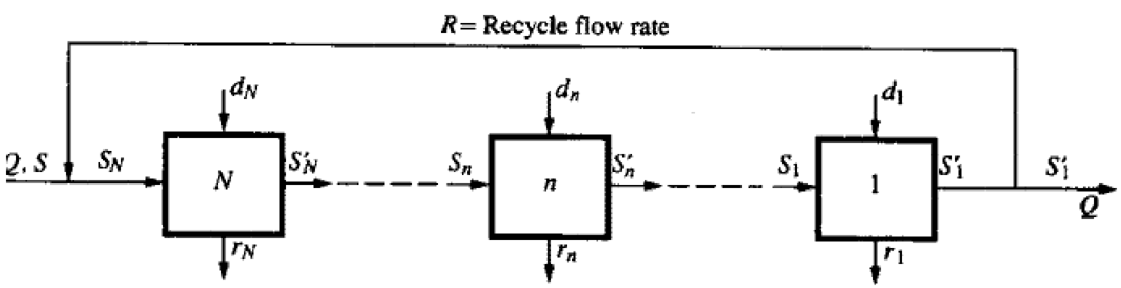


Fig. Recycle stream in a staged process

Known values;
S : concentration
Q : entering flow rate
R : recycle rate

Assume values;
S_n and S'_1

18.6 Dynamic programming for constrained optimizations

- The value of λ represents the sensitivity coefficient
- The DP process will be conducted with internal choices made to hold a constant value of λ through the successive tables.

Example 18.3 : Solve example 18.2 by treating it as a 4-stage DP

(Given)

- Each stage consisting of 300 s of pumping operation

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Example 18.3

Table Pumping energy is a stage

Volume at start	Volume at end: 15.8	16.2	16.6	17.0	17.4	26.4	26.8	27.2	27.6	28.0	33.8	34.2	34.6	35.0	35.4	40.0
0.0	3694.1	3883.5	4077.7	4276.6	4480.2											
15.8						2343.4	2523.6	2710.5	2904.0	3104.3						
16.2						2169.9	2343.4	2523.6	2710.5	2904.0						
16.6						2003.0	2169.9	2343.4	2523.6	2710.5						
17.0						1842.9	2003.0	2169.9	2343.4	2523.6						
17.4						1689.4	1842.9	2003.0	2169.9	2343.4						
26.4											1472.4	1635.9	1808.0	1988.7	2178.0	
26.8											1317.5	1472.4	1635.9	1808.0	1988.7	
27.2											1171.3	1317.5	1472.4	1635.9	1808.0	
27.6											1033.6	1171.3	1317.5	1472.4	1635.9	
28.0											904.5	1033.6	1171.3	1317.5	1472.4	
33.8																1265.0
34.2																1107.0
34.6																959.6
35.0																822.7
35.4																696.3

Example 18.3

Table Δp contribution in a stage

Volume at start	Volume at end															
	15.8	**16.2**	**16.6**	**17.0**	**17.4**	**26.4**	**26.8**	**27.2**	**27.6**	**28.0**	**33.8**	**34.2**	**34.6**	**35.0**	**35.4**	**40.0**
0.0	70142	71917	73693	75469	77245											
15.8						66323	68826	71329	73831	76334						
16.2						63821	66323	68826	71329	73831						
16.6						61318	63821	66323	68826	71329						
17.0						58815	61318	63821	66323	68826						
17.4						56312	58815	61318	63821	66323						
26.4											59693	62919	66146	69372	72599	
26.8											56466	59693	62919	66146	69372	
27.2											53239	56466	59693	62919	66146	
27.6											50013	53239	56466	59693	62919	
28.0											46786	50013	53239	56466	59693	
33.8																61211
34.2																57261
34.6																53312
35.0																49363
35.4																45414

Example 18.3 **Table** Cummulative energy and Δp from 600 to 1200 s

V at 600s	V at 900s	Cumulative energy					Cumulative Δp				λ
	33.8	1265.0	+	1472.4	=	2737.4	61211 +	59693	=	120,904	
											] -0.008
	34.2	1107.0	+	1635.9	=	2742.9	57261 +	62919	=	120,180	
											] -0.034
26.4	34.6	959.6	+	1808.0	=	2767.6	53312 +	66146	=	119,458	
											] -0.061
	35.0	822.7	+	1988.7	=	2811.4	49363 +	69372	=	118,735	
											] -0.087
	35.4	696.3	+	2178.0	=	2874.3	45414 +	72599	=	118,013	
	33.8	1265.0	+	1317.5	=	2582.5	61211 +	56466	=	117,677	
											] 0.0043
	34.2	1107.0	+	1472.4	=	2579.4	57261 +	59693	=	116,954	
											] -0.022
26.8	34.6	959.6	+	1635.9	=	2595.5	53312 +	62919	=	116,231	
											] -0.049
	35.0	822.7	+	1808	=	2630.7	49363 +	66146	=	115,509	
											] -0.075
	35.4	696.3	+	1988.7	=	2685.0	45414 +	69372	=	114,786	
	33.8	1265.0	+	1171.3	=	2436.3	61211 +	53239	=	114,450	
											] 0.0163
	34.2	1107.0	+	1317.5	=	2424.5	57261 +	56466	=	113,727	
											] -0.01
27.2	34.6	959.6	+	1472.4	=	2432.0	53312 +	59693	=	113,005	
											] -0.037
	35.0	822.7	+	1635.9	=	2458.6	49363 +	62919	=	112,282	
											] -0.063
	35.4	696.3	+	1808	=	2504.3	45414 +	66146	=	111,560	
	33.8	1265.0	+	1033.6	=	2298.6	61211 +	50013	=	111,224	
											] 0.0281
	34.2	1107.0	+	1171.3	=	2278.3	57261 +	53239	=	110,500	
											] 0.0017
27.6	34.6	959.6	+	1317.5	=	2277.1	53312 +	56466	=	109,778	
											] -0.025
	35.0	822.7	+	1472.4	=	2295.1	49363 +	59693	=	109,056	
											] -0.051
	35.4	696.3	+	1635.9	=	2332.2	45414 +	62919	=	108,333	
	33.8	1265.0	+	904.5	=	2169.5	61211 +	46786	=	107,997	
											] 0.04
	34.2	1107.0	+	1033.6	=	2140.6	57261 +	50013	=	107,274	
											] 0.0134
28.0	34.6	959.6	+	1171.3	=	2130.9	53312 +	53239	=	106,551	
											] -0.013
	35.0	822.7	+	1317.5	=	2140.2	49363 +	56466	=	105,829	
											] -0.039
	35.4	696.3	+	1472.4	=	2168.7	45414 +	59693	=	105,107	

Example 18.3

Table Optimal pumping plan

Time stage, s	Volumes in/leaving	Contribution to energy, kJ	Contribution to Δp
0 to 300	0/17.0	4,276.6	75,469
300 to 600	17.0/27.2	2,169.9	63,820
600 to 900	27.2/34.6	1,472.4	59,692
900 to 1200	34.6/40.0	959.6	53,312
	Total:	8,878.5	252,293