Optimal Design of Energy Systems (M2794.003400)

Chapter 9. SEARCH METHODS

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Chapter 9. SEARCH METHODS

9.1 Overview of search methods

- The major effort in the **optimization** was determining the values of the independent variables that provide the optimum.
- Search methods generally fall into categories;

  elimination

  hill-climbing

  - no one systematic procedure
  - ultimate approach if other optimization methods fail

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9.1 Overview of search methods

Single variable	a. Exhaustive b. Efficient
Multivariable, unconstrained	a. Lattice b. Univariate c. Steepest ascent
Multivariable, constrained	a. Penalty functions b. Search along a constraint

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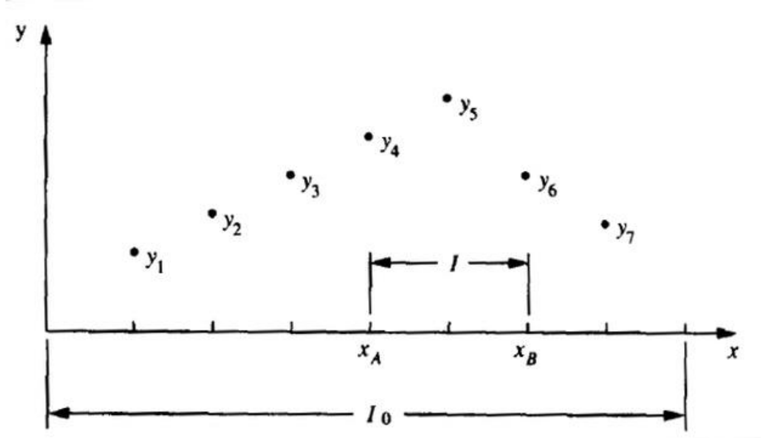
9.2 Interval of uncertainty

- In search methods, the **precise point** at which the optimum occurs **will never be known**
- The best that can be achieved is to specify the **interval of uncertainty**

9.3 Exhaustive search (linear search)

- The exhaustive search is most widely used
- $\underbrace{\text{Interval of interest}}_{I_0}$ is uniformly devided by (number of observation + 1)



number of observation: $n = 7$

devided interval: $\frac{1}{n+1} I_0$

Maximum lies : $y(x_\mathrm{A}) < y_\mathrm{max} < y(x_\mathrm{B})$

Interval of uncertainty: $I = \frac{2}{(n+1)} I_0 = \frac{2}{8} I_0$

Right next 2 sides of maximum y

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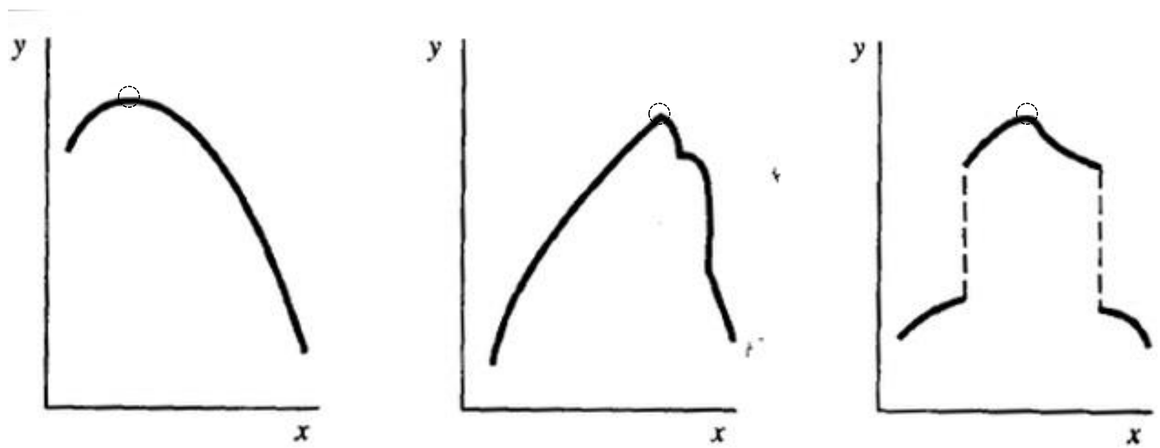
9.4 Unimodal functions

- Only one peak (or valley) in the interval of interest

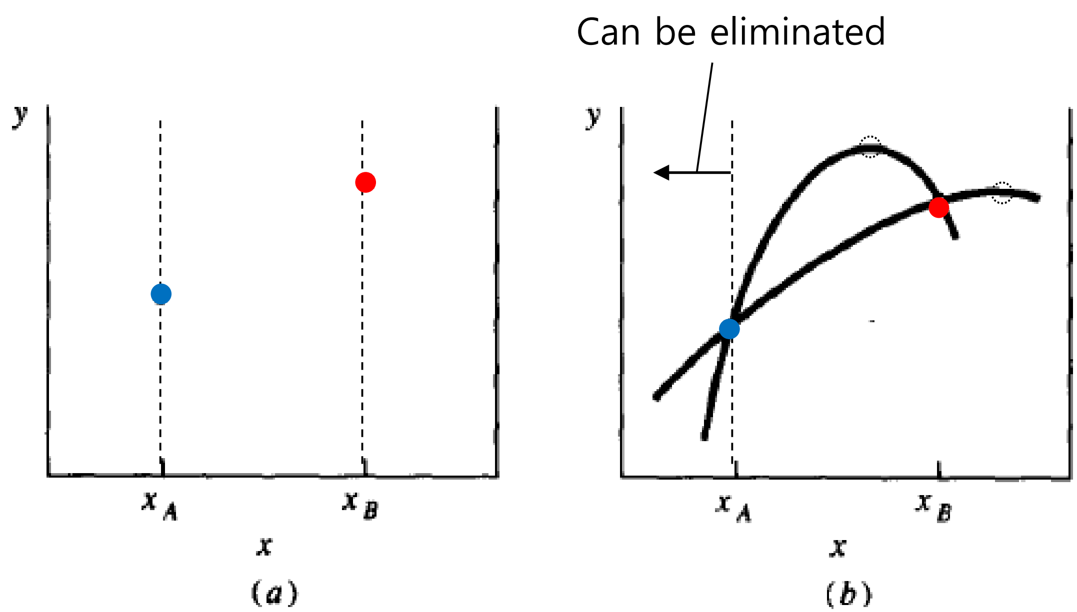
  - dichotomous search method
  - Fibonacci search method



(a) (b) (c)

9.5 Eliminating a section based on two tests

- It can be eliminated one side at two different position of an unimodal function.



$y(x_A) < y(x_B)$

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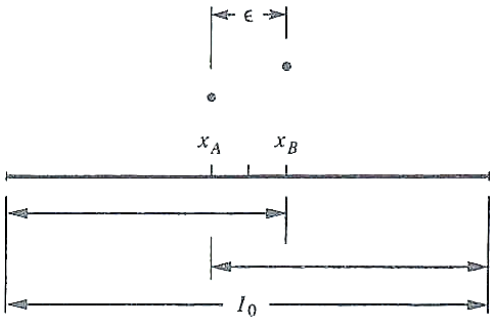
9.6 Dichotomous search

- Searching from the middle of the interval with a range, ε
- Comparing x_A, x_B, smaller part of the interval is eliminated



1st trial remaining interval : $I = \dfrac{I_0 + \varepsilon}{2}$

2nd trial remaining interval : $I = \dfrac{\dfrac{I_0 + \varepsilon}{2} + \varepsilon}{2} = \dfrac{I_0}{4} + \left(\varepsilon - \dfrac{1}{4}\varepsilon\right)$

3rd trial remaining interval : $I = \dfrac{\dfrac{\dfrac{I_0 + \varepsilon}{2} + \varepsilon}{2} + \varepsilon}{2} = \dfrac{I_0}{8} + \left(\varepsilon - \dfrac{1}{8}\varepsilon\right)$

n trial points (n=2,4,6,…) : $I = \dfrac{I_0}{2^{n/2}} + \varepsilon\left(1 - \dfrac{1}{2^{n/2}}\right)$

I : interval of uncertainty
I_0 : interval of interest
ε : space between two points

9.7 Fibonacci search

- What is Fibonacci series?

$$F_1 = 1, \quad F_2 = 1, \quad F_i = F_{i-2} + F_{i-1} \quad (i \geq 2)$$

$$F = 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \cdots$$

- Fibonacci series in nature

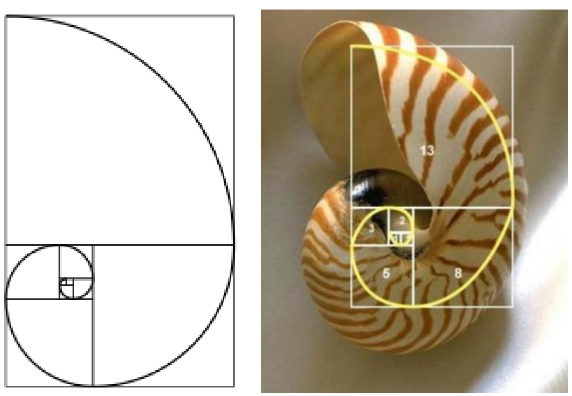


Fig. Fibonacci spiral and shell*

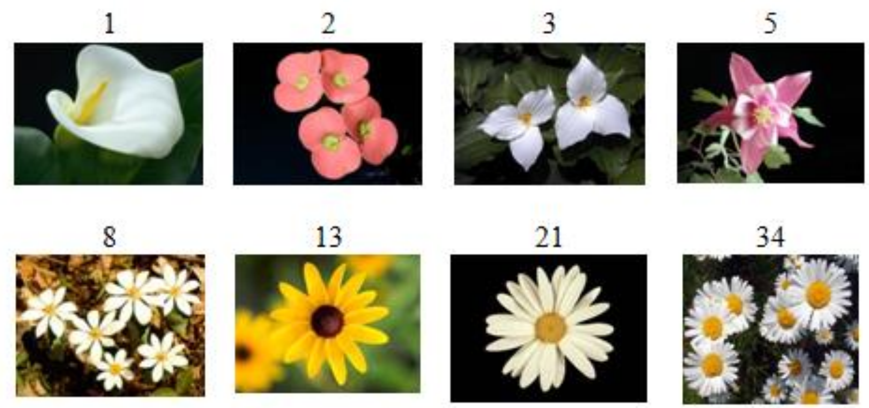


Fig. Number of flower petals and Fibonacci series**

*https://en.wikipedia.org/wiki/Golden_spiral,20170421 **http://developeriq.in/articles/2014,20170421

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9.7 Fibonacci search

- Applying Fibonacci series to search method

① Decide how many observations(n)

② Place the first observation in I_0 at a distance of $I_0\frac{F_{n-1}}{F_n}$ from both ends

③ Place the next observation in the interval of uncertainty at a position that is symmetric to the existing observation

④ Interval reduces according to Fibonacci series

$$I_1 = I_0\frac{F_{n-1}}{F_n} \qquad I_2 = I_1\frac{F_{n-2}}{F_{n-1}} = I_0\frac{F_{n-2}}{F_n} \qquad I_3 = I_2\frac{F_{n-3}}{F_{n-2}} = I_0\frac{F_{n-3}}{F_n} \qquad \cdots$$

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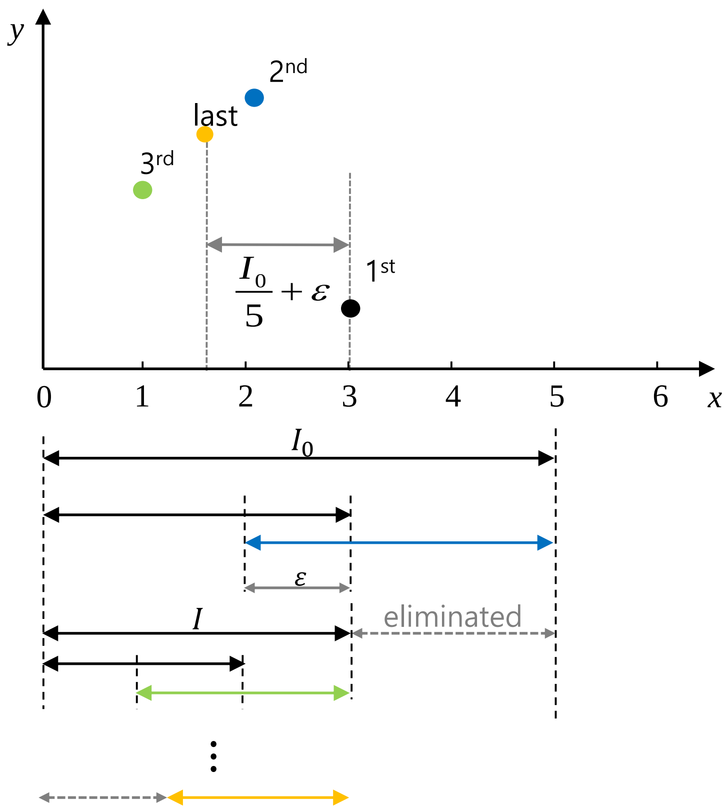
Example 9.1

Find the maximum of the function $y = -x^2 + 4x + 2$

in the interval $0 < x < 5$



Arbitrarily choose: $n = 4, \quad I_0 = 5$

1st : $x_1 = I_0 \dfrac{F_3}{F_4} = \dfrac{3}{5} I_0 = 3$

2nd : symmetric $0 \sim 5 \longrightarrow x_2 = 2$

eliminate $3<x<5$

3rd : symmetric $0 \sim 3 \longrightarrow x_3 = 1$

⋮

eliminate $0<x<1$

Final : $x = 2 - \varepsilon$

Interval of uncertainty

$$2 - \varepsilon \le x \le 3 \frac{I_0}{5} + \varepsilon = \frac{I_0}{F_n} + \varepsilon$$

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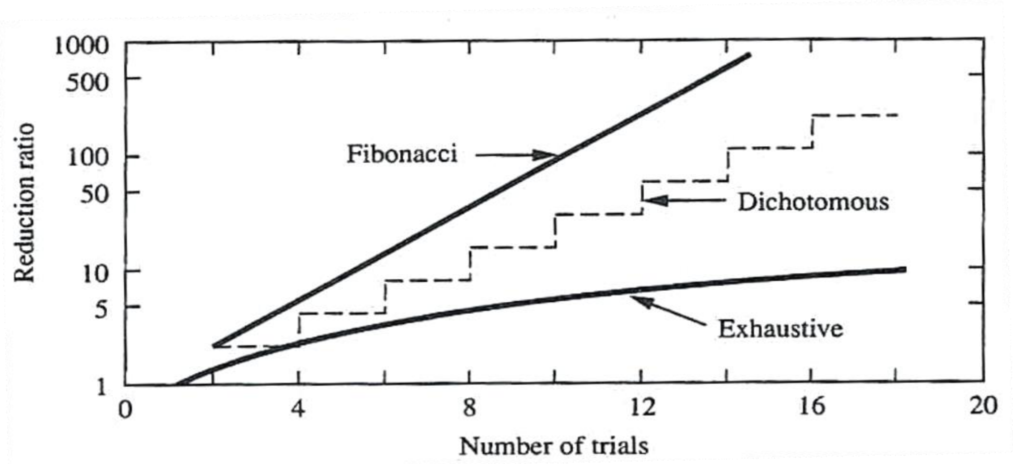
9.8 Comparative effectiveness of search methods

Reduction Ratio (RR) = $\dfrac{I_0}{I_n}$ single variable search

$$= \begin{cases} \dfrac{n+1}{2} & \text{exhaustive} \quad \text{O.K.} \\ 2^{\frac{n}{2}} & \text{dichotomous} \quad \text{good} \\ F_n & \text{Fibonacci} \quad \text{good} \end{cases}$$



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9.10 Multivariable, unconstrained optimization

- Single variable
  - Calculus — exhaustive
  - elimination
    - dichotomous
    - Finonacci

- Multivariable, unconstrained
  - hill-climbing
    - Lattice
    - Univariate

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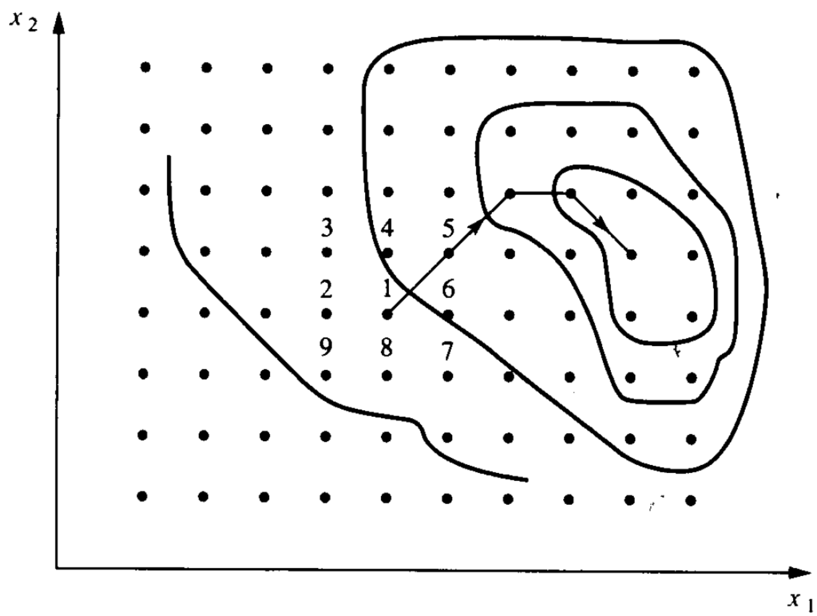
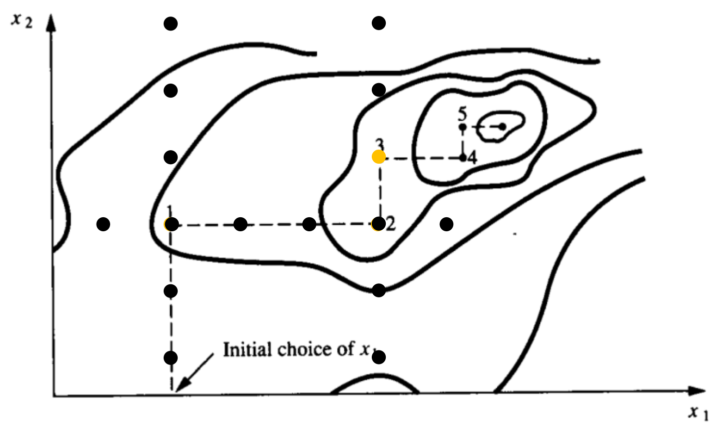
9.11 Lattice search

- Start at on point in the region of interest
- Check a number of points in a grid surrounding the central point
- Move the central point to maximum value of a grid
- If the central point is greater than other surrounding point: **coarse grid** $\longrightarrow$ **fine grid**



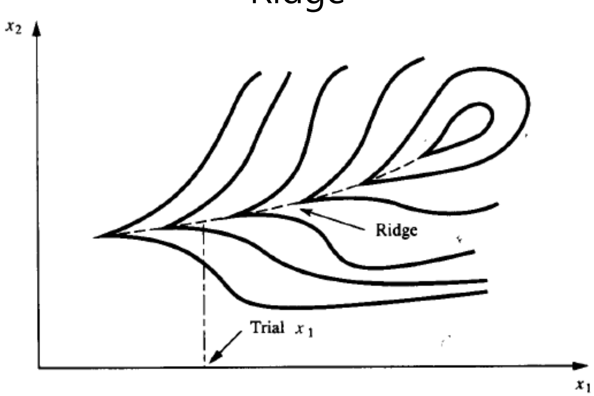
9.12 Univariate search

- Optimization with respect to **one variable** at a time

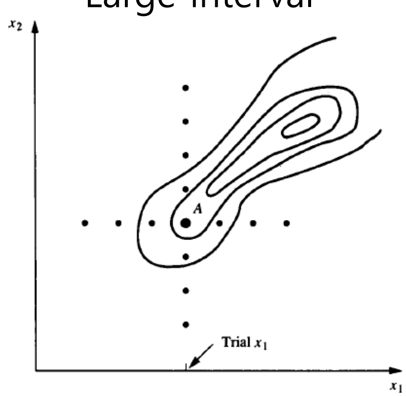


- Failure occurs

Ridge



Large interval



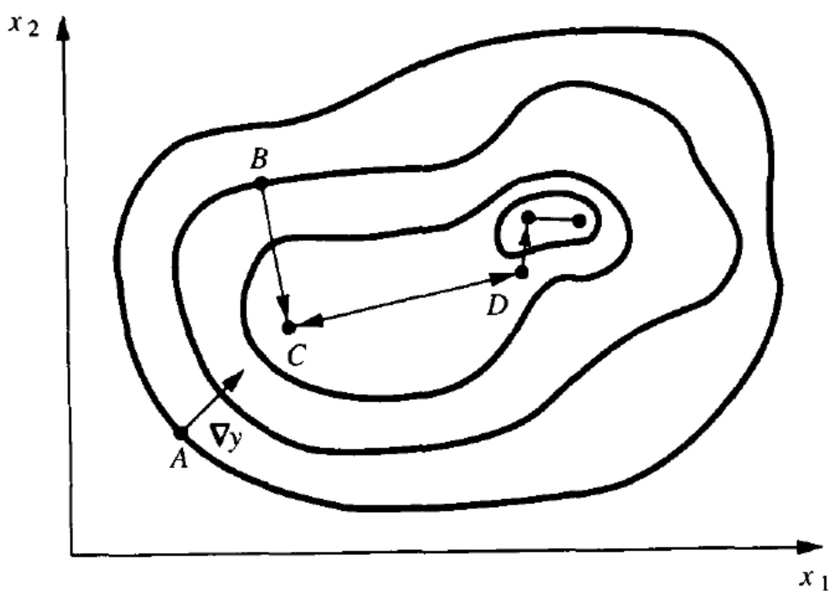
9.13 Steepest-ascent method

- Decide in which **direction** to move along the gradient
- Decide how far to move and then move that **distance**

$$\nabla y = \frac{\partial y}{\partial x_1}\widehat{i_1} + \frac{\partial y}{\partial x_2}\widehat{i_2}$$

$\widehat{i_1}$, $\widehat{i_2}$: unit vector in the x_1 and x_2

gradient vector (at A) is normal to the contour line (at A)



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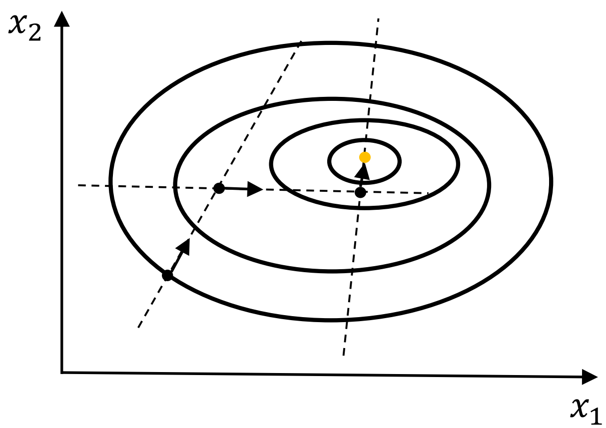
9.13 Steepest-ascent method

① trial point as near to the optimum as possible (otherwise, arbitrarily chosen)

② Gradient vector is normal to the contour line or surface and therefore indicates the direction of maximum rate of change

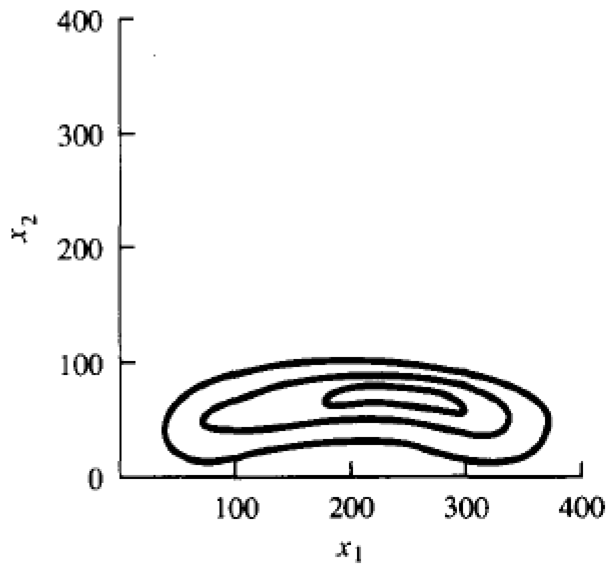
$$\frac{\Delta x_1}{\partial y/\partial x_1} = \cdots = \frac{\Delta x_n}{\partial y/\partial x_n} \Leftarrow \frac{\partial y}{\partial x_1} : \frac{\partial y}{\partial x_2} : \cdots : \frac{\partial y}{\partial x_n} = x_1 : x_2 : \cdots : x_n$$

③ in the direction of gradient, move until optimum is reached



9.14 Scales of the independent variables

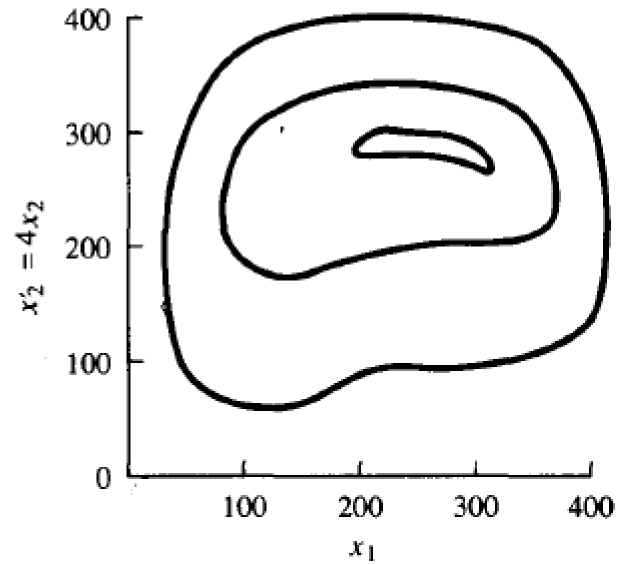
- Contours should be as spherical as possible to accelerate the convergence



(a) Original scale

$$\begin{cases} 0 < x_1 < 400 \\ 0 < x_2 < 100 \end{cases}$$

$$x_2' = cx_2$$



(b) Revised scale

$$\begin{cases} 0 < x_1 < 400 \\ 0 < x_2' < 400 \end{cases}$$

9.15 Constrained optimization

- The most frequent and most important ones encountered in the design of thermal systems

1) Conversion to unconstrained by use of penalty functions

2) Searching along the constraint

→ equality constraints only

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9.16 Penalty functions

$$y = y(x_1, x_2, \cdots x_n)$$ → maximum

if minimum

Subject to

$$\phi_1 = y(x_1, x_2, \cdots x_n) = 0$$

$$\vdots$$

$$\phi_m = y(x_1, x_2, \cdots x_n) = 0$$

New unconstrained function

$$Y = y - P_1\phi_1^2 - \cdots - P_m\phi_m^2$$

$$Y = y + P_1\phi_1^2 + \cdots + P_m\phi_m^2$$

P_i Relative weighting

too high – move very slowly

too small – terminate without satisfying the constraints

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9.17 Optimization by searching along a constraint-hemstitching

- Choose a trial point
- Driving toward the constraint(s) (fixed x1 or x2)
- On constraint(s), optimize along the constraint(s) (tangential move)

9.18 Driving toward the constraint(s)

$m < n$

m : the number of constraints

n : the number of variables

$n - m$: the number of remaining variables which should be solved

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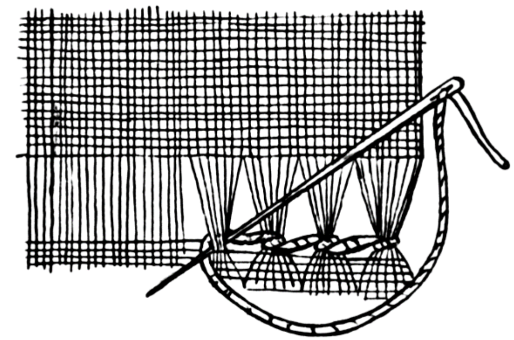
9.19 Hemstitching search when n-m=1

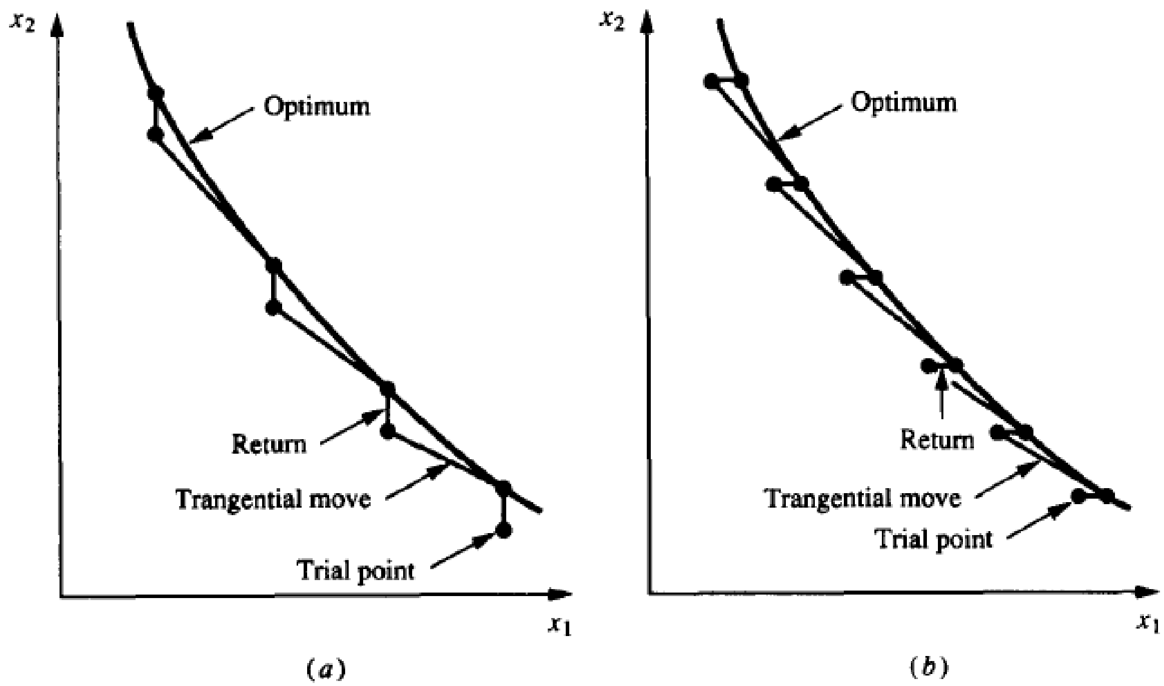
$$\left.\begin{array}{l}\#\text{ of constraints} = m \\ \#\text{ of variables} = n\end{array}\right\} \; n - m = 1$$



hemstitching

$x_1^2 x_2 = 8$

$\phi(x_2, x_2) = 0$



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9.19 Hemstitching search when n-m=1

- constraint

$$\phi(x_1, x_2) = 0$$

$$\Delta\phi = \frac{\partial\phi}{\partial x_1}\Delta x_1 + \frac{\partial\phi}{\partial x_2}\Delta x_2 = 0$$

$$\frac{\Delta x_1}{\Delta x_2} = -\frac{\partial\phi / \partial x_1}{\partial\phi / \partial x_2}$$

- objective function

$$\Delta y \approx \frac{\partial y}{\partial x_1}\Delta x_1 + \frac{\partial y}{\partial x_2}\Delta x_2$$

$$= \left(-\frac{\partial y}{\partial x_1}\frac{\partial\phi / \partial x_2}{\partial\phi / \partial x_1} + \frac{\partial y}{\partial x_2} \right)\Delta x_2 = G\Delta x_2$$

In minimization, if $G>0$, $\Delta x_2<0$
if $G<0$, $\Delta x_2>0$

In maximization, if $G>0$, $\Delta x_2>0$
if $G<0$, $\Delta x_2<0$

9.19 Hemstitching search when n-m=1

Three-variable problem where n=3, m=2

optimize $y = y(x_1, x_2, x_3)$

subject to $\phi_1(x_1, x_2, x_3) = 0$

$\phi_2(x_1, x_2, x_3) = 0$

On the constraints, (tangential move)

$$\Delta\phi_1 = \frac{\partial\phi_1}{\partial x_1}\Delta x_1 + \frac{\partial\phi_1}{\partial x_2}\Delta x_2 + \frac{\partial\phi_1}{\partial x_3}\Delta x_3 = 0$$

$$\Delta\phi_2 = \frac{\partial\phi_2}{\partial x_1}\Delta x_1 + \frac{\partial\phi_2}{\partial x_2}\Delta x_2 + \frac{\partial\phi_2}{\partial x_3}\Delta x_3 = 0$$

Eliminate Δx_1, Δx_2

$$\Delta y = \frac{\partial y}{\partial x_1}\Delta x_1 + \frac{\partial y}{\partial x_2}\Delta x_2 + \frac{\partial y}{\partial x_3}\Delta x_3$$

$$= G\Delta x_3$$

In minimization, if $G>0$, $\Delta x_3<0$
if $G<0$, $\Delta x_3>0$

In maximization, if $G>0$, $\Delta x_3>0$
if $G<0$, $\Delta x_3<0$

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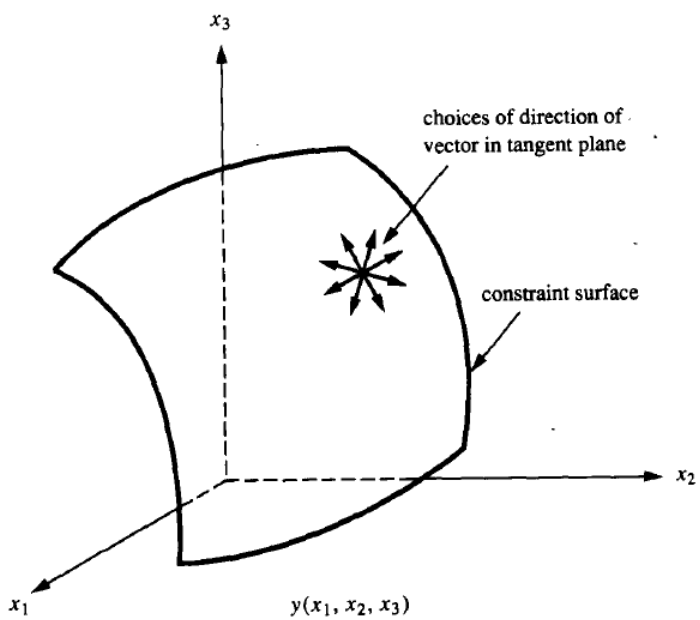
9.20 Moving tangent to a constraint in three dimensions

n=3, m=1



- maximum change of y

$$\Delta y = \frac{\partial y}{\partial x_1}\Delta x_1 + \frac{\partial y}{\partial x_2}\Delta x_2 + \frac{\partial y}{\partial x_3}\Delta x_3$$

- direction (tangent to a constraint)

$$\Delta \phi = \frac{\partial \phi}{\partial x_1}\Delta x_1 + \frac{\partial \phi}{\partial x_2}\Delta x_2 + \frac{\partial \phi}{\partial x_3}\Delta x_3 = 0$$

- distance

$$\Delta x_1^2 + \Delta x_2^2 + \Delta x_3^2 = r^2 = const.$$

- maximum

$$\Delta y = ?$$

9.20 Moving tangent to a constraint in three dimensions

Lagrange Multiplier Method

$$\frac{\partial y}{\partial x_1} - \lambda_1 (2\Delta x_1) - \lambda_2 \frac{\partial \phi}{\partial x_1} = 0 \qquad \cdots ①$$

$$\frac{\partial y}{\partial x_2} - \lambda_1 (2\Delta x_2) - \lambda_2 \frac{\partial \phi}{\partial x_2} = 0 \qquad \cdots ②$$

$$\frac{\partial y}{\partial x_3} - \lambda_1 (2\Delta x_3) - \lambda_2 \frac{\partial \phi}{\partial x_3} = 0 \qquad \cdots ③$$

$$① \times \frac{\partial \phi}{\partial x_1} + ② \times \frac{\partial \phi}{\partial x_2} + ③ \times \frac{\partial \phi}{\partial x_3}$$

$$\frac{\partial y}{\partial x_1}\frac{\partial \phi}{\partial x_1} + \frac{\partial y}{\partial x_2}\frac{\partial \phi}{\partial x_2} + \frac{\partial y}{\partial x_3}\frac{\partial \phi}{\partial x_3} - \lambda_2 \left[\left(\frac{\partial \phi}{\partial x_1} \right)^2 + \left(\frac{\partial \phi}{\partial x_2} \right)^2 + \left(\frac{\partial \phi}{\partial x_3} \right)^2 \right] = 0$$

$$\rightarrow \lambda_2$$

9.20 Moving tangent to a constraint in three dimensions

$$\frac{1}{2\lambda_1} = \frac{\Delta x_1}{\dfrac{\partial y}{\partial x_1} - \lambda_2 \dfrac{\partial \phi}{\partial x_1}} = \frac{\Delta x_2}{\dfrac{\partial y}{\partial x_2} - \lambda_2 \dfrac{\partial \phi}{\partial x_2}} = \frac{\Delta x_3}{\dfrac{\partial y}{\partial x_3} - \lambda_2 \dfrac{\partial \phi}{\partial x_3}}$$

$\Delta x_i =$ step size of on variable in the move

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9.21 Summary

1. Single variable
    - a. Exhaustive
    - b. Efficient
        - Dichotomous
        - Finonacci
2. Multivariable, unconstrained
    - a. Lattice
    - b. Univariate
    - c. Steepest ascent
3. Multivariable, constrained
    - a. Penalty functions
    - b. Search along a constraint