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457.646 Topics in Structural Reliability

In-Class Material: Class 06

II-6. Functions of Random Variables (See Supp. 02)

Consider $\mathbf{Y} = \mathbf{g}(\mathbf{X})$

(1) For input **X**: distribution model $(f_{\mathbf{X}}(\mathbf{x}))$ or expectations $(\mathbf{M}_{\mathbf{X}}, \mathbf{\Sigma}_{\mathbf{XX}})$ available

(2) For output **Y**: distribution model ( ) or expectations (,)?

<u>Examples:</u>

(1) Regional/inventory loss: $L = \sum_{i=1}^{n} V_i Dr_i$ → linear function

(2) Wind-induced pressure: $P = \frac{1}{2} C_\rho \rho V^2$

◎ Mathematical expectation of linear functions

$$Y_k = a_{k,0} + \sum_{i=1}^{n} a_{k,i} X_i, \quad k = 1, \ldots, m$$

① Algebraic formula $(n \le 3)$: See supp.

② Matrix formula:

For $\mathbf{Y} = \mathbf{A}_0 + \mathbf{A}\mathbf{X}$

where

$$\mathbf{Y} = \begin{Bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_m \end{Bmatrix}, \quad \mathbf{A}_0 = \begin{Bmatrix} a_{1,0} \\ a_{2,0} \\ \vdots \\ a_{m,0} \end{Bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \text{ and } \mathbf{X} = \begin{Bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{Bmatrix}$$

$\mathbf{M}_{\mathbf{Y}} =$

$\mathbf{\Sigma}_{\mathbf{YY}} =$

❖ Proof of Positive-definiteness of $\mathbf{\Sigma}_{\mathbf{XX}}$

Consider $Y = \mathbf{a}^{\mathrm{T}}\mathbf{X}$ ($\mathbf{A}_0 =$, $\mathbf{A} =$)

Using the formula above,

$\mathbf{\Sigma}_{\mathbf{YY}} = \sigma_Y^2 =$

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❖ Linear transformation for standardization, i.e., &

Suppose **X** has and

Find $\mathbf{Y} = \mathbf{A}_0 + \mathbf{A}\mathbf{X}$

such that $\mathbf{M}_{\mathbf{Y}} =$ and $\boldsymbol{\Sigma}_{\mathbf{YY}} =$

$\mathbf{M}_{\mathbf{Y}} = \mathbf{A}_0 + \mathbf{A}\mathbf{M}_{\mathbf{X}} =$ (1)

$\boldsymbol{\Sigma}_{\mathbf{YY}} = \mathbf{A}\boldsymbol{\Sigma}_{\mathbf{XX}}\mathbf{A}^{\mathrm{T}} =$ (2)

Since $\boldsymbol{\Sigma}_{\mathbf{XX}}$ is positive-definite, $\boldsymbol{\Sigma}_{\mathbf{XX}} = \mathbf{L}_{\Sigma}\mathbf{L}_{\Sigma}^{\mathrm{T}}$ (e.g. by ____________ decomposition)

Therefore, $= \mathbf{I}$ and

$\mathbf{A} =$ → Substitute to ( )

$\mathbf{A}_0 =$

In summary,

$\mathbf{Y} =$
$=$

Alternatively,

$$\boldsymbol{\Sigma}_{\mathbf{XX}} = \mathbf{D}_{\mathbf{X}}\mathbf{R}_{\mathbf{XX}}\mathbf{D}_{\mathbf{X}}$$
$$=$$
$$= \mathbf{L}_{\Sigma}\mathbf{L}_{\Sigma}^{\mathrm{T}}$$

Therefore, $\mathbf{L}_{\Sigma} =$ and $\mathbf{L}_{\Sigma}^{-1} =$

$\mathbf{Y} =$

→ This version is preferred because of numerical stability in decomposition ($|\rho| \leq 1$).

◎ Mathematical expectation of nonlinear functions

$Y_K = g_k(x),\ k = 1, \cdots, m$

Taylor series expansion around the mean point, $\mathbf{x} = \mathbf{M}_{\mathbf{X}}$

$$Y_K \approx g_k(\mathbf{M}_{\mathbf{X}}) + \left.\frac{\partial g_k}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{M}_{\mathbf{X}}} (\mathbf{x} - \mathbf{M}_{\mathbf{X}}) + \cdots + \cdots + \cdots$$

Matrix form

$$\mathbf{Y} \cong \mathbf{g}(\mathbf{M_X}) + \mathbf{J}_{\mathbf{Y},\mathbf{X}}\big|_{\mathbf{x}=\mathbf{M_X}} (\mathbf{X} - \mathbf{M_X})$$

① First-order approximation

(Scalar: See supp.)

$$\boxed{\begin{aligned} \mathbf{M}_{\mathbf{Y}}^{FO} &= \mathbf{g}(\qquad) \\ \mathbf{\Sigma}_{\mathbf{YY}}^{FO} &= \end{aligned}}$$

② Second-order approximation

⇒ Can use 2nd order approximation from Taylor series expansion

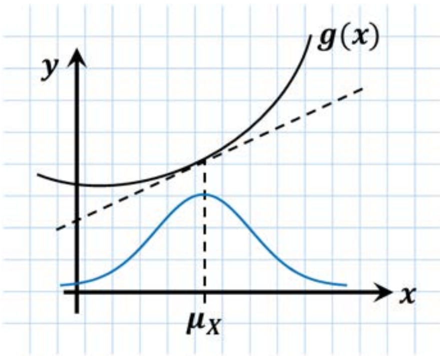
⇒ Not useful because higher-order moments are needed $(\gamma, \kappa, \cdots)$

③ Accuracy of FO/SO approximation

Sources of large errors in approx.

- σ_X
- Nonlinearity in $g(x)$

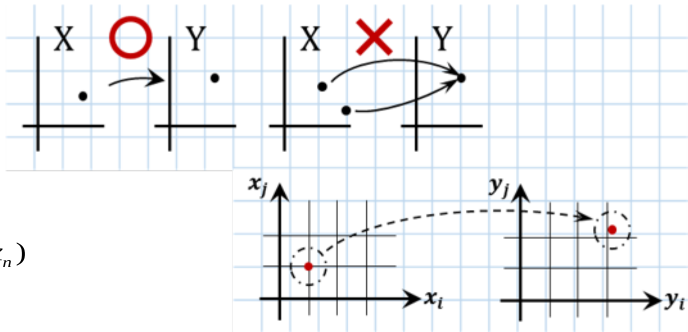
Example : $\mathbf{U} = \mathbf{K}^{-1}\mathbf{P}$ (Frame structure)



◎ Derived Distribution of Functions

Consider $\mathbf{Y} = \mathbf{g}(\mathbf{X})$ where $\mathbf{Y} = \{Y_1, \cdots, Y_m\}$ and $\mathbf{X} = \{X_1, \cdots, X_n\}$

Given: $f_{\mathbf{X}}(\mathbf{x})$ → $f_{\mathbf{Y}}(\mathbf{y})$?



① $m = n$, one-to-one mapping

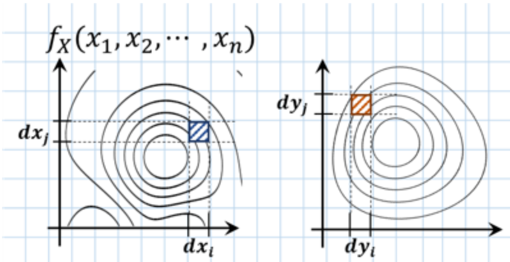
a) Discrete

$P_{\mathbf{Y}}(y_1, \cdots, y_n)$ $\qquad$ $P_{\mathbf{X}}(x_1, \cdots, x_n)$

b) Continuous

$f_{\mathbf{Y}}(y_1, \cdots, y_n)$ $\qquad$ $f_{\mathbf{X}}(x_1, \ldots, x_n)$

$$\boxed{\begin{aligned} f_{\mathbf{Y}}(\mathbf{y}) &= f_{\mathbf{X}}(\mathbf{x}) \cdot \left|\det \qquad \right| \\ &= f_{\mathbf{X}}(\mathbf{x}) \cdot \left|\det \qquad \right|^{-1} \end{aligned}}$$



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"Jacobian" $\mathbf{J}_{\mathbf{y},\mathbf{x}} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \cdots & \frac{\partial y_n}{\partial x_n} \end{bmatrix}$

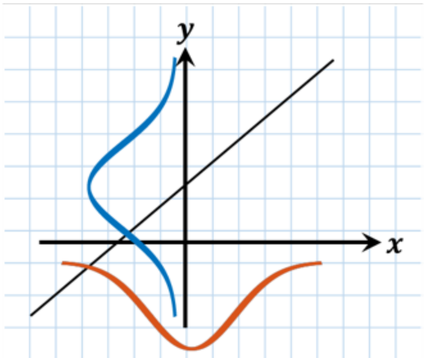
Consider $\mathbf{y} = \mathbf{g}(\mathbf{x}),\ \mathbf{x} = \mathbf{h}(\mathbf{y})$

※ $f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(\mathbf{h}(\mathbf{y}))\left|\det \mathbf{J}_{\mathbf{y},\mathbf{x}}(\mathbf{h}(\mathbf{y}))\right|^{-1}$

※ $m = n = 1$

$$f_Y(y) = f_X(x) \qquad\qquad = f_X(\quad)\left|\frac{dh(y)}{dy}\right|$$

Example: $X \sim N(0, 1^2)$



a) $Y = g(X) = aX + b$

One-to-one mapping?

$f_Y(y) = f_X(x)\cdot$

$=$

$=$

__________ Distribution

$\mu_Y =$

$\sigma_Y =$

b) T_1, T_2 ~ exponential r.v.'s (See supplement on "Other Distribution Models")

$$f_{T_1}(t_1) = \alpha \cdot \exp(-\alpha t_1),\ t_1 > 0$$
$$f_{T_2}(t_2) = \beta \cdot \exp(-\beta t_2),\ t_2 > 0$$

T_1, T_2 : statistically independent

Joint PDF of $\begin{cases} Y_1 = T_1 + T_2 \\ Y_2 = T_1 - T_2 \end{cases}$?

$$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{T}}(\mathbf{t})\left|\det \mathbf{J}_{y,t}\right|^{-1}$$

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$$\mathbf{J}_{\mathbf{y},\mathbf{t}} = \begin{bmatrix} \frac{\partial y_1}{\partial t_1} & \frac{\partial y_1}{\partial t_2} \\ \frac{\partial y_2}{\partial t_1} & \frac{\partial y_2}{\partial t_2} \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\left|\det \mathbf{J}_{\mathbf{y},\mathbf{t}}\right|^{-1} =$$

$$\therefore \quad f_{\mathbf{Y}}(\mathbf{y}) =$$

Inverse relationship

$$\begin{cases} T_1 = \frac{1}{2}(Y_1 + Y_2) \\ T_2 = \frac{1}{2}(Y_1 - Y_2) \end{cases}$$

$$\therefore \quad f_{\mathbf{Y}}(\mathbf{y}) = \frac{\alpha\beta}{2}\exp[-\frac{\alpha+\beta}{2}y_1 - \frac{\alpha-\beta}{2}y_2], \quad y_1 > 0, -y_1 < y_2 < y_1$$

- Range of **Y** derived from the condition $t_1, t_2 > 0 \;\; \& \;\; \mathbf{t} = \mathbf{h}(\mathbf{y})$