Seoul National University
Dept. of Civil and Environmental Engineering

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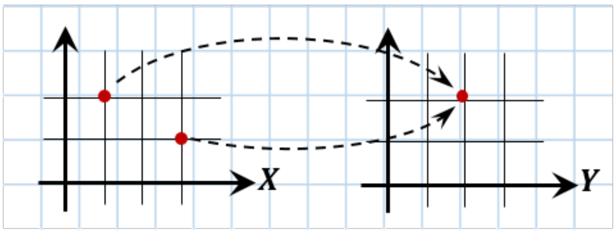
457.646 Topics in Structural Reliability

In-Class Material: Class 07

II-6. Functions of Random Variables (contd.)

◎ Derived Distribution of Functions (contd.)



② $m = n$, **but NOT one-to-one mapping**

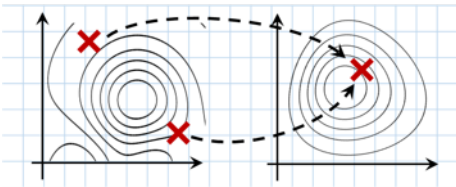
a) Discrete

$$P_{\mathbf{Y}}(y_1, \cdots, y_n) =$$

$$P_{\mathbf{X}}(x_1, \cdots, x_n)$$

roots of $\mathbf{y} = \mathbf{g}(\mathbf{x})$

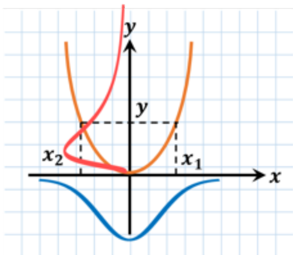
b) Continuous



$$f_Y(y) = \sum_{\substack{\text{all roots of} \\ \mathbf{y}=\mathbf{g}(\mathbf{x})}}$$

Example c)

$$Y = g(X) = X^2, \qquad X \sim N(0, 1^2)$$



$$\begin{cases} x_1 = h_1(y) = \\ x_2 = h_2(y) = \end{cases}$$

$$f_Y(y) = f_X(x_1)\left|\frac{dx_1}{dy}\right| + f_X(x_2)\left|\frac{dx_2}{dy}\right|$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x_1^2\right) \qquad + \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}x_2^2\right) \qquad =$$

③ $m < n$, **one-to one mapping**

$$\mathbf{Y}' \begin{cases} \mathbf{Y} \begin{cases} Y_1 = g_m(X_1, \cdots, X_n) \\ \vdots \qquad \vdots \\ Y_m = g_m(X_1, \cdots, X_n) \end{cases} \\ Y_{m+1} = \\ \vdots \\ Y_n = \end{cases} \qquad \mathbf{Y}' = \mathbf{g}'(\mathbf{X})$$

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Discrete

$$P_{\mathbf{Y}'}(\mathbf{y}') = P_{\mathbf{X}}(\mathbf{x})$$

Then,

$$P_{\mathbf{Y}}(\mathbf{y}) = \sum \cdots \sum P_{\mathbf{X}}(\mathbf{x})$$

a) Continuous

$$f_{\mathbf{Y}'}(\mathbf{y}')dy_1 \cdots dy_m \qquad = f_{\mathbf{X}}(\mathbf{x})dx_1 \cdots dx_m dx_{m+1} \cdots dx_n$$

$$f_{\mathbf{Y}'}(\mathbf{y}') = f_{\mathbf{X}}(\mathbf{x})\left|\det J_{\mathbf{Y}',\mathbf{X}}\right|^{-1}$$

$$= f_{\mathbf{X}}(\mathbf{x})\left|\det J_{\mathbf{Y},\mathbf{X}}\right|_{mxm}^{-1}$$

$$J_{Y',X} = \begin{bmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_m} & & \\ & & & & & \\ \frac{\partial y_m}{\partial x_1} & & \cdots & \frac{\partial y_m}{\partial x_m} & & \\ & & & & & \\ & & & & & \end{bmatrix}$$

$$\therefore \quad f_{\mathbf{Y}}(\mathbf{y}) = \int_{x_{m+1}} \cdots \int_{x_n} f_{\mathbf{X}}(\mathbf{x})\left|\det J_{\mathbf{Y},\mathbf{X}}\right|_{m\times m}^{-1} dx_{m+1} \cdots dx_n$$

Example d)

$Y = T_1 + T_2$ ←**contd. From Example b)**

$f_Y(y)$?

$$\mathbf{Y}' \begin{cases} Y_1 = T_1 + T_2 \\ Y_2 = \quad\;\; T_2 \end{cases}$$

$$f_{\mathbf{Y}'}(\mathbf{y}') = f_{\mathbf{T}}(\mathbf{t})\left|\det J_{\mathbf{Y}',\mathbf{T}}\right|^{-1} \qquad \left|\det J_{\mathbf{Y},\mathbf{T}}\right|_{1\times 1}^{-1} =$$

$$= f_{\mathbf{T}}(\mathbf{t})\left|\det J_{\mathbf{Y}',\mathbf{T}}\right|_{1\times 1}^{-1}$$

$$=$$

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$$f_{\mathbf{Y}}(\mathbf{y}) = f_{Y_1}(y_1) = \int \qquad\qquad dt_2$$

$$= \int f_{T_1}(\qquad) f_{T_2}(\qquad) dt_2$$

$$= \frac{\alpha\beta}{\alpha-\beta}[\exp(-\beta y) - \exp(-\alpha y)],\ y > 0$$

When $\alpha = \beta$, using l'Hopitals rule,

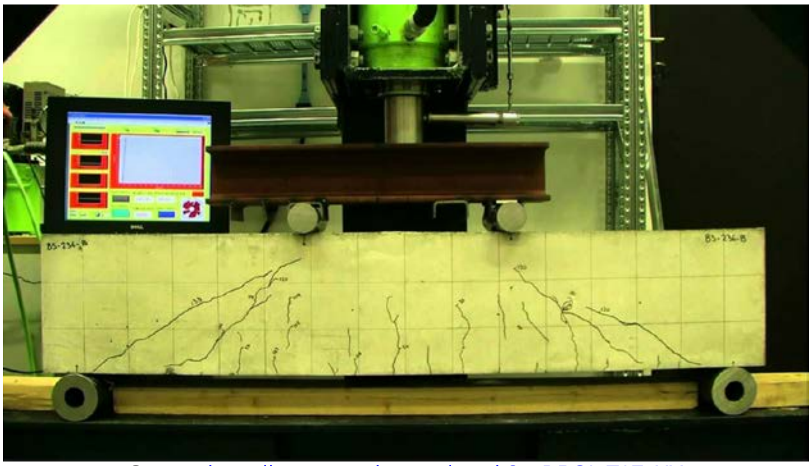
$$\lim_{\beta\to\alpha} f_Y(y) = \lim_{\beta\to\alpha} \frac{\frac{\partial(\ )}{\partial\beta}}{\frac{\partial(\ )}{\partial\beta}} = \alpha^2 y \exp(-\alpha y),\ y > 0$$

④ $m < n$ **, NOT one-to one mapping**

Ⅲ. Structural Reliability (Component)

◎ Structural Reliability Analysis (contd.)

e.g. Shear failure of RC beam w/o stirrups



Source: https://www.youtube.com/watch?v=DPQIpT1ZvXY

"Limit-state" function

$$g(\mathbf{X}) = V_c - V_d$$

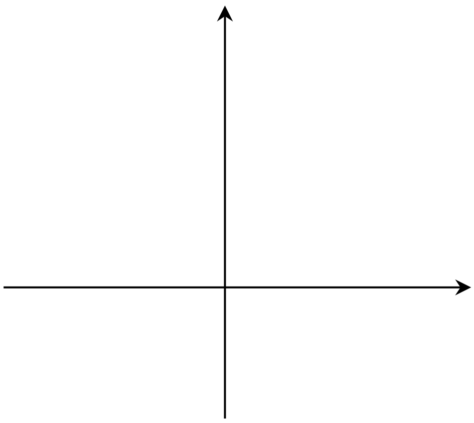
$$= \frac{1}{6}\sqrt{f_c^{'}} b_w d + \varepsilon - V_d \le 0$$

where $X = \{ f_c^{'}, b_w, d, \varepsilon, V_d, \cdots \}$ random variables

Failure Probability

$P_f = P(g(\mathbf{x}) \qquad)$

$=$



"<u>Structural</u> Reliability Analysis"

(Anatomical + Systematic)

Three important tasks for structural reliability analysis:

1)

2)

3)

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◎ Joint Probability Distribution Models

① Joint Normal $\mathbf{X} \sim N(\mathbf{M_X}, \mathbf{\Sigma_{XX}})$

a) Joint PDF

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det \mathbf{\Sigma}}} \exp\left[-\frac{1}{2}(\mathbf{x}-\mathbf{M_X})^{\mathrm{T}} \mathbf{\Sigma}_{\mathbf{XX}}^{-1}(\mathbf{x}-\mathbf{M_X})\right]$$

$n=1$ $\quad f_X(x) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]$ Uni-variate normal PDF (See supp.)

$n=2$ $\quad f_{X_1X_2}(x_1,x_2) = f(\qquad\qquad\qquad)$ Bi-variate normal PDF (See supp.)

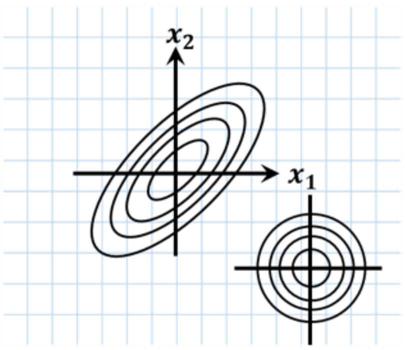
b) Properties

- Joint distribution completely defined by
- All lower order distribution are
- $\mathbf{X} = \left\{\qquad\right\}$ $\quad \mathbf{M_X} = \left\{\qquad\right\}$ $\quad \sum_{\mathbf{XX}} = \left\{\qquad\qquad\right\}$

Given $\mathbf{X_2} = \mathbf{x}_2$, then $\mathbf{X_1} \sim N(\mathbf{M}_{1|2}, \mathbf{\Sigma}_{1,1|2})$

Conditional mean and covariance

$$\begin{cases} \mathbf{M}_{1|2} = \mathbf{M}_1 + \mathbf{\Sigma}_{1,2}\mathbf{\Sigma}_{2,2}^{-1}(\mathbf{x}_2 - \mathbf{M}_2) \\ \mathbf{\Sigma}_{1,1|2} = \mathbf{\Sigma}_{1,1} - \mathbf{\Sigma}_{1,2}\mathbf{\Sigma}_{2,2}^{-1}\mathbf{\Sigma}_{2,1} \end{cases}$$



e.g. $n=2$, i.e. $\mathbf{X} = \begin{Bmatrix}\mathbf{X_1}\\ \mathbf{X_2}\end{Bmatrix} = \begin{Bmatrix}X_1\\ X_2\end{Bmatrix}$

$X_1 \sim N(\mu_{1\|2}, \sigma_{1\|2}^2)$	if $\rho = 0$ (“ ”)
$\mu_{1\|2} = \mu_1 + \rho\sigma_1\left(\frac{x_2-\mu_2}{\sigma_2}\right)$	$\mu_{1\|2} =$
$\sigma_{1\|2}^2 = \sigma_1^2(1-\rho^2)$	$\sigma_{1\|2}^2 =$

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- Uncorrelated ( ) s.i for jointly normal
  (in general, $\rho = 0$ □ $s.i$)

- Linear functions of $\mathbf{X} \sim N(\mathbf{M}, \mathbf{\Sigma})$ → follow ____________

$$\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{A_0}$$

$$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(\mathbf{x}) \cdot \qquad J_{\mathbf{Y},\mathbf{X}} = \qquad \therefore \det =$$

$$f_{\mathbf{Y}}(\mathbf{y}) \propto \exp\left[-\frac{1}{2}(\mathbf{x}-\mathbf{M}_{\mathbf{x}})^{\mathrm{T}}\mathbf{\Sigma}_{\mathbf{xx}}^{-1}(\mathbf{x}-\mathbf{M}_{\mathbf{x}})\right]$$

In summary, $\mathbf{X} \sim N(\mathbf{M}_{\mathbf{X}}, \mathbf{\Sigma}_{\mathbf{XX}})$

$$\Rightarrow \mathbf{Y} \sim N(\mathbf{M}_{\mathbf{Y}}, \mathbf{\Sigma}_{\mathbf{YY}})$$

$$\mathbf{M}_{\mathbf{Y}} =$$

$$\mathbf{\Sigma}_{\mathbf{YY}} =$$

c) Standard Normal

For univariate, 'standard normal' means, $\mu =$ $\quad, \sigma =$

$\therefore$ For jointly normal,

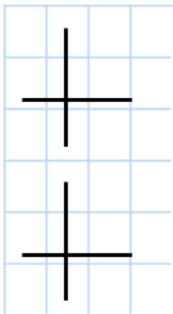
$$\mathbf{M}_{\mathbf{X}} =$$

$$\mathbf{\Sigma}_{\mathbf{XX}} =$$

$$\mathbf{Z} \sim N(\mathbf{0}, \quad) \qquad \varphi_n(\mathbf{z}, \quad) = \frac{1}{(2\pi)^{n/2}\sqrt{\det}} \cdot \exp\left[-\frac{1}{2}\mathbf{z}^{\mathrm{T}}\mathbf{R}_{\mathbf{XX}}\mathbf{z}\right]$$

$$\mathbf{U} \sim N(\mathbf{0}, \quad) \qquad \varphi_n(\mathbf{u}, \quad) = \frac{1}{(2\pi)^{n/2}\sqrt{\det}} \cdot \exp\left[-\frac{1}{2} \qquad \right]$$

$$= \prod_{i=1}^{n}$$



U used for FORM/SORM

For normal,

$$\begin{cases} \mathbf{x} = \mathbf{D}\mathbf{L}\mathbf{u} + \mathbf{M} \\ \mathbf{u} = \mathbf{L}^{-1}\mathbf{D}^{-1}(\mathbf{x} - \mathbf{M}) \end{cases}$$