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457.646 Topics in Structural Reliability

In-Class Material: Class 08

III. Structural Reliability (Component)

◎ Joint Probability Distribution Models

② Joint Lognormal

$X_1, \cdots, X_n$ are jointly lognormal if $\ln X_1, \cdots, \ln X_n$ are jointly __________

a) Parameters

$$\lambda_i = E[\qquad] = \ln \mu_i - 0.5 \ln(1+\delta_i^2)$$

$$\xi_i^2 = Var[\qquad] = \ln(1+\delta_i^2) \ (\cong \delta_i^2 \text{ for } \delta \quad 1)$$

$$\rho_{\ln X_i, \ln X_j} = \frac{1}{\xi_i \xi_j} \ln(1+\rho_{ij}\delta_i\delta_j)$$

b) Properties

- Completely defined in terms of (   ) & (   )
- All lower order distribution are jointly
- Conditional distribution are jointly
- Uncorrelated ⇄ S.I.
- Product / Quotient of jointly lognormal r.v.'s follows
- $\rho_{X_i, \ln X_j} = \frac{1}{\xi_i} \delta_j \rho_{ij}$

③ General Joint Distribution Forms

e.g. Johnson & Kotz (1976)

$\Rightarrow$ on multivariate prob. distribution models

④ Joint Distribution by conditioning (e.g. Bayesian Networks)

$$f(x_1, \cdots, x_n) = f(x_n | x_1, \cdots, x_{n-1}) \times$$

⑤ Joint Distribution model with : Prescribed marginals: $\qquad , i = 1, \cdots, n$ and

correlation coefficient matrix :

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- Read CRC Ch.14
- **See Liu & Kiureghian (1986)** a) Morgenstern

  b) Nataf

※ "Copula": formula to construct joint PDF with marginal distributions
(Review by Jongmin Park (SNU): Term Project Report in 2014)

a) Morgenstern distribution

$$\boxed{F_{\mathbf{X}}(\mathbf{x}) = \prod_{i=1}^{n} F_{X_i}(x_i) \cdot \left\{1 + \sum_{i<j} \alpha_{ij}[1 - F_{X_i}(x_i)][1 - F_{X_j}(x_j)]\right\}}$$

Q) Can we derive $F_{X_i}(x_i)$ from $F_{\mathbf{X}}(\mathbf{x})$?

i.e. $x_2, x_3, \cdots, x_n \to$ then $F_{\mathbf{X}}(\mathbf{x}) =$?

Q) Can we describe dependence using α_{ij} ?

$F_{X_i X_j}(x_i, x_j) =$

$f_{X_i X_j}(x_i, x_j) = \underline{\qquad\qquad}$

$$= f_{X_i}(x_i) \cdot f_{X_j}(x_j) \cdot \left\{1 + \alpha_{ij}[1 - 2F_{X_i}(x_i)][1 - 2F_{X_j}(x_j)]\right\}$$

$\Rightarrow \qquad \le \alpha_{ij} \le$

$$\begin{cases} \alpha_{ij} = 0 \\ \alpha_{ij} \ne 0 \end{cases}$$

Therefore, α_{ij} is a parameter that represents (corr coeff.)

But α_{ij} ρ_{ij}

Lin & Der Kiureghian (1986) showed

$$\rho_{ij} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\frac{x_i - \mu_i}{\sigma_i}\right)\left(\frac{x_j - \mu_j}{\sigma_j}\right) f_{X_i X_j}(x_i, x_j)\, dx_i dx_j$$

$$= 4\alpha_{ij} Q_i Q_j \qquad \Rightarrow \qquad |\rho_{ij}| \le 0.30$$

Where $Q_i = \int_{-\infty}^{\infty} \left[\left(\frac{x_i - \mu_i}{\sigma_i}\right) F_{X_i}(x_i)\right] f_{X_i}(x_i) dx_i \approx 0.28$

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Table 1: selected distribution

Table 2: Q_i

Table 3 : maximum $|\rho_{ij}|$

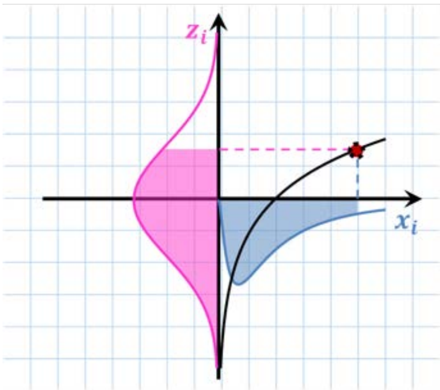
$\Rightarrow$ In summary, using Morgenstern's model, you cannot describe X_i, X_j whose $|\rho_{ij}| > 0.30$

b) Nataf model (Nataf, 1962) ("Gaussian Copula")

$$\mathbf{X} \sim F_{X_i}(x_i),\ i = 1, \cdots, n \qquad \mathbf{R} = [\rho_{ij}] \quad \underset{\overset{(2)}{Nataf}}{\overset{(1)}{\rightleftarrows}} \quad \mathbf{Z} \sim N(\mathbf{0}, \mathbf{R}') \qquad \mathbf{R}' = [\rho'_{ij}]$$

Transformation to $\mathbf{Z}$

$$Z_i =$$



Why?

$$f_{Z_i}(z_i) = f_{X_i}(x_i) \cdot \left|\frac{dx_i}{dz_i}\right|$$

$$f_{Z_i}(z_i) \cdot \qquad = f_{X_i}(x_i) \cdot$$

$$\Phi(\quad) = F_{X_i}(\quad)$$