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457.646 Topics in Structural Reliability

In-Class Material: Class 20

“ ” (cf.)

Ⅴ. Structural Reliability under Model & Stastical Uncertainties

(Ref.: “Analysis of Structural Reliability under Model and Statistical Uncertainties: A Bayesian Approach” ~ eTL)

◎ Formulation of Reliability Problems under Epistemic Uncertainties

① Reliability Problem with Aleatoric uncertainties (only)

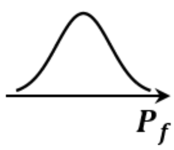
$$P_f = \int f_{\mathbf{x}}(\mathbf{x})d\mathbf{x}$$

$\mathbf{x}$: r.v’s representing aleatoric uncertainties in the problem

→ Use component and/or system reliability method

② Reliability Problem under Aleatoric & Epistemic certainties

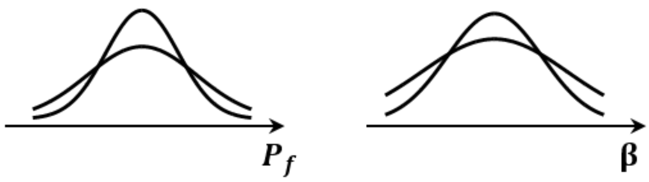
$$P_f(\boldsymbol{\theta}) = \int_{\cup\cap g(\mathbf{x};\ \ )\le 0} f_{\mathbf{x}}(\mathbf{x};\ \ )d\mathbf{x} \Rightarrow \beta(\boldsymbol{\theta}) = \Phi^{-1}[\qquad]$$



$$\boldsymbol{\theta} = [\boldsymbol{\theta}_f \quad \boldsymbol{\theta}_g]$$

uncertain parameters

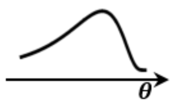
⇒ P_f & β become __________ due to uncertainty in $\boldsymbol{\theta}_f$ and/or $\boldsymbol{\theta}_g$



cf. $\beta(\boldsymbol{\theta})$, $P_f(\boldsymbol{\theta})$ ⇒ _________ reliability index given value of uncertain parameters

◎ Three approaches for estimating reliability under epistemic uncertainties

Suppose $f_{|\boldsymbol{\theta}|}(\boldsymbol{\theta})$ is available,



① Point estimate of Reliability: $P_f(\boldsymbol{\theta})$ at $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}$

$\hat{\boldsymbol{\theta}}$: point estimate (representative) of $\boldsymbol{\theta}$

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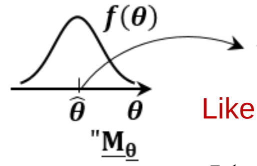
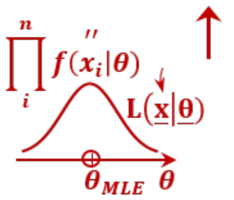
e.g. $\begin{cases} \hat{\boldsymbol{\theta}} = \mathbf{M}_{\boldsymbol{\theta}} = \int \boldsymbol{\theta} f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\, d\boldsymbol{\theta} & \text{Bayesian} \\ \hat{\boldsymbol{\theta}} = \boldsymbol{\theta}_{MLE} = \arg\max\limits_{\boldsymbol{\theta}} L(\mathbf{x};\boldsymbol{\theta}) & \text{Non-Bayesian} \end{cases}$



Likelihood function

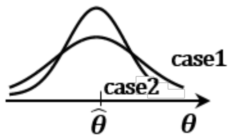
$L(\mathbf{x};\boldsymbol{\theta}) \propto P(x,\cdots,x_n|\boldsymbol{\theta})$



$\Rightarrow$ $P_f(\hat{\boldsymbol{\theta}})$, $\beta(\hat{\boldsymbol{\theta}})$: Perform reliability analysis with $\boldsymbol{\theta} =$ fixed

Note i) $P_f(\mathbf{M}_{\boldsymbol{\theta}}) = \mathbf{M}^{FO}_{P_f(\boldsymbol{\theta})}$

ii) Variability in $\boldsymbol{\theta}$ not considered



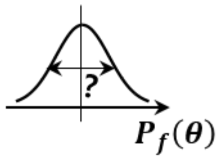
② "Predictive" Reliability

$$\tilde{P}_f = E_{\boldsymbol{\theta}}\left[P_f(\boldsymbol{\theta})\right] = \int P_f(\boldsymbol{\theta}) \cdot f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\, d\boldsymbol{\theta}$$

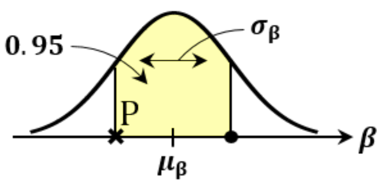
$$\tilde{\beta} = \Phi^{-1}[\qquad]$$

→ incorporates variability in $\boldsymbol{\theta}$

→ but still point estimate, i.e. does not measure variability in $P_f(\boldsymbol{\theta})$ caused by that in $\boldsymbol{\theta}$



③ Bounds on Reliability (Confidence Intervals)



$100 \times p(\%)$ confident that β is b/w x and o

First, find mean and variance of $\beta(\boldsymbol{\theta})$

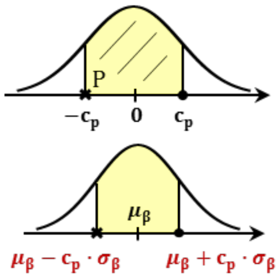
reliability analysis $\qquad$ $f_{\boldsymbol{\theta}}(\boldsymbol{\theta})$

$$\begin{cases} \mu_\beta \cong \mu_\beta^{FO} = \beta(\mathbf{M}_{\boldsymbol{\theta}}) \\ \sigma_\beta{}^2 \cong (\sigma_\beta{}^2)^{FO} = \nabla_{\boldsymbol{\theta}}\beta(\mathbf{M}_{\boldsymbol{\theta}})\boldsymbol{\Sigma}_{\boldsymbol{\theta\theta}}\nabla_{\boldsymbol{\theta}}\beta(\mathbf{M}_{\boldsymbol{\theta}})^T \end{cases}$$

Parameter sensitivity (e.g. FORM)

Second, assume $\beta \sim N\left(\mu_\beta, \sigma_\beta\right)$

Std. normal



$$u_\beta = \frac{\beta - \mu_\beta}{\sigma_\beta}$$

P	c_p
0.70	1.04
0.80	1.28
0.90	1.64
0.95	1.96
0.99	2.58

$$\langle \beta \rangle_{100\times p(\%)} = \mu_\beta \pm c_p \sigma_\beta$$

(if $\tilde{\beta}$ available, $\tilde{\beta} \pm c_p \sigma_\beta$)

$$\langle P_f \rangle_{100\times p(\%)} = \Phi\left[-\left(\tilde{\beta} \pm c_p \sigma_\beta\right)\right] \qquad P_f = \Phi(-\beta)$$

Then, $f_{\boldsymbol{\theta}_f}\left(\boldsymbol{\theta}_f\right)$, $f_{\boldsymbol{\theta}_g}\left(\boldsymbol{\theta}_g\right)$??

(Review) Rel. Analysis under Epistemic Uncertainties (Model or Statistical)

① Point Estimate $\quad P_f\left(\hat{\theta}\right)$, $\beta\left(\hat{\theta}\right)$

② Predictive Reliability $\quad \tilde{P}_f = E_\theta\left[P_f(\theta)\right]$

③ Bounds $\quad \langle \beta \rangle_{100\times p(\%)} = \mu_\beta \pm c_p \sigma_\beta$

$f_{\theta_f}\left(\theta_f\right)$? $f_{\theta_g}\left(\theta_g\right)$?

◎ Bayesian Parameter Estimation

$$f(\boldsymbol{\theta}) = c \cdot L(\boldsymbol{\theta}) \cdot p(\boldsymbol{\theta})$$

cf. Bayes rule

$$\underset{f}{P(A|B)} = \underset{c}{\frac{1}{P(B)}} \cdot \underset{L}{P(B|A)} \cdot \underset{p}{P(A)}$$

① $P(\boldsymbol{\theta})$: ( ) distribution

- represents state of our knowledge ( ) making observations (objective information)
- may incorporate ( ) info. such as “engineering judgment”

② $L(\boldsymbol{\theta})$: (   ) function

- represents (   ) information gained from the observation
- function (   ) to conditional prob. of the observation given $\boldsymbol{\theta}$

$$L(\boldsymbol{\theta}) \propto P(E_{obs} | \boldsymbol{\theta})$$

③ c: (   ) factor

- makes $c \cdot L(\boldsymbol{\theta}) \cdot p(\boldsymbol{\theta})$ a valid PDF

i.e. $\int_{\boldsymbol{\theta}} f(\boldsymbol{\theta}) = \int_{\boldsymbol{\theta}} c \cdot L(\boldsymbol{\theta}) \cdot P(\boldsymbol{\theta}) d\boldsymbol{\theta} =$

$\therefore \; c =$

④ $f(\boldsymbol{\theta})$: (   ) distribution

- represents updated knowledge about $\boldsymbol{\theta}$
- subjective   +   objective
-

;

;

rare observation available

as more observations are made

◎ Computation of c and posterior statistics

$$\left.\begin{aligned} c &= \left[\int L(\boldsymbol{\theta}) \cdot p(\boldsymbol{\theta}) \cdot d\boldsymbol{\theta} \right]^{-1} \\ \mathbf{M}(\boldsymbol{\theta}) &= \int \boldsymbol{\theta} \cdot f(\boldsymbol{\theta}) d\boldsymbol{\theta} = \int \boldsymbol{\theta} \cdot c \cdot L(\boldsymbol{\theta}) \cdot p(\boldsymbol{\theta}) \cdot d\boldsymbol{\theta} \\ \boldsymbol{\Sigma}_{\boldsymbol{\theta\theta}} &= \int \boldsymbol{\theta\theta}^T f(\boldsymbol{\theta}) d\boldsymbol{\theta} - \mathbf{M}(\boldsymbol{\theta}) \mathbf{M}(\boldsymbol{\theta})^T \end{aligned}\right\} \text{multi-fold integrals}$$

How?

$$\left\{\begin{array}{l} \text{Convenient forms for special distribution (directly update statistics “conjugate”)} \\ \text{Special numerical algorithms (Geyskens et al. 1993)} \\ \text{Sampling methods (MCS, importance sampling, } \cdots \text{)} \end{array}\right.$$