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**457.646 Topics in Structural Reliability**

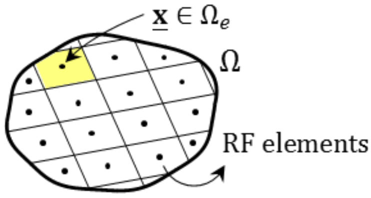
**In-Class Material: Class 26**

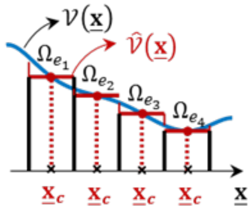
## ◎ Discrete Representation of RFs (Summary: Sudret & ADK 2000; 2002 PEM)

① Mid-point method

$$v(\mathbf{x}) \simeq \hat{v}(\mathbf{x}) = v(\mathbf{x}_c), \ \mathbf{x} \in \Omega_e$$

(constant in each $\Omega_e$)




- Represented by a constant r.v.

  over each RF element

- Positive definiteness problem of $\mathbf{R}$ … if RF element size is small relative to $\theta$

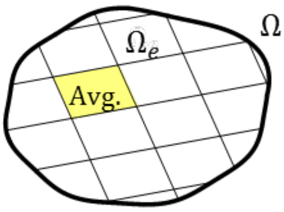
Recommended size of RF element size

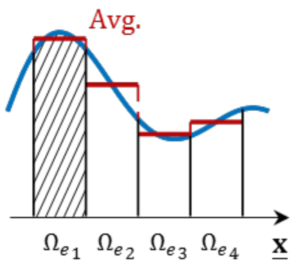
$$\frac{\theta}{10} \sim \frac{\theta}{15} \leq \text{ RF size } \leq \frac{\theta}{3} \sim \frac{\theta}{5}$$

Numerical stability (Positive definiteness) | Accurate representation

② Spatial averaging method

$$\hat{v}(\mathbf{x}) = \frac{\int_{\Omega_e} v(\mathbf{x}) d\Omega}{\int_{\Omega_e} d\Omega}, \quad \mathbf{x} \in \Omega_e$$




- Represented by a <u>single</u> r.v per $\Omega_e$
- Variances are ( ) → _____-estimate $P_f$
- Positive definiteness problem

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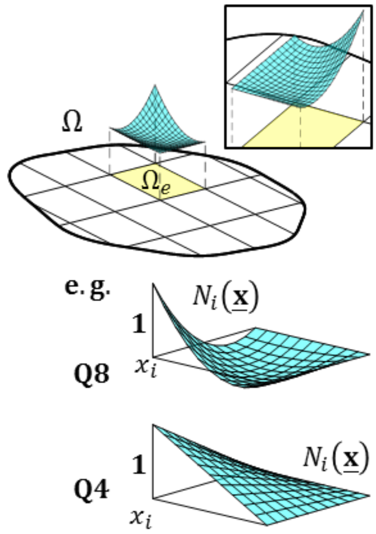
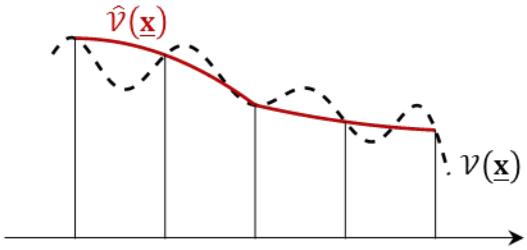
③ Shape function method (←motivated by FE people)

$$v(\mathbf{x}) \simeq \hat{v}(\mathbf{x}) = \sum_{\substack{\text{element}\\\text{nodes}}} N_i(\mathbf{x}) v(\mathbf{x}_i)$$




• Represented by continuous function




$$N_i(\mathbf{x}_j) = \delta_{ij}$$

to guarantee $\hat{v}(\mathbf{x}_i) = v(\mathbf{x}_i)$

④ Karhunen-Loève (KL) expansion (Gaussian RFs)

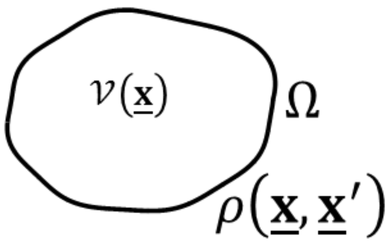
→ Describe RF in terms of finite # of shape functions

defined over ________ domain

(no geometric discretization)

→ Discretization based on

______________ structure $\rho(\mathbf{x}, \mathbf{x}')$




Goal: Want to descrive $\rho(\mathbf{x}, \mathbf{x}')$ by

$$\rho(\mathbf{x}, \mathbf{x}') = \sum_{i=1}^{\infty} \lambda_i \varphi_i(\mathbf{x}) \varphi_i(\mathbf{x}')$$

Orthogonal shape (base) functions

Can find $\lambda$, $\varphi$ by solving an integral eigenvalue problem, i.e.

$$\int_{\Omega} \rho(\mathbf{x}, \mathbf{x}') \varphi_i(\mathbf{x}') d\mathbf{x}' = \lambda_i \varphi_i(\mathbf{x})$$ (Fredholem integral eqn – 2nd kind)

Note $\rho(\mathbf{x}, \mathbf{x}')$ is bounded, symmetric, (+) definite.

If so, one can find

$\varphi_i(\mathbf{x})$: orthogonal $\int \varphi_i(\mathbf{x}) \varphi_j(\mathbf{x}) d\mathbf{x} = \delta_{ij}$

$\lambda_i$: real & positive

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Can drop $\lambda_i$'s if $\lambda_r \cong 0$

Then using $\varphi_i(\mathbf{x})$, and $\lambda_i$, $i=1,\ldots,r$, one can describe Gaussian RF $v(x)$ by

KL expansion of Gaussian RF

$$\boxed{v(\mathbf{x}) \simeq \hat{v}(\mathbf{x}) = \mu(\mathbf{x}) + \sigma(\mathbf{x})\sum_{i=1}^{r}(u_i\sqrt{\lambda_i}\varphi_i(\mathbf{x})), \quad x \in \Omega} \Rightarrow v(\mathbf{x}) \Rightarrow \{u_1,\cdots,u_r\}$$

$u_i \rightarrow N(0,1),\ u_i$ s.i

Let's check!

i. Gaussian? Yes, function of $u_i$'s

ii. $E[\hat{v}(\mathbf{x})] = \mu(\mathbf{x})$? $E[\hat{v}(\mathbf{x})] =$

iii. $Var[\hat{v}(\mathbf{x})] = E[( \qquad )^2]$

$$= E[ \qquad \sum_{i=1}^{r}\sum_{j=1}^{r} \qquad ]$$

$$= \sigma^2(\mathbf{x})\sum_{i=1}^{r}\sum_{j=1}^{r} \qquad \sqrt{\lambda_i}\sqrt{\lambda_j}\varphi_i(\mathbf{x})\varphi_j(\mathbf{x})$$

$$= \sigma^2(\mathbf{x})\sum_{i=1}^{r}\lambda_i\varphi_i^2(\mathbf{x})$$

$$= \sigma^2(\mathbf{x})$$

(because $\rho(\mathbf{x},\mathbf{x}) = \qquad = \qquad$ )

iv. $\rho_{\hat{v}\hat{v}}(\mathbf{x},\mathbf{x}') \overset{?}{=} \rho(\mathbf{x},\mathbf{x}')$

$$= E[(\hat{v}(\mathbf{x}) - \mu(\mathbf{x}))(\hat{v}(\mathbf{x}') - \mu(\mathbf{x}'))] / \sigma(\mathbf{x})\sigma(\mathbf{x}')$$

$$= E[\sum_{i=1}^{r}\sum_{j=}^{r} u_i\sqrt{\lambda_i}\varphi_i(\mathbf{x})u_j\sqrt{\lambda_j}\varphi_j(\mathbf{x}')]$$

$$= \sum_{i=1}^{r}\sum_{j=1}^{r} E[ \qquad ]\sqrt{\lambda_i}\sqrt{\lambda_j}\varphi_i(\mathbf{x})\varphi_i(\mathbf{x}')$$

$$= \sum_{i=1}^{r}\lambda_i\varphi_i(\mathbf{x})\varphi_i(\mathbf{x}')$$

$$= \varphi(\mathbf{x},\mathbf{x}')$$

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- # of RV's:
- Represented by function
- No necessary
- Most efficient (in terms of # of )
- Requires solution of an integral eigenvalue problem.

⑤ Orthogonal expansion (eigen-expansion, but correlated rv's)

⑥ Optimal linear estimation (OLE)~ linear regression

⑦ Expansion OLE

⋮ See Sudret & ADK (2000)

**◎ Nataf RF**

$v(\mathbf{x}) \Rightarrow F(v, \mathbf{x})$, $\rho_{ZZ}(\mathbf{x}, \mathbf{x}')$

$v(\mathbf{x}) = F_v^{-1}\{\Phi(\hat{Z}(\mathbf{x}))\}$, $Z(\mathbf{x}) \sim N(\mathbf{0}, \rho_{ZZ}(\mathbf{x}, \mathbf{x}'))$ ($Z(\mathbf{x}) \rightarrow$ Gaussian RF)

⇒ Construct $Z(\mathbf{x})$ and discrete to $\hat{Z}(\mathbf{x})$

⇒ $v(\mathbf{x}) = F^{-1}\{\Phi(\hat{Z}(\mathbf{x}))\}$

**VIII-4. Response Surface Method (CRC Ch.19 & Mike Tipping's chapter)**

**◎ Reliability Analysis, Uncertainty Quantification & Response Surface**

Reliability Analysis

$P_f = \int_{g(\mathbf{x})\le 0} f_{\mathbf{x}}(\mathbf{x})d\mathbf{x}$ → e.g. FORM/SORM $g(\mathbf{x}_i)$, $\nabla g(\mathbf{x}_i)$

→ e.g. Sampling $q_i = I(\mathbf{x}_i)$ or $\dfrac{I(\mathbf{x}_i)\cdot f(\mathbf{x}_i)}{h(\mathbf{x}_i)}$

where $I(\mathbf{x}_i) = \begin{cases} 1 & g(\mathbf{x}_i) \le 0 \\ 0 & g(\mathbf{x}_i) > 0 \end{cases}$

Uncertainty Quantification

"Process of determining the effect of input uncertainties"

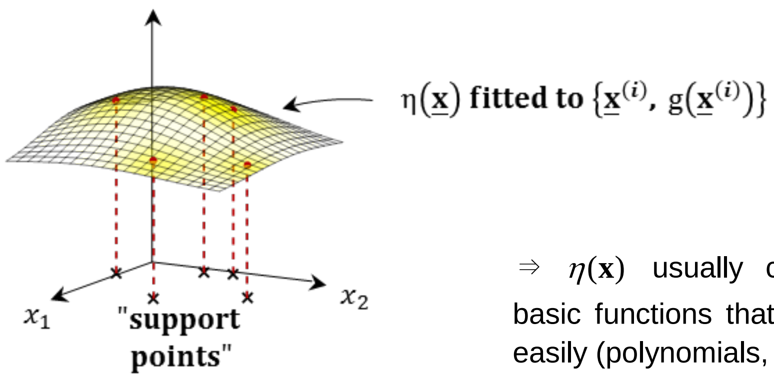
on response metrics of interest (Eldred et al. 2008)

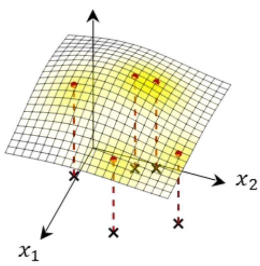
e.g. $E[g(\mathbf{x})^m] = \int_{\mathbf{x}} g(\mathbf{x})^m f_{\mathbf{x}}(\mathbf{x})d\mathbf{x}$

① $g(\mathbf{x})$ Sometimes

$\begin{cases} \text{Computationally costly for MCS} \\ \text{No analytical gradients but many RVs} \\ \qquad \Rightarrow \text{FORM/SORM difficult} \\ \text{Experiments expensive (statistical analysis of experiment data infeasible)} \end{cases}$

② Idea: $g(\mathbf{x}) \simeq \eta(\mathbf{x})$ ($\eta(\mathbf{x})$ ← "response surface" or "surrogate" model)




⇒ $\eta(\mathbf{x})$ usually constructed in terms of basic functions that can be computed more easily (polynomials, exp, ⋯ )

⇒ Should fit $g(\mathbf{x}^{(i)})$ sufficiently well especially in the region that contributes most to $P_f$ or $E[g(\mathbf{x})^m]$

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③ History

- Box and Wilson (1954): influential
- Applied mostly in chemical, industrial eng. etc.

  (Mostly for “experimental design”)
- Rackwitz (1982) ⇒ Use RS for Structural Reliability Analysis
- Has been applied to random field, nonlinear structural dynamics, etc.