Seoul National University
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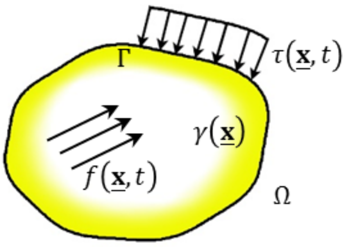
457.646 Topics in Structural Reliability

In-Class Material: Class 28

VII-4. Finite Element Reliability Analysis (Haukaas, 2006)

→ summary and good findings

◎ Equations of Motion and Randomness



"Weak" form of equilibrium:

$$\int_{\Omega} \delta u_i \gamma \ddot{u}_i d\Omega + \int_{\Omega} \delta u_{i,j} \sigma_{ij} d\Omega - \int_{\Omega} \delta u_i f_i d\Omega - \int_{\Gamma} \delta u_i \tau_i d\Gamma = 0$$

γ : density, $\ddot{u}_i$: acc, $u_{i,j}$: strain, σ_{ij} : stress, f_i : body force, τ_i : traction

① Basic random fields

$C_{ijkl}(\mathbf{x})$, $\gamma(\mathbf{x})$: material properties (constants)

Tensor of material elastic constants, $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

$f_i(\mathbf{x},t)$, $\tau_i(\mathbf{x};t)$: loads

Ω, Γ : geometry

$\Rightarrow$ Discretized to a random vector $\mathbf{v}$

② Derived response is a function of $\mathbf{v}$

$u_i(\mathbf{x},t,\mathbf{v})$: displacement

$\varepsilon_{ij}(\mathbf{x},t,\mathbf{v})$: strain

$\varepsilon_{ij}^P(\mathbf{x},t,\mathbf{v})$: plastic strain

$\sigma_{ij}(\mathbf{x},t,\mathbf{v})$: stress

$\vdots$

$\mathbf{S}_i(\mathbf{x},t,\mathbf{v})$: generic response vector

$g(S(\mathbf{v}),\mathbf{v}) \le 0$

③ FE models and r.v's

i. Nonlinear & Dynamic problem

$$\mathbf{M}(\mathbf{v})\ddot{\mathbf{u}}(t,\mathbf{v})+\mathbf{C}(\mathbf{v})\dot{\mathbf{u}}(t,\mathbf{v})+\mathbf{R}(\mathbf{u}(t,\mathbf{v}),\mathbf{v})=\mathbf{P}(t,\mathbf{v})$$

ii. Static problem

$$\mathbf{R}(\mathbf{u}(t,\mathbf{v}),\mathbf{v})=\mathbf{P}(t,\mathbf{v})$$

iii. Linear Static problem

$$\mathbf{K}(\mathbf{v})\cdot\mathbf{u}(\mathbf{v})=\mathbf{P}(\mathbf{v})$$

④ FE reliability analysis

i. MCS $\mathbf{v}_i,\ \ i=1,\cdots,N$

ii. Importance Sampling

iii. Response Surface $g\approx\eta(\mathbf{x})$

iv. Form (HLRF)

Initialize $\mathbf{u}_1=\mathbf{u}(\mathbf{v}_1)$

↓

$\mathbf{v}_i=\mathbf{v}(\mathbf{u}_i)$ skip if $i=1$

$G(\mathbf{u}_i)=g(\mathbf{S}(\mathbf{v}_i),\mathbf{v}_i)$

$$\nabla_{\mathbf{u}}G(\mathbf{u}_i)=\nabla_{\mathbf{v}}g(\mathbf{v})J_{\mathbf{v},\mathbf{u}}$$
$$=(\nabla_{\mathbf{s}}g\cdot J_{\mathbf{s},\mathbf{v}}+\nabla_{\mathbf{v}}g)\cdot J_{\mathbf{v},\mathbf{u}}$$

$\xleftarrow{\mathbf{S}(\mathbf{v}_i),\,J_{\mathbf{s},\mathbf{v}}}$ FE code

↓ The same procedure

⋮

e.g. FERUM-ABAQUS

(Young Joo, Lee, 2012)

◎ Gradient $J_{\mathbf{s},\mathbf{v}}$?

e.g. $\dfrac{\partial u_i}{\partial E},\dfrac{\partial \sigma_i}{\partial P},\cdots$

Methods to get sensitivity $J_{\mathbf{s},\mathbf{v}}$

e.g. Linear Static Problem (suppose there is only one r.v. $\mathbf{v}=v$)

$$\underbrace{\mathbf{K}(v)}_{\text{Stiffness}}\cdot\underbrace{\mathbf{u}(v)}_{\text{displacement}}=\underbrace{\mathbf{P}(v)}_{\text{loads}}\quad\rightarrow\quad\mathbf{u}=\mathbf{K}^{-1}(v)\cdot\mathbf{P}(v)$$

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① Finite Difference Method ("FFD" option of FERUM)

$\mathbf{u}(v) = \mathbf{K}^{-1}(v) \cdot \mathbf{P}(v)$ original FE

$\mathbf{u}(v+\Delta v) = \mathbf{K}^{-1}(v+\Delta v) \cdot \mathbf{P}(v+\Delta v)$ (i.e. additional FE for each v_i in $\mathbf{v}$)

$$\frac{\partial \mathbf{u}}{\partial v} \cong \frac{\mathbf{u}(v+\Delta v) - \mathbf{u}(v)}{\Delta v}$$

⇒ Need to solve FE again (for each r.v)

⇒ Can cause numerical errors

② Perturbation Method

$$\mathbf{K}\mathbf{u} = \mathbf{P}$$

$$\Delta \mathbf{K} = \mathbf{K}(v+\Delta v) - \mathbf{K}(v)$$
$$\Delta \mathbf{P} = \mathbf{P}(v+\Delta v) - \mathbf{P}(v)$$

$$(\mathbf{K} + \Delta\mathbf{K})(\mathbf{u} + \Delta\mathbf{u}) = \mathbf{P} + \Delta\mathbf{P}$$

$$\mathbf{K}\mathbf{u} + \mathbf{K}\Delta\mathbf{u} + \Delta\mathbf{K}\mathbf{u} + \Delta\mathbf{K}\Delta\mathbf{u} = \mathbf{P} + \Delta\mathbf{P}$$

$$\therefore \Delta\mathbf{u} \cong \mathbf{K}^{-1}(\Delta\mathbf{P} - \Delta\mathbf{K}\mathbf{u})$$

⇒ Do not have to re-solve FE

⇒ Error ($\Delta\mathbf{K}\Delta\mathbf{u} \approx 0$)

③ Direct Differentiation Method ('DDM' option for FERUM)

$$\mathbf{K}\mathbf{u} = \mathbf{P}$$

$$\frac{\partial \mathbf{K}}{\partial v}\mathbf{u} + \mathbf{K}\frac{\partial \mathbf{u}}{\partial v} = \frac{\partial \mathbf{P}}{\partial v}$$

$$\frac{\partial \mathbf{u}}{\partial v} = \mathbf{K}^{-1}\left(\frac{\partial \mathbf{P}}{\partial v} - \frac{\partial \mathbf{K}}{\partial v}\mathbf{u}\right)$$

→ Do not need to solve FEM again

→ No error

→ $\dfrac{\partial \mathbf{K}}{\partial v} = \sum_e \dfrac{\partial \mathbf{K}^e}{\partial v}$ ($\dfrac{\partial \mathbf{K}^e}{\partial v}$ ← direct stiffness method)

→ Nonlinear static, nonlinear dynamic

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④ Adjoint method

➔ Tutorial by Prof. Andrew M. Bradley at Stanford University: http://cs.stanford.edu/~ambrad/adjoint_tutorial.pdf)

$x \in \mathbb{R}^{n_x}$, $p \in \mathbb{R}^{n_p}$, $f(x(p)):\ \mathbb{R}^{n_x} \rightarrow \mathbb{R}$

Subject to $h(x(p), p) = 0$ for $h:\ \mathbb{R}^{n_x} \times \mathbb{R}^{n_p} \rightarrow \mathbb{R}$

e.g. h=0 → PDE equilibrium (mass ∈ p, displacement ∈ x, member force = f)

$d_p f$? (total derivative of f w.r.t p)

Consider the Lagrangian

$$L(x, p, \lambda) = f(x(p)) + \lambda^{\mathrm{T}} h(x(p), p)$$

$$\begin{aligned} d_p f &= d_p L \qquad (\because \text{ only on } h = \qquad\qquad) \\ &= \partial_x f d_p x + d_p \lambda^{\mathrm{T}} h + \lambda^{\mathrm{T}}(\partial_x h d_p x + \partial_p h) \\ &= f_x x_p + \lambda^{\mathrm{T}}(h_x x_p + h_p) \qquad (\because \qquad\qquad) \\ &= (f_x + \lambda^T h_x) x_p + \lambda^{\mathrm{T}} h_p \end{aligned}$$

Choose λ such that $h_x^{\mathrm{T}} \lambda = -f_x^{\mathrm{T}}$ (“adjoint equation”) → λ*

Then we can avoid calculating ( )

Then compute $d_p f$ as ___________

⇒ Used for RBTO of structures under stochastic excitations (Chun, Song and Paulino, 2016)

VI. Simulation methods (contd.)

◎ Latin Hypercube Sampling (Mckay et al. 1979)

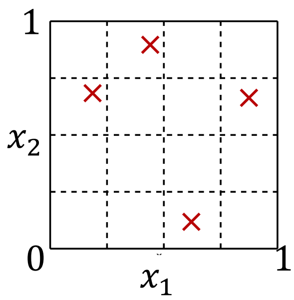
Extension of “Latin Square” – appearing exactly once in each row and exactly once in each column)

(←) 7x7 Latin Square stained glass honoring R.A. Fisher’s work on DOE

Evenly distribute sampling points to promote early convergence

e.g. $\mathbf{X} = \{X_1, X_2\}$ uniform (0,1), s.i

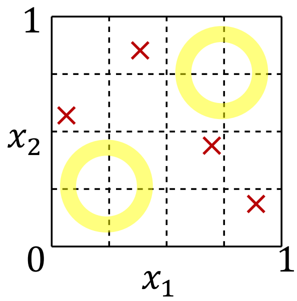
⇒ 4 samples



- Brute force MCS:

  Samples are generated independently

  No memory



- Latin Hypercube Sampling:

  There is only one sample in each row and column

  0(w/ memory)

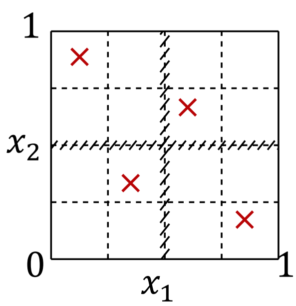
Possible LHS combination?

$$\prod_{n=0}^{M-1}(M-n)^{N-1} \quad \left.\right\} \quad M=4,\ N=2 \quad \therefore 24 \text{ cases}$$

- Orthogonal Sampling:

  LHS + subspace sampled

  w/ same frequency



choose LHS combinations that satisfy orthogonal sampling conditions

Example) Y.S. Kim et al. (2009)

→ Seismic Performance Assessment of Interdependent Lifeline Systems

⇒ Generated random samples of post-disaster conditions of network components using LHS

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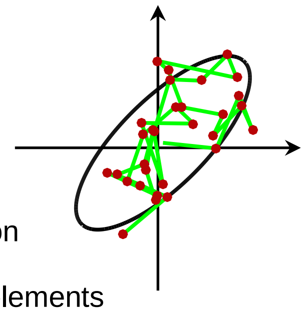
◎ Markov Chain Monte Carlo Simulation (MCMC)

$P(\mathbf{Z}^{(m+1)} | \mathbf{Z}^{(m)})$ transition prob.

→ Use MCS to generate samples as a Markov chain (good for high-dimensional problem)

① Metropolis-Hastings algorithm (Hastings, 1970)

~Accept/reject w/ probability (see next page)



② Gibbs sampling (Geman & Geman 1984)

See next page: Sample "one" element each time based on

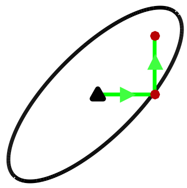
Conditional distribution given the outcomes of the other elements

e.g. $P(Z_1, Z_2, Z_3)$ $\mathbf{Z}$

sample $Z_1^{\tau+1}$ by $P(Z_1 | Z_2^{\tau}, Z_3^{\tau})$

$Z_2^{\tau+1}$ by $P(Z_2 | Z_1^{\tau}, Z_3^{\tau})$

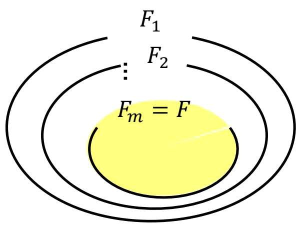
$Z_3^{\tau+1}$ by $P(Z_3 | Z_1^{\tau}, Z_2^{\tau})$



③ Subset Simulation (Au & Beck, 2001)

$F_1 \supset F_2 \supset \cdots \supset F_m = F$ event of interest

$P(F) = P(F_m)$ too low



e.g. $F_i = \{D > C_i\}$

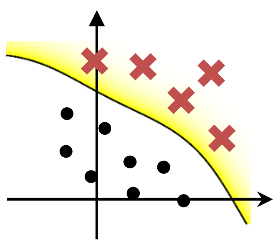
$C_1 < C_2 < \cdots < C_m = C$

$$P(F) = P(F_m) = P(\bigcap_{i=1}^{m} F_i)$$

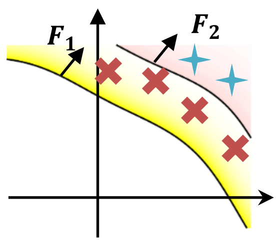
$$= P(F_m | \bigcap_{i=1}^{m-1} F_i) \cdot P(\bigcap_{i=1}^{m-1} F_i)$$

$$= P(F_m | f_{m-1}) \cdot P(\bigcap_{i=1}^{m-1} F_i)$$

$$= \underbrace{P(F_1) \times P(F_2 | F_1)}_{\text{Each larger than } P(F)} \times \cdots \times P(F_m | F_1 \cdots F_{m-1})$$



$P(F_1)$



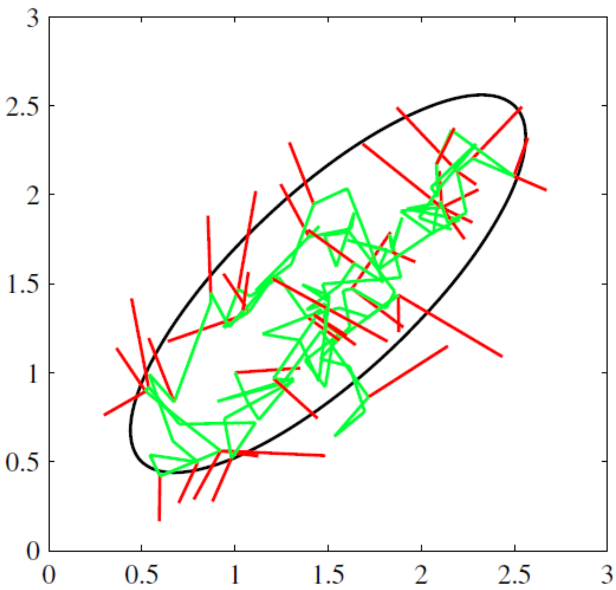
$P(F_2|F_1)$

Each larger than $P(F)$

Use MCMC algorithm to compute $P(F_{i+1} | F_i)$

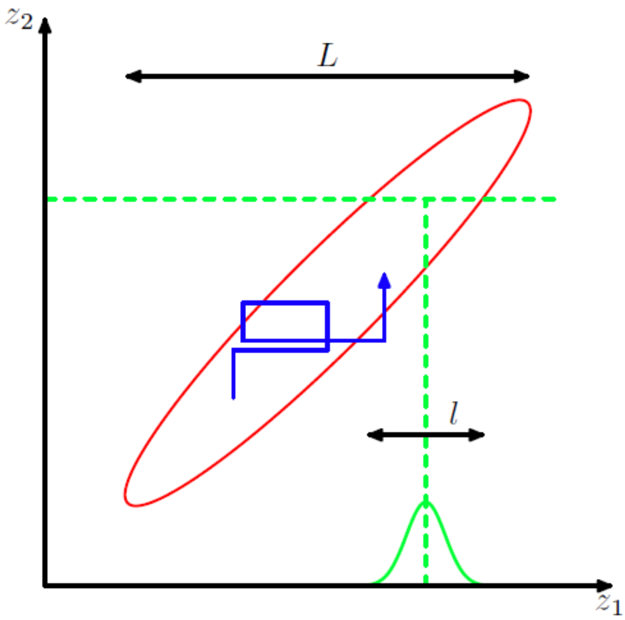
*** Generating samples of a bi-variate Gaussian distribution using Metropolis algorithm**

A simple illustration using Metropolis algorithm to sample from a Gaussian distribution whose one standard-deviation contour is shown by the ellipse. The proposal distribution is an isotropic Gaussian distribution whose standard deviation is 0.2. Steps that are accepted are shown as green lines, and rejected steps are shown in red. A total of 150 candidate samples are generated, of which 43 are rejected.



*** Generating samples of a bi-variate Gaussian distribution using Gibbs sampling**

Illustration of Gibbs sampling by alternate updates of two variables whose distribution is a correlated Gaussian. The step size is governed by the standard deviation of the conditional distribution (green curve), and is $O(l)$, leading to slow progress in the direction of elongation of the joint distribution (red ellipse). The number of steps needed to obtain an independent sample from the distribution is $O((L/l)^2)$.



Reference: “Pattern Recognition and Machine Learning” by Christopher M. Bishop (2006)

◎ Extrapolation-based MCS (Naess et al. 2009)

$g(\lambda) = g - \mu_g(1-\lambda)$ $\qquad$ $\lambda = 0:$ $\quad g(\lambda) = g - \mu_g$ $\quad P_f \simeq 50\%$

$0 \le \lambda \le 1$ $\qquad$ $\lambda = 1:$ $\quad g(\lambda) = g$ $\quad P_f \ll 1$

Generate samples $\{g_1, \cdots, g_n\}$ and use to estimate

$\tilde{P}_f(\lambda) = \dfrac{N_f(\lambda)}{N}$ while varying λ

Fitted to $\underset{\lambda\to1}{\cong} q(\lambda)\cdot\exp\{-a(\lambda-b)^c\}$ (can assume constant q), i.e.

$$\tilde{P}_f(\lambda) \underset{\lambda\to1}{\cong} q^*\cdot\exp\{-a^*(\lambda-b^*)^{c^*}\}$$

Find a, b, c, q by fitting and extrapolate as $\tilde{P}_f(\lambda)$ as $\lambda \to 1$

$\Rightarrow$ Has been applied to component/system (Naess et al. 2009)

and large-size system problems (Naess et al. 2010)

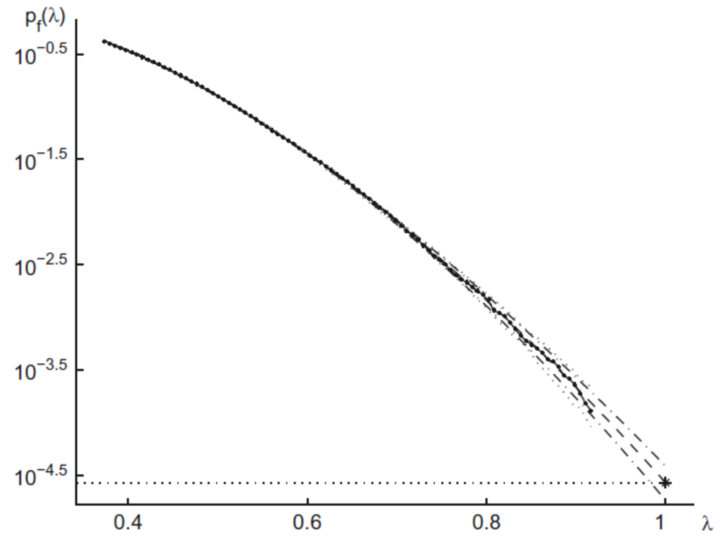


Fig. 9. Plot of $\log \hat{p}_f(\lambda_j)$ for Example 4: Monte Carlo $(\cdot)$; fitted optimal curve $(--)$; reanchored empirical confidence band $(\cdots)$; fitted confidence band $(-\cdot)$. $\log q = -0.303$, $a = 16.231$, $b = 0.252$, $c = 1.591$.

Many thanks for your hard work in this semester to learn theories of structural reliability and their applications. I wish you the very best on your course work, research and future career.

2016

Best,
Junho