

# Electromagnetics:

Time-Varying Fields

Maxwell's Equations

(7-1, 7-2, 7-3, 7-4, 7-5)

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# Electromagnetostatic Model vs. Maxwell's Equations

Electrostatic model:

$$\nabla \times \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\mathbf{D} = \epsilon \mathbf{E}$$

Magnetostatic model:

$$\nabla \times \mathbf{H} = \mathbf{J} = \sigma \mathbf{E}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{H} = \frac{1}{\mu} \mathbf{B}$$

Maxwell's equations (incl. time-varying):

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

*Faraday's law*

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

*Ampère's law*

$$\nabla \cdot \mathbf{D} = \rho$$

*Gauss's law*

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

*Constitutive relations*

$$\mathbf{H} = \frac{1}{\mu} \mathbf{B} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}$$

*Generalized electromagnetism!*

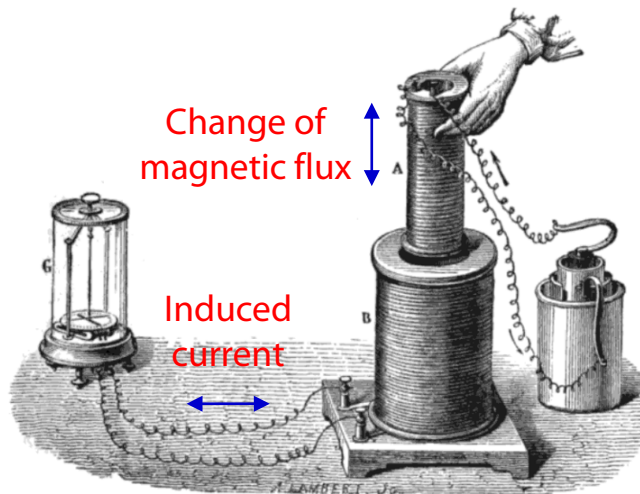
# Faraday's Law of Electromagnetic Induction

Faraday discovered in 1831:

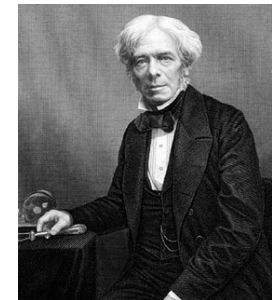
→ *The electromotive force induced in a closed circuit is equal to the negative rate of increase of the magnetic flux linking the circuit.*

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s}$$

← *Negative sign: "Lenz's law"*



Source: [http://en.wikipedia.org/wiki/Faraday's\\_law\\_of\\_induction](http://en.wikipedia.org/wiki/Faraday's_law_of_induction)



Michael Faraday  
(1791–1867)

# Faraday's Law: Fundamental Postulate

Faraday's law of induction:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s}$$

Fundamental postulate:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

→ *The electric field intensity in a region of time-varying magnetic flux is therefore nonconservative and cannot be expressed as the gradient of a scalar potential any more.*

# A Stationary Circuit in a Time-Varying Magnetic Field

For a stationary circuit:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \leftarrow \text{Fundamental postulate}$$

$$\rightarrow \oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s}$$

$$\rightarrow \oint_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} = -\frac{d\Phi}{dt}$$

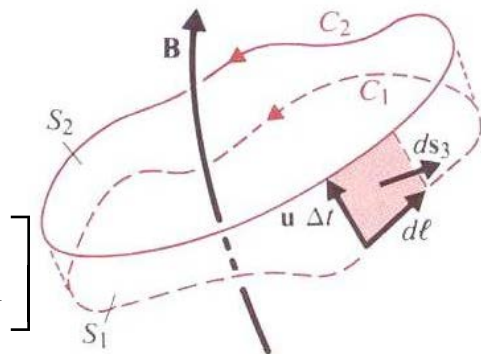
*→ The postulate holds for a stationary circuit with Faraday's experimental law!*

# A Moving Circuit in a Time-Varying Magnetic Field (1)

For a moving circuit:

$$-\frac{d\Phi}{dt} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} \quad \leftarrow \text{Flux change rate}$$

$$= -\lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left[ \int_{S_2} \mathbf{B}(t + \Delta t) \cdot d\mathbf{s}_2 - \int_{S_1} \mathbf{B}(t) \cdot d\mathbf{s}_1 \right]$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$$\rightarrow \mathbf{B}(t + \Delta t) = \mathbf{B}(t) + \frac{\partial \mathbf{B}(t)}{\partial t} \Delta t + H.O.T.$$

$$\rightarrow \frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} + \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left[ \int_{S_2} \mathbf{B} \cdot d\mathbf{s}_2 - \int_{S_1} \mathbf{B} \cdot d\mathbf{s}_1 + H.O.T. \right]$$

$$\rightarrow \int_V \nabla \cdot \mathbf{B} \, dv = \int_{S_2} \mathbf{B} \cdot d\mathbf{s}_2 - \int_{S_1} \mathbf{B} \cdot d\mathbf{s}_1 + \int_{S_3} \mathbf{B} \cdot d\mathbf{s}_3 = 0$$

$$\leftarrow ds_3 = d\mathbf{l} \times \mathbf{u} \Delta t$$

$$\rightarrow \int_{S_2} \mathbf{B} \cdot d\mathbf{s}_2 - \int_{S_1} \mathbf{B} \cdot d\mathbf{s}_1 = -\Delta t \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

$$\rightarrow \frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} - \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

# A Moving Circuit in a Time-Varying Magnetic Field (2)

For a moving circuit:

$$\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} - \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}$$

$$\rightarrow -\frac{d\Phi}{dt} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} + \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = \oint_C \mathbf{E}' \cdot d\mathbf{l}$$

$$\rightarrow \oint_C (\mathbf{E}' - \mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

$$\rightarrow \mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B} \quad \leftarrow \mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

*← Is this really true?*

$$\rightarrow \oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

*(Only valid for a nonrelativistic charge, i.e.,  $u \ll c$ )*

$$\rightarrow \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\mathbf{E}' = \gamma(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \frac{\gamma^2}{c^2(\gamma + 1)}(\mathbf{E} \cdot \mathbf{u})\mathbf{u}$$

$$\leftarrow \gamma = (1 - u^2/c^2)^{-1/2}$$

*→ The postulate still holds for a moving circuit with Faraday's experimental law!*

*→ Special theory of relativity*

# Ampère's Law for Time-Varying EM Fields

What Ampère had discovered for *steady* currents:

$$\nabla \times \mathbf{H} = \mathbf{J}$$

*However, the principle of conservation of charge must be satisfied at all times:*

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$$

Ampère's law for time-varying fields:

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 \stackrel{?}{=} \nabla \cdot \mathbf{J} \leftarrow \text{Is this always true?}$$

*This must be corrected as the following:*

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = \nabla \cdot \left( \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right)$$

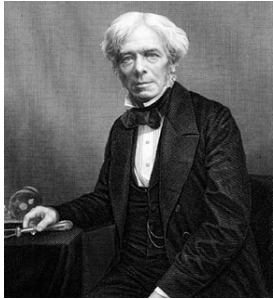
$$\rightarrow \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

*Displacement current density*

*"The major contribution of J. C. Maxwell"*



# Maxwell's Equations



Michael Faraday  
(1791–1867)

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Andre Marie Ampere  
(1775 - 1835)



Carl Friedrich Gauss  
(1777 - 1855)

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} && \text{Faraday's law} \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} && \text{Ampère's law} \\ \nabla \cdot \mathbf{D} &= \rho && \text{Gauss's law} \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned}$$

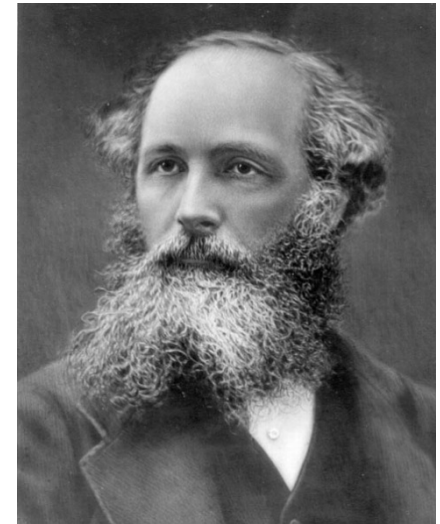
$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad \text{Constitutive relations}$$

$$\mathbf{H} = \frac{1}{\mu} \mathbf{B} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M} \quad \text{relations}$$

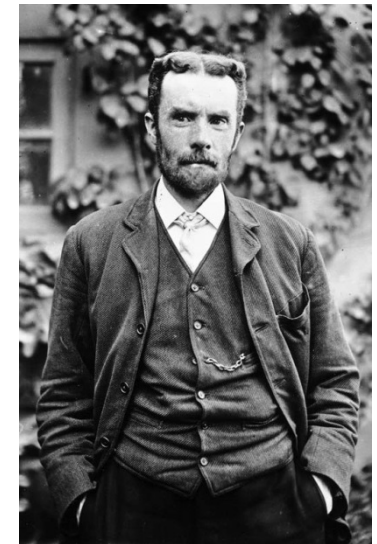
*The two divergence equations can be derived from the two curl equations!*

*→ 12 unknowns & 12 equations!*

**Ready for electromagnetics!**



James Clerk Maxwell  
(1831–1879)



Oliver Heaviside  
(1850–1925)

# Differential/Integral Form of Maxwell's Equations

Differential form vs. integral form:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\rightarrow \oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

$$\rightarrow \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left( \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s}$$

$$\rightarrow \oint_S \mathbf{D} \cdot d\mathbf{s} = \int_V \rho dv$$

$$\rightarrow \oint_S \mathbf{B} \cdot d\mathbf{s} = 0$$

# Potential Functions

Vector magnetic potential:

$$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A} \quad (\text{T})$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \rightarrow \quad \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} (\nabla \times \mathbf{A})$$

$$\rightarrow \nabla \times \left( \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

Scalar electric potential:

$$\rightarrow \mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V$$

$$\rightarrow \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \quad (\text{V/m})$$

For static fields:

$$V = \frac{1}{4\pi\epsilon_0} \int_{v'} \frac{\rho}{R} dv'$$

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_{v'} \frac{\mathbf{J}}{R} dv'$$

*→ Still valid for time-varying fields?*

# Wave Equation for Vector Potential

For a homogeneous medium:

$$\begin{aligned}\nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \rightarrow \quad \nabla \times \mu \mathbf{H} = \mu \mathbf{J} + \mu \frac{\partial \mathbf{D}}{\partial t} \\ \rightarrow \quad \nabla \times \nabla \times \mathbf{A} &= \mu \mathbf{J} + \mu \epsilon \frac{\partial}{\partial t} \left( -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \right) \\ \rightarrow \quad \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} &= \mu \mathbf{J} - \nabla \left( \mu \epsilon \frac{\partial V}{\partial t} \right) - \mu \epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} \\ \rightarrow \quad \nabla^2 \mathbf{A} - \mu \epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\mu \mathbf{J} + \nabla \left( \nabla \cdot \mathbf{A} + \mu \epsilon \frac{\partial V}{\partial t} \right)\end{aligned}$$

*Recall Helmholtz's theorem:*

$$\rightarrow \nabla \times \mathbf{A} = \mathbf{B}$$

$$\rightarrow \nabla \cdot \mathbf{A} = ? \quad \leftarrow \quad \boxed{\nabla \cdot \mathbf{A} = -\mu \epsilon \frac{\partial V}{\partial t}}$$

*$\rightarrow$  Lorentz condition (or Lorentz gauge)*

Wave equation in Lorentz condition:

$$\boxed{\nabla^2 \mathbf{A} - \mu \epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}}$$

# Wave Equation for Scalar Potential

For a homogeneous medium:

$$\begin{aligned}\nabla \cdot \mathbf{D} &= \rho \quad \rightarrow \quad \nabla \cdot \epsilon \mathbf{E} = \rho \\ &\rightarrow \quad -\nabla \cdot \epsilon \left( \nabla V + \frac{\partial \mathbf{A}}{\partial t} \right) = \rho \\ &\rightarrow \quad \nabla^2 V + \frac{\partial}{\partial t} (\nabla \cdot \mathbf{A}) = -\frac{\rho}{\epsilon}\end{aligned}$$

*→ Recall the Lorentz condition*

$$\rightarrow \boxed{\nabla \cdot \mathbf{A} + \mu \epsilon \frac{\partial V}{\partial t} = 0}$$

Wave equation in Lorentz condition:

$$\boxed{\nabla^2 V - \mu \epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}}$$

*cf. Coulomb gauge (Radiation or transverse gauge):*

$$\rightarrow \nabla \cdot \mathbf{A} = 0$$

$$\rightarrow \boxed{\nabla^2 V = -\frac{\rho}{\epsilon}}$$

# Electromagnetic Boundary Conditions

Continuity conditions:

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \rightarrow \oint_C \mathbf{E} \cdot d\mathbf{l} = - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

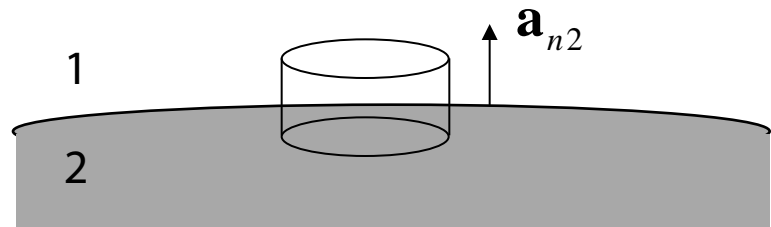
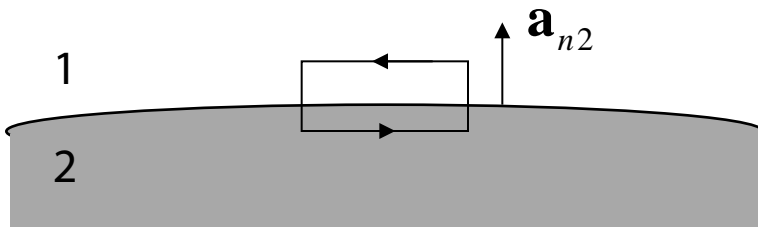
$$\rightarrow E_{1t} = E_{2t} \quad (\text{V/m})$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \rightarrow \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left( \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s}$$

$$\rightarrow \mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \quad (\text{A/m})$$

$$\nabla \cdot \mathbf{D} = \rho \rightarrow \mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \quad (\text{C/m}^2)$$

$$\nabla \cdot \mathbf{B} = 0 \rightarrow B_{1n} = B_{2n} \quad (\text{T})$$



# Interface between Two Lossless Linear Media

Lossless non-conductive medium:  $\sigma = 0$

*→ No free charges & no surface currents*

General BCs

$$\begin{aligned}\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} &= \mathbf{0} \quad \rightarrow \quad E_{1t} = E_{2t} \\ \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} &= \mathbf{J} \quad \rightarrow \quad \mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \\ \nabla \cdot \mathbf{D} &= \rho \quad \rightarrow \quad \mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \\ \nabla \cdot \mathbf{B} &= 0 \quad \rightarrow \quad B_{1n} = B_{2n}\end{aligned}$$

BCs between two lossless media

$$\begin{aligned}E_{1t} = E_{2t} &\rightarrow \frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2} \\ H_{1t} = H_{2t} &\rightarrow \frac{B_{1t}}{\mu_1} = \frac{B_{2t}}{\mu_2} \\ D_{1n} = D_{2n} &\rightarrow \epsilon_1 E_{1n} = \epsilon_2 E_{2n} \\ B_{1n} = B_{2n} &\rightarrow \mu_1 H_{1n} = \mu_2 H_{2n}\end{aligned}$$

# Interface between a Dielectric and a PEC

Perfect electric conductor:  $\sigma = \infty$

→ *No fields exist in the interior, i.e.,  $\mathbf{E} = \mathbf{D} = \mathbf{B} = \mathbf{H} = 0$ ,*

*“for time-varying fields”!*

→ *Any charges the perfect conductor may have will reside on the surface only!*

General BCs

$$\begin{aligned} \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 &\rightarrow E_{1t} = E_{2t} \\ \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} &\rightarrow \mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \\ \nabla \cdot \mathbf{D} = \rho &\rightarrow \mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \\ \nabla \cdot \mathbf{B} = 0 &\rightarrow B_{1n} = B_{2n} \end{aligned}$$



Medium 1  
(Dielectric)

Medium 2  
(PEC)

$$E_{1t} = 0$$

$$E_{2t} = 0$$

$$\mathbf{a}_{n2} \times \mathbf{H}_1 = \mathbf{J}_s$$

$$H_{2t} = 0$$

$$\mathbf{a}_{n2} \cdot \mathbf{D}_1 = \rho_s$$

$$D_{2n} = 0$$

$$B_{1n} = 0$$

$$B_{2n} = 0$$