

Electromagnetics:

Time-Varying Fields

Maxwell's Equations

(7-6, 7-7)

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Farewell to Electrostatic Model

Electrostatic model:

$$\nabla \times \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\mathbf{D} = \epsilon \mathbf{E}$$

Magnetostatic model:

$$\nabla \times \mathbf{H} = \mathbf{J}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{B} = \mu \mathbf{H}$$

Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \textit{Faraday's law}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \textit{Ampère's law}$$

$$\nabla \cdot \mathbf{D} = \rho \quad \textit{Gauss's law}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad \textit{Constitutive relations}$$

$$\mathbf{H} = \frac{1}{\mu} \mathbf{B} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}$$

Solution of Wave Equations for Potentials (1)

Wave equation for a time-varying distributed charges:

$$\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}$$

Homogeneous wave equation:

$$\nabla^2 V_h - \mu\epsilon \frac{\partial^2 V_h}{\partial t^2} = 0$$

Spherical symmetry:

$$\rightarrow \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial V_{h0}}{\partial R} \right) - \mu\epsilon \frac{\partial^2 V_{h0}}{\partial T^2} = 0$$

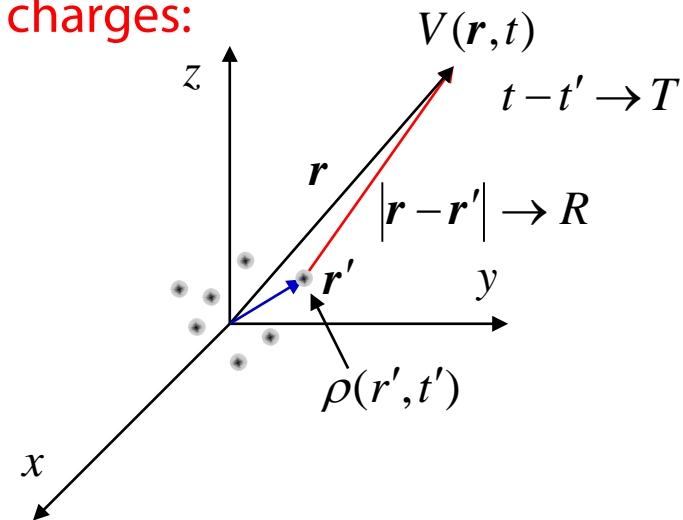
$$\rightarrow V_{h0}(R, T) \equiv \frac{1}{R} U(R, T) \quad \rightarrow \frac{\partial^2 U}{\partial R^2} - \mu\epsilon \frac{\partial^2 U}{\partial T^2} = 0$$

$$\rightarrow U(R, T) = f(T - R/u) \text{ or } f(T + R/u), \quad \boxed{u = 1/\sqrt{\mu\epsilon}}$$

*Physically not useful
for the moment!*

Phase velocity

$$\rightarrow \boxed{V_{h0}(R, T) = \frac{1}{R} f(T - R/u)}$$



Solution of Wave Equations for Potentials (2)

Wave equation for a time-varying distributed charges:

$$\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}$$

Green's function:

Dirac delta functions

$$(\nabla^2 - \mu\epsilon \frac{\partial^2}{\partial t^2})G = \delta(t-t')\delta(\mathbf{r}-\mathbf{r}') \text{ for } \mathbf{r} \rightarrow \mathbf{r}', t \rightarrow t'$$

$$(\nabla^2 - \mu\epsilon \frac{\partial^2}{\partial t^2})G = 0 \text{ for } \mathbf{r} \neq \mathbf{r}', t \neq t'$$

$$\rightarrow V = \iiint G(\mathbf{r}, \mathbf{r}', t, t') \left[-\frac{\rho(\mathbf{r}', t')}{\epsilon} \right] d^3\mathbf{r}' dt'$$

Valid for homogeneous eq.

For $\mathbf{r} \neq \mathbf{r}', t \neq t'$: $V_{h0}(R, T) = \frac{1}{R} f(T - R/u)$

$$\rightarrow G_{trial}(\mathbf{r}, \mathbf{r}', t, t') = \frac{f(t-t' - |\mathbf{r}-\mathbf{r}'|/u)}{|\mathbf{r}-\mathbf{r}'|}$$

Solution of Wave Equations for Potentials (3)

For $\mathbf{r} \rightarrow \mathbf{r}'$, $t \neq t'$:

$$\begin{aligned}
 (\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2})G_{trial} &= \lim_{\mathbf{r} \rightarrow \mathbf{r}'} \left[(\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2}) \frac{f(t - t' - |\mathbf{r} - \mathbf{r}'|/u)}{|\mathbf{r} - \mathbf{r}'|} \right] \\
 &= \lim_{\mathbf{r} \rightarrow \mathbf{r}'} \left[(\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2}) \frac{f(t - t')}{|\mathbf{r} - \mathbf{r}'|} \right] \\
 &= \lim_{\mathbf{r} \rightarrow \mathbf{r}'} \left\{ \left[f(t - t') \nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) - \frac{\mu\varepsilon}{|\mathbf{r} - \mathbf{r}'|} \frac{d^2 f(t - t')}{dt^2} \right] \right\}
 \end{aligned}$$

→ Negligible compared to the 1st term

$$= \lim_{\mathbf{r} \rightarrow \mathbf{r}'} \left[f(t - t') \nabla^2 \left(\frac{1}{|\mathbf{r} - \mathbf{r}'|} \right) \right] = -f(t - t') 4\pi \delta(\mathbf{r} - \mathbf{r}')$$

Recall:

$$\leftarrow \nabla^2 V = -\frac{q}{\varepsilon_0} \delta(\mathbf{r} - \mathbf{r}')$$

In result:

$$(\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2})G_{trial} = -4\pi f(t - t') \delta(\mathbf{r} - \mathbf{r}')$$

Solution of Wave Equations for Potentials (4)

For $\mathbf{r} \rightarrow \mathbf{r}', t \neq t'$:

$$(\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2})G_{trial} = -4\pi f(t-t')\delta(\mathbf{r}-\mathbf{r}')$$

For $\mathbf{r} \rightarrow \mathbf{r}', t \rightarrow t'$:

$$(\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2})G = \delta(t-t')\delta(\mathbf{r}-\mathbf{r}') \rightarrow G_{trial} = G$$

if $f(\xi) = -\frac{1}{4\pi}\delta(\xi)$

$$\rightarrow G_{trial}(\mathbf{r}, \mathbf{r}', t, t') = \frac{f(t-t'-|\mathbf{r}-\mathbf{r}'|/u)}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{4\pi} \frac{\delta(t-t'-|\mathbf{r}-\mathbf{r}'|/u)}{|\mathbf{r}-\mathbf{r}'|} = G$$

Finally:

$$V = \iint G(\mathbf{r}, \mathbf{r}', t, t') \left[-\frac{\rho(\mathbf{r}', t')}{\varepsilon}\right] d^3\mathbf{r}' dt'$$

Green's function

$$= \iint -\frac{1}{4\pi} \frac{\delta(t-t'-|\mathbf{r}-\mathbf{r}'|/u)}{|\mathbf{r}-\mathbf{r}'|} \left[-\frac{\rho(\mathbf{r}', t')}{\varepsilon}\right] d^3\mathbf{r}' dt'$$

$$= \frac{1}{4\pi\varepsilon} \int \frac{\rho(\mathbf{r}', t-|\mathbf{r}-\mathbf{r}'|/u)}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}' \rightarrow \text{Retarded scalar potential}$$

\rightarrow Is this a general solution?

Retarded Vector Potential

Similarly, for a vector magnetic potential:

$$\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/u)}{|\mathbf{r} - \mathbf{r}'|} d^3 \mathbf{r}'$$

“It takes time for electromagnetic waves to travel in order that the effects of the time-varying charges or currents are to appear at distant points.”

Source-Free Wave Equations

Source free, simple medium: $\rho = 0$ & $\mathbf{J} = 0$ ($\sigma = 0$)

→ *Linear, isotropic and homogeneous*

Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{B} = \mu \mathbf{H} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M}$$



$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{H} = 0$$

$$\nabla \times \nabla \times \mathbf{E} = -\mu \frac{\partial}{\partial t} (\nabla \times \mathbf{H}) = -\mu \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$$= \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

$$\rightarrow \nabla^2 \mathbf{E} - \mu \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$u = 1 / \sqrt{\mu \varepsilon}$$

→ *Phase velocity*

$$\rightarrow \nabla^2 \mathbf{E} - \frac{1}{u^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\rightarrow \nabla^2 \mathbf{H} - \frac{1}{u^2} \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

→ *Homogeneous vector wave eqs.* 8

Time-Harmonic Fields (1)

Maxwell's equations: *Linear differential equations*

→ *Principle of superposition*

Wave equation for a scalar potential:

$$\nabla^2 V - \mu\varepsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon}$$

For a monochromatic charge density:

$$\begin{aligned}\rho(\mathbf{r}, t) &= \frac{1}{2} [\rho(\mathbf{r}) e^{j\omega t} + c.c.] \\ &= \text{Re}[\rho(\mathbf{r}) e^{j\omega t}]\end{aligned}$$

(*c.c.*: complex conjugate)

Let a trial solution be:

$$\begin{aligned}V(\mathbf{r}, t) &= \frac{1}{2} [V(\mathbf{r}) e^{j\omega t} + c.c.] \\ &= \text{Re}[V(\mathbf{r}) e^{j\omega t}]\end{aligned}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

Time-Harmonic Fields (2)

Wave equation:

$$\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}$$

Recall the principle of superposition!

$$\rightarrow (\nabla^2 - \mu\epsilon \frac{\partial^2}{\partial t^2}) \left\{ \frac{1}{2} [V(\mathbf{r})e^{j\omega t} + c.c.] \right\} = -\frac{1}{\epsilon} \frac{1}{2} [\rho(\mathbf{r})e^{j\omega t} + c.c.]$$

$$\rightarrow (\nabla^2 - \mu\epsilon \frac{\partial^2}{\partial t^2}) [V(\mathbf{r})e^{j\omega t}] = -\frac{1}{\epsilon} [\rho(\mathbf{r})e^{j\omega t}]$$

Note: $\frac{\partial}{\partial t} \rightarrow j\omega$

$$\rightarrow \nabla^2 V(\mathbf{r}) + k^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon}, \quad k = \omega\sqrt{\mu\epsilon} = \omega/u$$

Recall Green's function:

Phase retardation

H.W.

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon} \int \frac{\rho(\mathbf{r}') e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'$$

Similarly:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') e^{-jk|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} d^3\mathbf{r}'$$

Time-Harmonic Fields (3)

Maxwell's equations: *Linear differential equations*

→ *Principle of superposition*

→ *Phasor notation for monochromatic fields:*

$$\begin{aligned}\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{H} &= \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \\ \nabla \cdot \mathbf{D} &= \rho \\ \nabla \cdot \mathbf{B} &= 0\end{aligned}$$

→

$$\begin{aligned}\nabla \times \mathbf{E} &= -j\omega \mathbf{B} \\ \nabla \times \mathbf{H} &= \mathbf{J} + j\omega \mathbf{D} \\ \nabla \cdot \mathbf{D} &= \rho \\ \nabla \cdot \mathbf{B} &= 0\end{aligned}$$

or

$$\begin{aligned}\nabla \times \mathbf{E} &= -j\omega \mu \mathbf{H} \\ \nabla \times \mathbf{H} &= j\omega \epsilon \mathbf{E} \\ \nabla \cdot \mathbf{E} &= 0 \\ \nabla \cdot \mathbf{H} &= 0\end{aligned}$$

Source-free, simple medium

↓

For non-monochromatic fields:

$$\begin{aligned}\mathbf{E}(\mathbf{r}, t) &= \operatorname{Re} \sum_i \boldsymbol{\psi}_i(\mathbf{r}) e^{j\omega_i t} \\ \text{or} \\ \mathbf{E}(\mathbf{r}, t) &= \operatorname{Re} \int \boldsymbol{\psi}(\mathbf{r}, \omega) e^{j\omega t} d\omega\end{aligned}$$

$$\begin{aligned}\nabla^2 \mathbf{E} + k^2 \mathbf{E} &= 0 \\ \nabla^2 \mathbf{H} + k^2 \mathbf{H} &= 0\end{aligned}$$

→ *Homogeneous Helmholtz's eqs.*

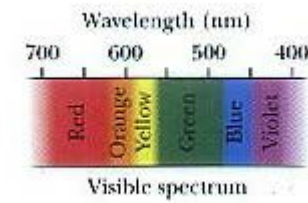
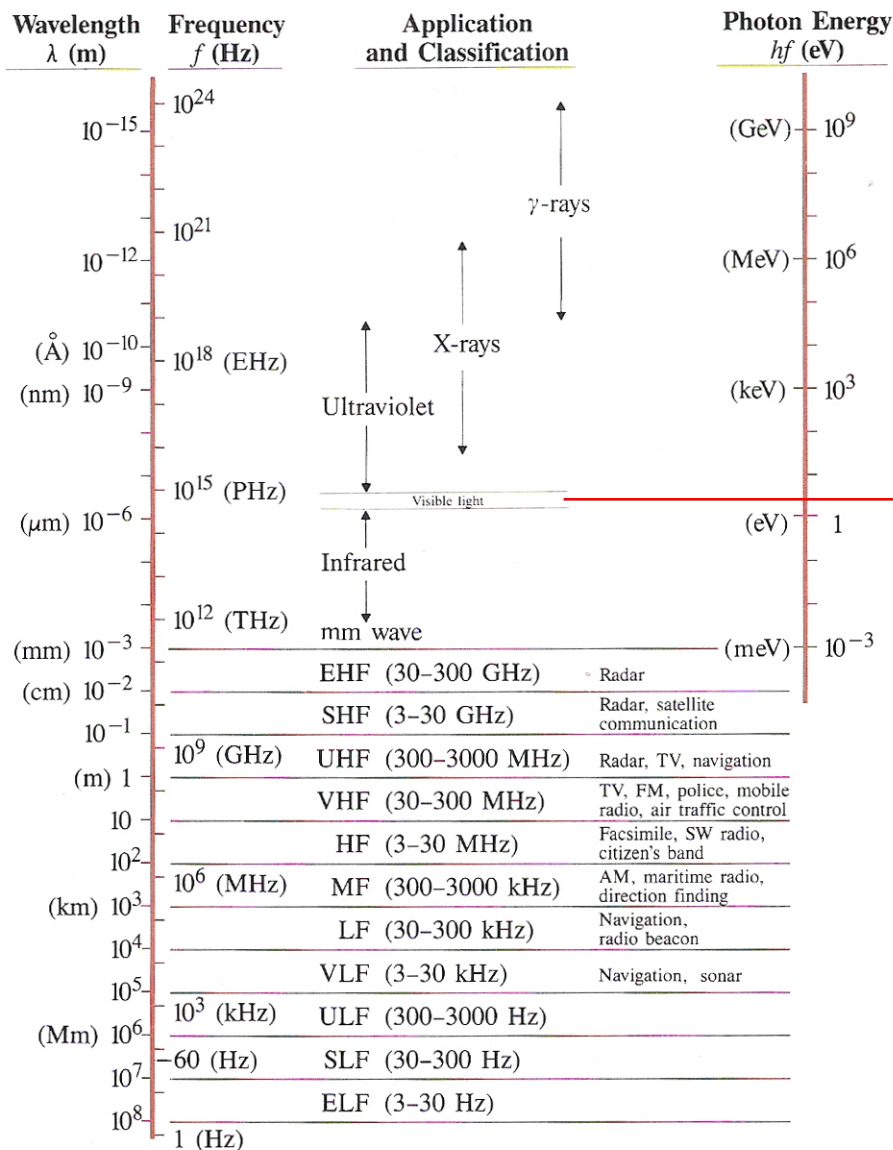
Principle of Duality

For a source-free, simple medium: $(\mathbf{E}, \mathbf{H}) \leftarrow \text{Solution of Maxwell's eqs.}$

$$\begin{aligned} \rightarrow \mathbf{E}' &= \eta \mathbf{H} \\ \rightarrow \mathbf{H}' &= -\frac{1}{\eta} \mathbf{E} \end{aligned} \quad \leftarrow \eta = \sqrt{\mu / \varepsilon} \quad \leftarrow \text{Intrinsic impedance}$$

Principle of duality: **H.W.**

Electromagnetic Spectrum



Visible light (400 nm ~ 700 nm)