

Electromagnetics:

Plane Waves in Lossless Media

(8-1, 8-2)

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Scalar and Vector Potentials

Wave equations for a homogeneous medium:

$$\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}$$

$$\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu\mathbf{J}$$

→ *Recall the Lorentz condition*

$$\leftarrow \nabla \cdot \mathbf{A} + \mu\epsilon \frac{\partial V}{\partial t} = 0$$

Fields from potentials:

$$\rightarrow \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

$$\rightarrow \mathbf{B} = \nabla \times \mathbf{A}$$

← *Formalism by Maxwell*

Plane Waves in Lossless Media (1)

Source-free, lossless, & uniform isotropic:

$$\rho = 0, \mathbf{J} = 0, \sigma = 0, \mu = \text{const.}, \& \ \varepsilon = \text{const.}$$

Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{B} = \mu \mathbf{H} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M}$$



$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{H} = 0$$

$$\rightarrow \nabla^2 \mathbf{E} - \mu \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\rightarrow \nabla^2 \mathbf{H} - \mu \varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

Homogeneous wave eqs.

Plane Waves in Lossless Media (2)

Homogeneous wave equation:

$$\nabla^2 \mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

For a monochromatic wave:

← ω : fixed to one value

$$\rightarrow \mathbf{E} = \frac{1}{2} [\mathbf{E}(\mathbf{r}) e^{j\omega t} + c.c.] = \text{Re}[\mathbf{E}(\mathbf{r}) e^{j\omega t}]$$

For a monochromatic “plane” wave:

← \mathbf{k} : fixed to one value

$$\rightarrow \mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} = \mathbf{E}_0 e^{-j(k_x x + k_y y + k_z z)}$$

$$\rightarrow \nabla^2 [\mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}] = \mathbf{E}_0 \nabla^2 (e^{-j\mathbf{k}\cdot\mathbf{r}}) = -k^2 \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} \quad \text{Note: } \nabla \rightarrow -j\mathbf{k}$$

$$\leftarrow k^2 = |\mathbf{k}|^2 = k_x^2 + k_y^2 + k_z^2$$

$$\rightarrow \frac{\partial^2}{\partial t^2} [\mathbf{E}(\mathbf{r}) e^{j\omega t}] = -\omega^2 \mathbf{E}(\mathbf{r}) e^{j\omega t} \quad \text{Note: } \frac{\partial}{\partial t} \rightarrow j\omega$$

In result:

$$\nabla^2 \mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \rightarrow -k^2 + \omega^2 \mu\epsilon = 0 \rightarrow k = \pm \omega \sqrt{\mu\epsilon}$$

Note that \mathbf{E}_0 and \mathbf{k} are still arbitrary!

$$\rightarrow \mathbf{E} = \frac{1}{2} [\mathbf{E}_0 e^{j(\omega t - \mathbf{k}\cdot\mathbf{r})} + c.c.] = \text{Re}[\mathbf{E}_0 e^{j(\omega t - \mathbf{k}\cdot\mathbf{r})}]$$

Plane Waves in Lossless Media (3)

Recall source-free, lossless, & uniform isotropic media:

$$\rho = 0, \mathbf{J} = 0, \sigma = 0, \mu = \text{const.}, \& \ \varepsilon = \text{const.}$$

Relation between \mathbf{E} & \mathbf{k} or \mathbf{H} & \mathbf{k} :

For an electric field:

$$\rightarrow \nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot (f\mathbf{A}) = \nabla f \cdot \mathbf{A} + f(\nabla \cdot \mathbf{A})$$

$$\rightarrow \nabla \cdot [\mathbf{E}_0 e^{j(\omega t - \mathbf{k} \cdot \mathbf{r})}] = (\nabla \cdot \mathbf{E}_0) e^{j(\omega t - \mathbf{k} \cdot \mathbf{r})} + \mathbf{E}_0 \cdot \nabla [e^{j(\omega t - \mathbf{k} \cdot \mathbf{r})}]$$

$$\rightarrow \nabla \cdot [\mathbf{E}_0 e^{j(\omega t - \mathbf{k} \cdot \mathbf{r})}] = \mathbf{E}_0 \cdot [-j\mathbf{k} e^{j(\omega t - \mathbf{k} \cdot \mathbf{r})}] = 0 \quad \text{Note: } \nabla \rightarrow -j\mathbf{k}$$

$$\rightarrow \mathbf{E}_0 \cdot \mathbf{k} = 0 \quad \leftarrow \mathbf{E} \text{ is perpendicular to } \mathbf{k}!$$

For a magnetic field:

Similarly:

$$\nabla^2 \mathbf{H} - \mu\varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0 \quad \rightarrow \mathbf{H} = \frac{1}{2} [\mathbf{H}_0 e^{j(\omega t - \mathbf{k} \cdot \mathbf{r})} + c.c.] = \text{Re}[\mathbf{H}_0 e^{j(\omega t - \mathbf{k} \cdot \mathbf{r})}]$$

$$\rightarrow \nabla \cdot \mathbf{H} = 0 \rightarrow \mathbf{H}_0 \cdot \mathbf{k} = 0 \quad \leftarrow \mathbf{H} \text{ is also perpendicular to } \mathbf{k}!$$

Plane Waves in Lossless Media (4)

Relation between \mathbf{E} & \mathbf{H} :

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad \rightarrow \quad \mathbf{H}(\mathbf{r}) = -\frac{1}{j\omega\mu} \nabla \times \mathbf{E}(\mathbf{r})$$

$$\rightarrow \nabla \times \mathbf{E}(\mathbf{r}) = \nabla \times (\mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}}) \quad \boxed{\nabla \times (f\mathbf{A}) = \nabla f \times \mathbf{A} + f(\nabla \times \mathbf{A})}$$

$$= (\cancel{\nabla \times \mathbf{E}_0}) e^{-j\mathbf{k}\cdot\mathbf{r}} + \nabla(e^{-j\mathbf{k}\cdot\mathbf{r}}) \times \mathbf{E}_0$$

$$= [-j\mathbf{k} e^{-j\mathbf{k}\cdot\mathbf{r}}] \times \mathbf{E}_0 \quad \boxed{\text{Note: } \nabla \rightarrow -j\mathbf{k}}$$

$$= -j e^{-j\mathbf{k}\cdot\mathbf{r}} (\mathbf{k} \times \mathbf{E}_0)$$

$$\rightarrow \mathbf{H}(\mathbf{r}) = \mathbf{H}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} = -\frac{1}{j\omega\mu} \nabla \times \mathbf{E}(\mathbf{r})$$

$$= -\frac{1}{j\omega\mu} (-j e^{-j\mathbf{k}\cdot\mathbf{r}}) (\mathbf{k} \times \mathbf{E}_0) = \frac{1}{\omega\mu} (\mathbf{k} \times \mathbf{E}_0) e^{-j\mathbf{k}\cdot\mathbf{r}}$$

$$\rightarrow \mathbf{H}_0 = \frac{1}{\omega\mu} (\mathbf{k} \times \mathbf{E}_0) \quad \leftarrow \mathbf{H} \text{ is perpendicular to } \mathbf{E}_0 \text{ as well as } \mathbf{k}!$$

Plane Waves in Lossless Media (4)

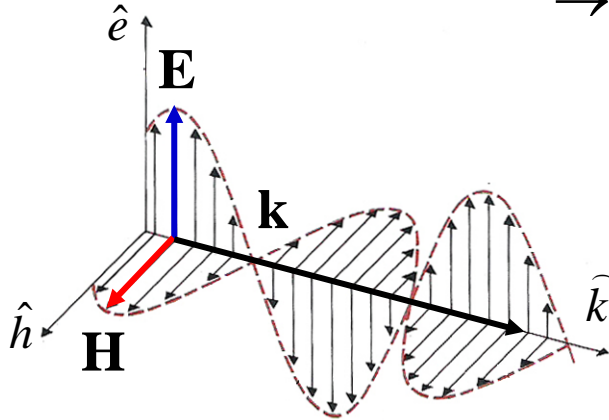
Relations among \mathbf{E} , \mathbf{H} & \mathbf{k} :

$$\nabla \cdot \mathbf{E} = 0 \quad \rightarrow \quad -j\mathbf{k} \cdot \mathbf{E} = 0 \quad \rightarrow \quad \mathbf{k} \perp \mathbf{E}$$

$$\nabla \cdot \mathbf{H} = 0 \quad \rightarrow \quad -j\mathbf{k} \cdot \mathbf{H} = 0 \quad \rightarrow \quad \mathbf{k} \perp \mathbf{H}$$

$$\mathbf{H} = -\frac{1}{j\omega\mu}(\nabla \times \mathbf{E}) \quad \rightarrow \quad \mathbf{H} = -\frac{1}{j\omega\mu}(-j\mathbf{k} \times \mathbf{E}) \quad \rightarrow \quad \mathbf{k} \perp \mathbf{H} \text{ \& \ } \mathbf{H} \perp \mathbf{E}$$

$$\rightarrow \quad \mathbf{k} \propto \mathbf{E} \times \mathbf{H} \quad \leftarrow \text{Transverse electromagnetic (TEM) wave}$$



Plane Waves in Lossless Media (5)

Travelling wave:

$$\mathbf{E} = \text{Re}[\mathbf{E}_0 e^{j(\omega t - \mathbf{k} \cdot \mathbf{r})}], \quad \mathbf{H} = \text{Re}[\mathbf{H}_0 e^{j(\omega t - \mathbf{k} \cdot \mathbf{r})}]$$

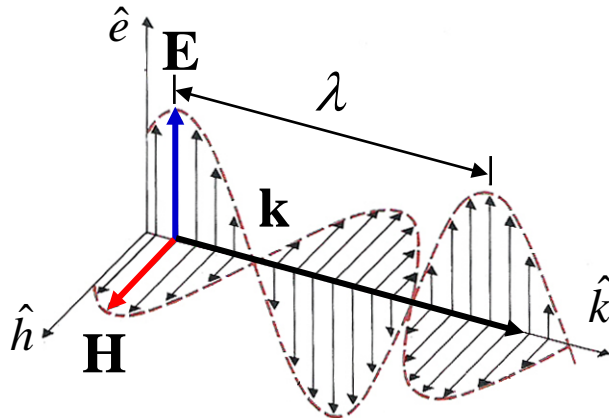
Phase velocity: $\omega t - \mathbf{k} \cdot \mathbf{r} = \text{const.}$

$$u_p = \frac{dr}{dt} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}}$$

For free space:

$$\rightarrow \frac{1}{\sqrt{\mu_0 \epsilon_0}} \equiv c \cong 3 \times 10^8 \text{ (m/s)}$$

Intrinsic impedance: $\left| \frac{\mathbf{E}}{\mathbf{H}} \right| = \left| \frac{\omega\mu}{\mathbf{k}} \right| = \left| \frac{\omega\mu}{\omega\sqrt{\mu\epsilon}} \right| = \sqrt{\frac{\mu}{\epsilon}} \rightarrow \sqrt{\frac{\mu_0}{\epsilon_0}} \cong 377 \cong 120\pi \text{ (\Omega)}$



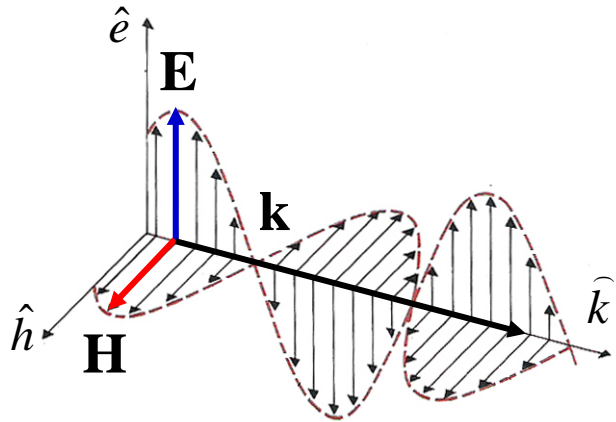
Significance of parameters:

Wavelength λ : Minimum travelling distance to lead to the same position of the wave

Wavenumber k : $2\pi \times$ number of wavelength per unit distance

$$\rightarrow k = \frac{2\pi}{\lambda} \quad \rightarrow u_p = \frac{\omega}{k} = \frac{2\pi f}{2\pi / \lambda} = f\lambda$$

Polarization of Plane Waves (1)



For a fixed value of \mathbf{k} :

$$\mathbf{E} = \text{Re}[\mathbf{E}_0 e^{j(\omega t - \mathbf{k} \cdot \mathbf{r})}], \quad \mathbf{H} = \text{Re}[\mathbf{H}_0 e^{j(\omega t - \mathbf{k} \cdot \mathbf{r})}]$$

Note that the direction of the E-field vector (polarization) and its phase are still arbitrary.

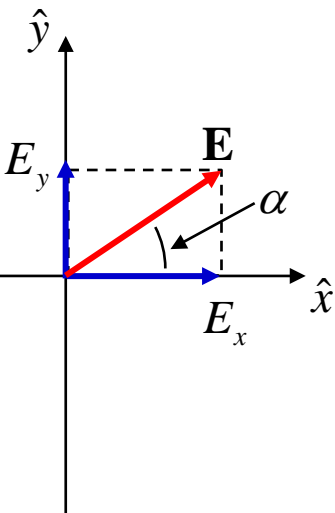
Let $\mathbf{k} \parallel z$ -axis: $\rightarrow \mathbf{E} = \text{Re}(E_x \hat{x} + E_y \hat{y}) = \text{Re}[E_{0,x} e^{j(\omega t - kz)} \hat{x} + E_{0,y} e^{j(\omega t - kz)} \hat{y}]$

$$\rightarrow E_{0,x} = |E_{0,x}| e^{j\phi_{0,x}}, \quad E_{0,y} = |E_{0,y}| e^{j\phi_{0,y}}$$

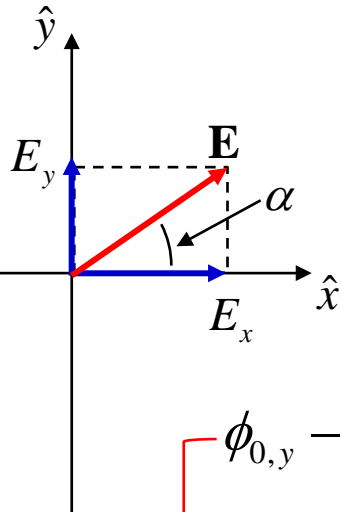
$$\rightarrow \mathbf{E} = \text{Re}[|E_{0,x}| e^{j(\omega t - kz + \phi_{0,x})} \hat{x} + |E_{0,y}| e^{j(\omega t - kz + \phi_{0,y})} \hat{y}]$$

Instantaneous angle of \mathbf{E} against x -axis:

$$\alpha = \tan^{-1} \frac{\text{Re}[|E_{0,y}| e^{j(\omega t - kz + \phi_{0,y})}]}{\text{Re}[|E_{0,x}| e^{j(\omega t - kz + \phi_{0,x})}]}$$



Polarization of Plane Waves (2)



Instantaneous angle of \mathbf{E} against x -axis:

$$\alpha = \tan^{-1} \frac{\operatorname{Re}\left[|E_{0,y}| e^{j(\omega t - kz + \phi_{0,y})}\right]}{\operatorname{Re}\left[|E_{0,x}| e^{j(\omega t - kz + \phi_{0,x})}\right]}$$

$$\phi_{0,y} - \phi_{0,x} = 0, \pi : \rightarrow \alpha = \pm \tan^{-1} \left| \frac{E_{0,y}}{E_{0,x}} \right| \leftarrow \text{Linearly polarized}$$

$$\phi_{0,y} - \phi_{0,x} = \pm \frac{\pi}{2}, |E_{0,x}| = |E_{0,y}| :$$

$$\rightarrow \mathbf{E} = |E_{0,x}| [\cos(\omega t - kz + \phi_{0,x}) \hat{x} \mp \sin(\omega t - kz + \phi_{0,x}) \hat{y}]$$

$$\rightarrow \alpha = \mp(\omega t - kz + \phi_{0,x}) \leftarrow \text{Circularly polarized (Left-hand / Right-hand)}$$

Otherwise: \leftarrow Elliptically polarized

Proof? \rightarrow More suitable for homework!