

Electromagnetics:

Plane Waves in Lossy Media

Group Velocity

(8-3, 8-4)

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EM Waves in Lossless Media

Source-free, linear, isotropic, homogeneous & **lossless**:

$$\rho = 0, \mathbf{J} = 0, \sigma = 0, \mu = \text{const.}, \text{ \& } \varepsilon = \text{const.}$$

Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{B} = \mu \mathbf{H} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M}$$



$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{H} = 0$$

$$\rightarrow \nabla^2 \mathbf{E} - \mu \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\rightarrow \nabla^2 \mathbf{H} - \mu \varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

Homogeneous wave eqs.

EM Waves in "Lossy" Media (1)

Source-free, linear, isotropic, homogeneous & **lossy**:

$$\rho = ?, \mathbf{J} = ?, \mu = \text{const.}, \varepsilon = \text{const.}, \& \sigma \neq 0$$

Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\mathbf{D} = \varepsilon \mathbf{E} = \varepsilon_0 \mathbf{E} + \mathbf{P}$$

$$\mathbf{B} = \mu \mathbf{H} = \mu_0 \mathbf{H} + \mu_0 \mathbf{M}$$

$$\mathbf{J} = \sigma \mathbf{E} \quad \leftarrow \text{Conduction current}$$

$$\rightarrow \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} = \sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

For a monochromatic wave:

$$\sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \rightarrow (\sigma + j\omega\varepsilon) \mathbf{E} = j\omega \left(\varepsilon + \frac{\sigma}{j\omega} \right) \mathbf{E} \\ = j\omega \varepsilon_c \mathbf{E}$$

Complex permittivity

If we use the complex permittivity: $\mathbf{J} \rightarrow 0!$

For the charge density:

$$\nabla \cdot \mathbf{E} = 0 \quad \rightarrow \nabla \cdot \mathbf{D} = 0 \quad \rightarrow \rho = 0$$

EM Waves in "Lossy" Media (2)

From Faraday's law: $\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$

$$\rightarrow \nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$$

$$\rightarrow -\mu \frac{\partial}{\partial t} \nabla \times \mathbf{H} = -\mu \frac{\partial}{\partial t} \left(\mathbf{J} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \right)$$

$$= -\mu \frac{\partial}{\partial t} \left(\sigma \mathbf{E} + \varepsilon \frac{\partial \mathbf{E}}{\partial t} \right) = -\mu \sigma \frac{\partial \mathbf{E}}{\partial t} - \mu \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

Homogeneous wave equation for lossy media:

$$\rightarrow \nabla^2 \mathbf{E} - \mu \sigma \frac{\partial \mathbf{E}}{\partial t} - \mu \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

With the complex permittivity notation:

$$e^{j\omega t} \rightarrow \nabla^2 \mathbf{E} - \mu \varepsilon_c \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad \text{with } \varepsilon_c = \varepsilon + \frac{\sigma}{j\omega}$$

Be careful with the complex permittivity and field notation!

$$e^{-j\omega t} \rightarrow \nabla^2 \mathbf{E}^* - \mu^* \varepsilon_c^* \frac{\partial^2 \mathbf{E}^*}{\partial t^2} = 0$$

Plane Waves in Lossy Media

For a monochromatic “plane” wave:

$$e^{j\omega t} \rightarrow \nabla^2 \mathbf{E} - \mu\epsilon_c \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

$$\rightarrow \mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{-j\mathbf{k}_c \cdot \mathbf{r}} = \mathbf{E}_0 e^{-j(k_{cx}x + k_{cy}y + k_{cz}z)}$$

$$\nabla \rightarrow -j\mathbf{k}_c, \quad \frac{\partial}{\partial t} \rightarrow j\omega$$

$$(-j\mathbf{k}_c) \cdot (-j\mathbf{k}_c) - \mu\epsilon_c (j\omega)^2 = -k_c^2 + \mu\epsilon_c \omega^2 = 0$$

$$\rightarrow k_c = \pm \omega(\mu\epsilon_c)^{1/2} \quad \text{Propagating along } +r$$

$$\rightarrow k_c = \omega \sqrt{\mu \left(\epsilon + \frac{\sigma}{j\omega} \right)^{1/2}} = \omega \sqrt{\mu\epsilon} \left(1 - j \frac{\sigma}{\omega\epsilon} \right)^{1/2}$$

$$k_c = \beta - j\alpha = \omega \sqrt{\mu\epsilon} \left(1 - j \frac{\sigma}{\omega\epsilon} \right)^{1/2}$$

Phase constant

Attenuation constant

Low-Loss Dielectrics

Nonzero but low conductivity: $\sigma \neq 0$, $\sigma / \omega \epsilon \ll 1$

Taylor series expansion:

$$k_c = \beta - j\alpha = \omega \sqrt{\mu \epsilon} \left(1 - j \frac{\sigma}{\omega \epsilon}\right)^{1/2}$$

$$f = (1+x)^{1/2} = 1 + x \left. \frac{df}{dx} \right|_{x=0} + \frac{x^2}{2!} \left. \frac{d^2 f}{dx^2} \right|_{x=0} + \dots$$

$$\cong \omega \sqrt{\mu \epsilon} \left[1 - j \frac{1}{2} \frac{\sigma}{\omega \epsilon} + \frac{1}{8} \left(\frac{\sigma}{\omega \epsilon}\right)^2\right]$$

Phase constant $\rightarrow \beta \cong \omega \sqrt{\mu \epsilon} \left[1 + \frac{1}{8} \left(\frac{\sigma}{\omega \epsilon}\right)^2\right]$

Attenuation constant $\rightarrow \alpha \cong \omega \sqrt{\mu \epsilon} \left(\frac{1}{2} \frac{\sigma}{\omega \epsilon}\right) = \frac{\sigma}{2} \sqrt{\frac{\mu}{\epsilon}}$

Intrinsic impedance $\rightarrow \eta_c = \sqrt{\frac{\mu}{\epsilon_c}} = \sqrt{\frac{\mu}{\epsilon}} \left(1 - j \frac{\sigma}{\omega \epsilon}\right)^{-1/2} \cong \sqrt{\frac{\mu}{\epsilon}} \left(1 + j \frac{\sigma}{2\omega \epsilon}\right)$

Phase velocity $\rightarrow u_p = \frac{\omega}{\beta} \cong \frac{1}{\sqrt{\mu \epsilon}} \left[1 - \frac{1}{8} \left(\frac{\sigma}{\omega \epsilon}\right)^2\right]$

Good Conductors

Nonzero and high conductivity: $\sigma \neq 0$, $\sigma / \omega \epsilon \gg 1$

Dominant

$$k_c = \beta - j\alpha = \omega \sqrt{\mu \epsilon} \left(1 - j \frac{\sigma}{\omega \epsilon}\right)^{1/2}$$

$$(-j)^{1/2} \rightarrow (e^{-j\frac{\pi}{2}})^{1/2} = e^{-j\frac{\pi}{4}} = \frac{1-j}{\sqrt{2}}$$

$$\cong \omega \sqrt{\mu \epsilon} \left(-j \frac{\sigma}{\omega \epsilon}\right)^{1/2} = \sqrt{\omega \mu \sigma} (-j)^{1/2} = \frac{1-j}{\sqrt{2}} \sqrt{\omega \mu \sigma}$$

Phase constant
Attenuation constant

$$\rightarrow \beta = \alpha \cong \sqrt{\frac{\omega \mu \sigma}{2}}$$

Intrinsic impedance

$$\rightarrow \eta_c = \sqrt{\frac{\mu}{\epsilon_c}} \cong \sqrt{\frac{\mu}{\epsilon}} \left(-j \frac{\sigma}{\omega \epsilon}\right)^{-1/2} = \frac{1+j}{\sqrt{2}} \sqrt{\frac{\omega \mu}{\sigma}} = (1+j) \frac{\alpha}{\sigma}$$

Magnetic field lags behind electric field by $\pi/4$!

Phase velocity

$$\rightarrow u_p = \frac{\omega}{\beta} \cong \sqrt{\frac{2\omega}{\mu \sigma}}$$

Skin Depth

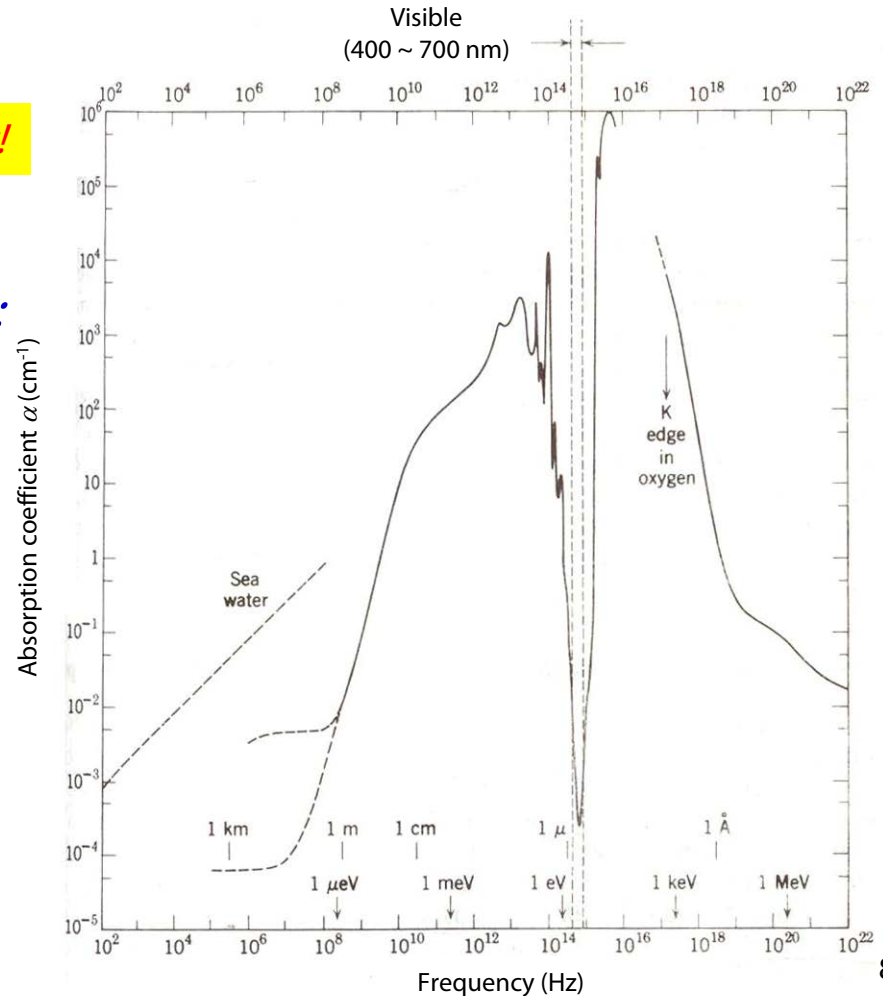
Propagation distance for $1/e$ attenuation for good conductors:

$$\rightarrow \delta = \frac{1}{\alpha} \approx \sqrt{\frac{2}{\omega\mu\sigma}}$$

Strongly frequency-dependent!

However, σ is also frequency dependent!

cf. Absorption coefficient of water:



J. D. Jackson, Classical Electrodynamics, 3rd ed. Wiley, 1999.

Ionized Gases (1)

Plasmas: Ionised gases with equal electron and ion densities

Electrons much lighter than ions \rightarrow as free electron gases

Recall Newton's law of motion II: $\mathbf{F} = m\mathbf{a}$

$$\rightarrow -e\mathbf{E} = m \frac{d^2\mathbf{x}}{dt^2} = -m\omega^2\mathbf{x} \quad \text{for a monochromatic wave at } \omega$$

Displacement: $\rightarrow \mathbf{x} = \frac{e}{m\omega^2}\mathbf{E}$

Electric dipole moment: $\rightarrow \mathbf{p} = -e\mathbf{x}$ *N : Density of number of electrons*

Polarization vector: $\rightarrow \mathbf{P} = N\mathbf{p} = -\frac{Ne^2}{m\omega^2}\mathbf{E}$

Ionized Gases (2)

Polarization vector: $\mathbf{P} = N\mathbf{p} = -\frac{Ne^2}{m\omega^2}\mathbf{E}$

Electric displacement vector:

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P} \quad \leftarrow \text{Constitutive relation}$$

$$= \varepsilon_0\left(1 - \frac{Ne^2}{m\omega^2\varepsilon_0}\right)\mathbf{E} = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right)\mathbf{E} \quad \text{with } \omega_p = \sqrt{\frac{Ne^2}{m\varepsilon_0}}$$

Equivalent permittivity:

$$\varepsilon_p = \varepsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right) = \varepsilon_0\left(1 - \frac{f_p^2}{f^2}\right) \quad \leftarrow f_p \cong 9\sqrt{N}$$

*Plasma frequency
or cutoff frequency*

$$k_p = \beta - j\alpha = \omega\sqrt{\mu\varepsilon_p}$$

Freq. response: $\begin{cases} f < f_p \rightarrow \text{No transmission} \rightarrow \text{Back reflection by ionospheric layers} \\ f > f_p \rightarrow \text{Un-attenuated transmission} \end{cases}$

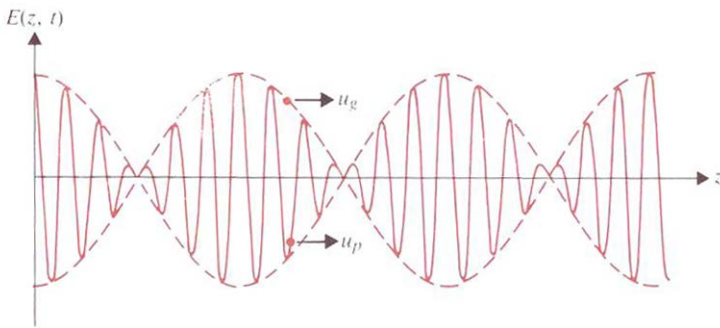
Group velocity

Phase velocity: $u_p = \frac{\omega}{\beta}$ Material dispersion: $\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu(\omega)\varepsilon(\omega)}$

Consider a dual-tone wave packet: $\omega_0 + \Delta\omega$ & $\omega_0 - \Delta\omega$ ($\Delta\omega \ll \omega_0$)

$$E(z, t) = E_0 \cos[(\omega_0 + \Delta\omega)t - (\beta_0 + \Delta\beta)z] + E_0 \cos[(\omega_0 - \Delta\omega)t - (\beta_0 - \Delta\beta)z]$$

$$= 2E_0 \underbrace{\cos(t\Delta\omega - z\Delta\beta)}_{\substack{\text{Slowly oscillating} \\ \rightarrow \text{Envelope}}} \underbrace{\cos(\omega_0 t - \beta_0 z)}_{\substack{\text{Rapidly oscillating} \\ \rightarrow \text{Carrier}}}$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Phase: $\omega_0 t - \beta_0 z = \text{const.}$

Phase velocity: $\rightarrow u_p = \frac{dz}{dt} = \frac{\omega_0}{\beta_0}$

Envelope: $\Delta\omega t - \Delta\beta z = \text{const.}$

Group velocity: $\rightarrow u_g = \frac{dz}{dt} = \frac{\Delta\omega}{\Delta\beta} \rightarrow \frac{d\omega}{d\beta}$

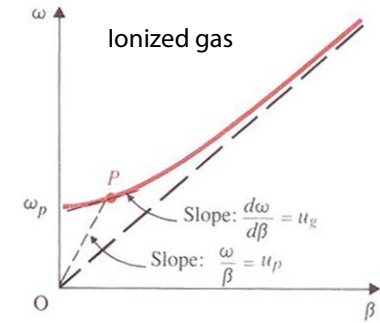
Dispersion

Group velocity: $u_g = \frac{d\omega}{d\beta}$

$u_p = \frac{\omega}{\beta}$

$$\frac{d\beta}{d\omega} = \frac{d}{d\omega} \left(\frac{\omega}{u_p} \right) = \frac{1}{u_p} - \frac{\omega}{u_p^2} \frac{du_p}{d\omega}$$

$$\rightarrow u_g = \frac{1}{\frac{1}{u_p} - \frac{\omega}{u_p^2} \frac{du_p}{d\omega}} = \frac{u_p}{1 - \frac{\omega}{u_p} \frac{du_p}{d\omega}}$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Dispersion:

$$\left\{ \begin{array}{l} \frac{du_p}{d\omega} = 0 \rightarrow u_g = u_p \rightarrow \text{No dispersion} \\ \frac{d\omega}{du_p} < 0 \rightarrow u_g < u_p \rightarrow \text{Normal dispersion} \\ \frac{d\omega}{du_p} > 0 \rightarrow u_g > u_p \rightarrow \text{Abnormal dispersion} \end{array} \right.$$

