Electromagnetics:

Oblique Incidence at a Plane Conducting Boundary Normal Incidence at a Plane Dielectric Boundary (8-7, 8-8)

Yoonchan Jeong

School of Electrical Engineering, Seoul National University

Tel: +82 (0)2 880 1623, Fax: +82 (0)2 873 9953

Email: yoonchan@snu.ac.kr

Electromagnetic Boundary Conditions

Continuity conditions:

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \rightarrow \oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} \rightarrow E_{1t} = E_{2t}$$

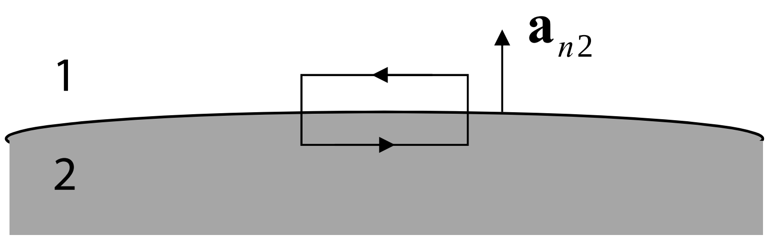
$$\rightarrow \mathbf{a}_{n2} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$$

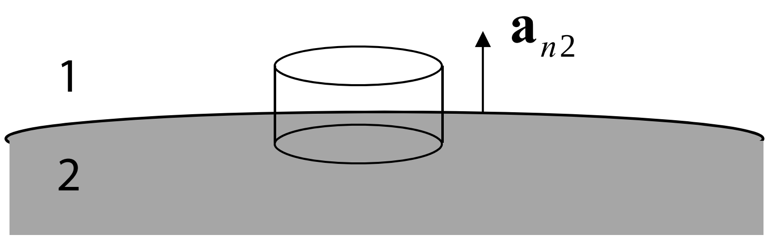
$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \rightarrow \oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s}$$

$$\rightarrow \mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$$

$$\nabla \cdot \mathbf{D} = \rho \rightarrow \mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$$

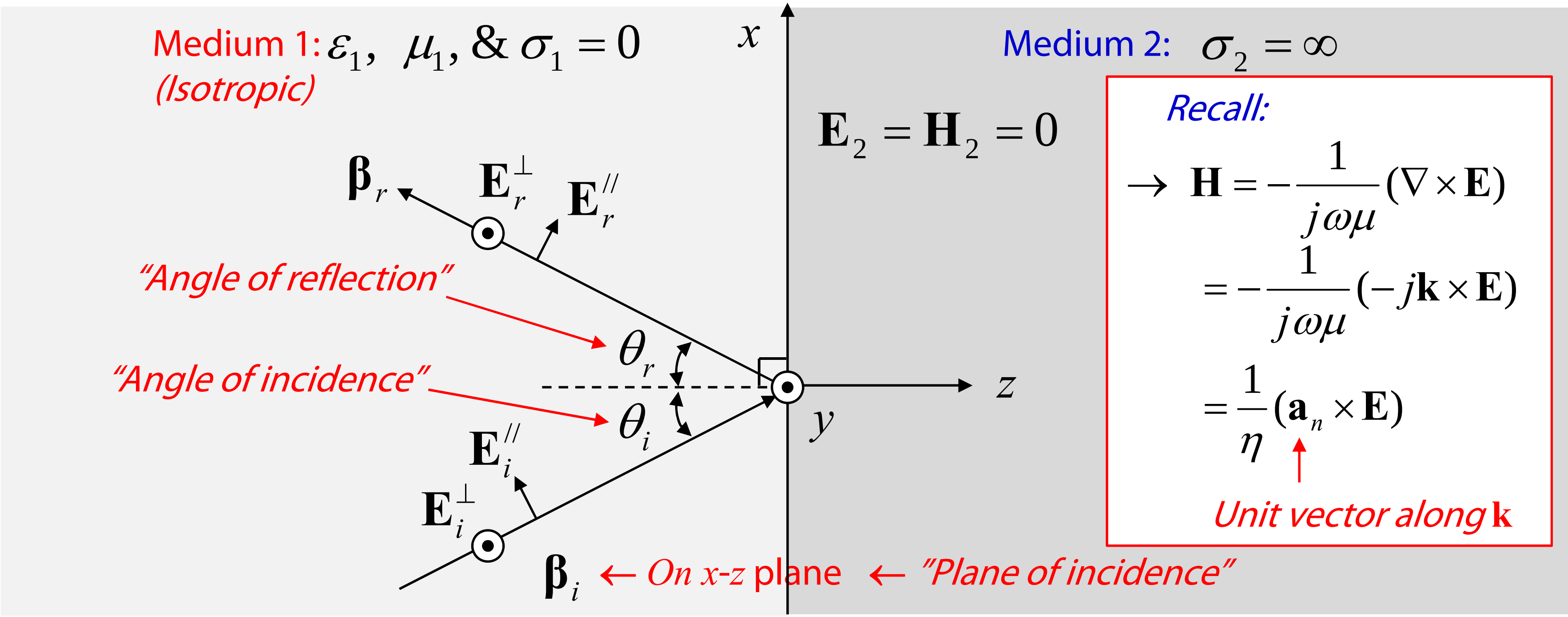
$$\nabla \cdot \mathbf{B} = 0 \rightarrow B_{1n} = B_{2n} \rightarrow \mathbf{a}_{n2} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$$





Oblique Incidence at a Plane Conducting Boundary

For plane waves: $E,\ H \ \propto\ e^{-j\boldsymbol{\beta}\cdot\mathbf{r}}$



$$= \beta_i \mathbf{a}_{ni} = \beta_1(\mathbf{a}_x \sin\theta_i + \mathbf{a}_z \cos\theta_i)$$

→ Still orthogonal!

Boundary condition for E-field (z = 0): $\mathbf{E}_{1t} = \mathbf{E}_{2t} \ \rightarrow\ \mathbf{E}_i^{\perp} + \mathbf{E}_{it}^{//} + \mathbf{E}_r^{\perp} + \mathbf{E}_{rt}^{//} = 0$

Perpendicular polarization (TE)

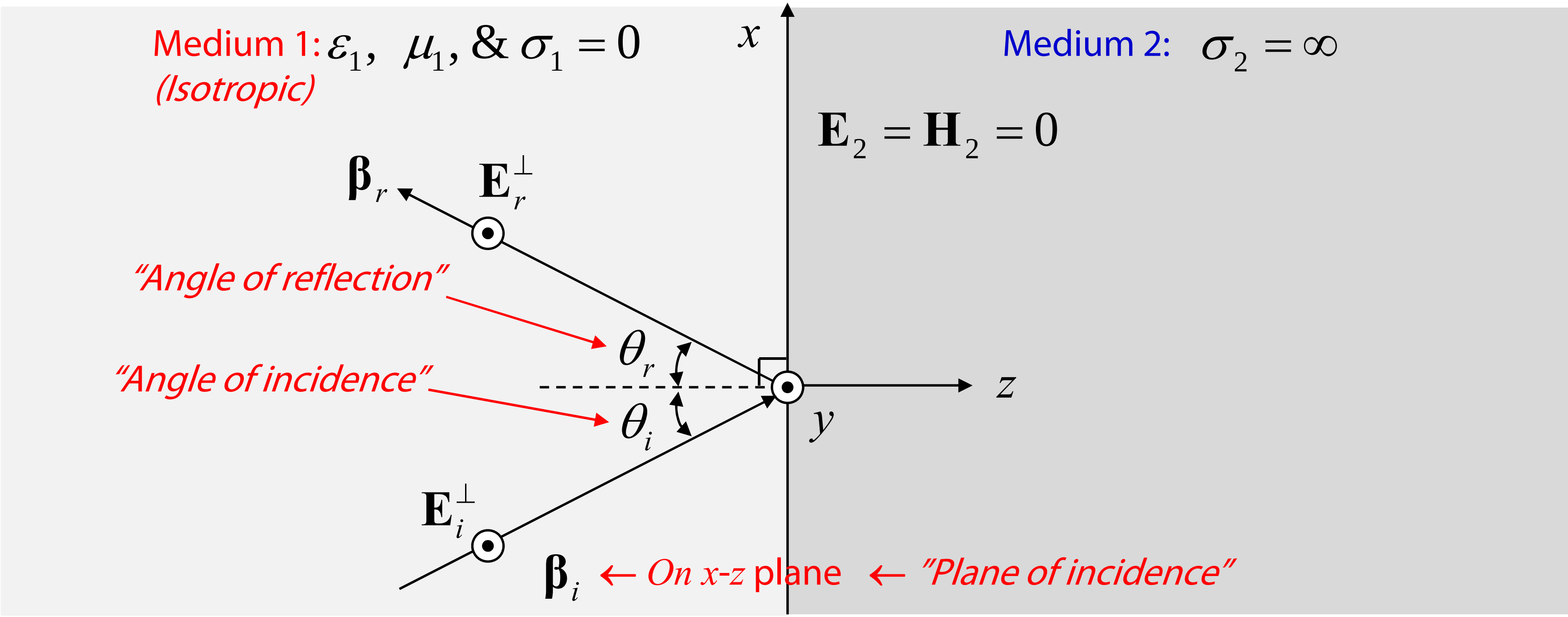
$$\mathbf{E}_i^{\perp} + \mathbf{E}_r^{\perp} = 0 \ (z = 0)$$

Parallel polarization (TM)

$$\mathbf{E}_{it}^{//} + \mathbf{E}_{rt}^{//} = 0 \ (z = 0)$$

Oblique Incidence: Perpendicular Polarization (TE) (1)

For plane waves: $E,\ H \propto e^{-j\boldsymbol{\beta}\cdot\mathbf{r}}$



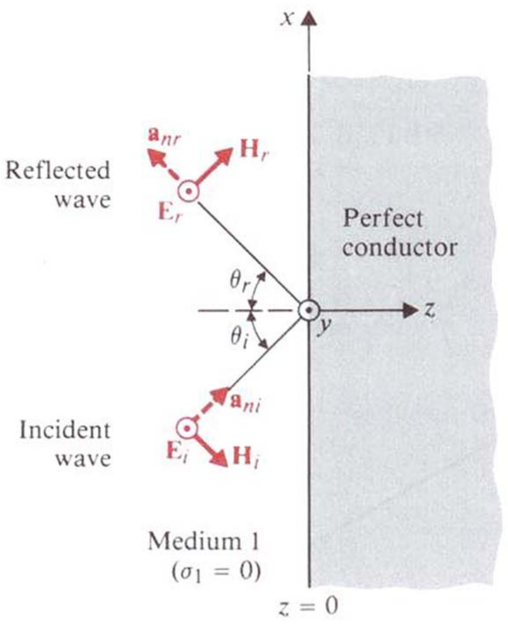
$$= \beta_i \mathbf{a}_{ni} = \beta_1(\mathbf{a}_x \sin\theta_i + \mathbf{a}_z \cos\theta_i)$$

For the reflected field:

$$\rightarrow\ \boldsymbol{\beta}_r = \beta_r \mathbf{a}_{nr} = \beta_1(\mathbf{a}_x \sin\theta_r - \mathbf{a}_z \cos\theta_r)$$ ← *Still on the plane of incidence*

Why?

Oblique Incidence: Perpendicular Polarization (TE) (2)



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

In medium 1 ($z < 0$):

Given

$$\mathbf{E}_i(x,z) = \mathbf{a}_y E_{i0} e^{-j\boldsymbol{\beta}_i \cdot \mathbf{r}} = \mathbf{a}_y E_{i0} e^{-j\beta_1 (x\sin\theta_i + z\cos\theta_i)}$$

$$\mathbf{H}_i(x,z) = \frac{1}{\eta_1}[\mathbf{a}_{ni} \times \mathbf{E}_i(x,z)]$$

$$= \frac{E_{i0}}{\eta_1}(-\mathbf{a}_x \cos\theta_i + \mathbf{a}_z \sin\theta_i) e^{-j\beta_1 (x\sin\theta_i + z\cos\theta_i)}$$

Unknown

$$\mathbf{E}_r(x,z) = \mathbf{a}_y E_{r0} e^{-j\boldsymbol{\beta}_r \cdot \mathbf{r}} = \mathbf{a}_y E_{r0} e^{-j\beta_1 (x\sin\theta_r - z\cos\theta_r)}$$

$$\mathbf{H}_r(x,z) = \frac{1}{\eta_1}\mathbf{a}_{nr} \times \mathbf{E}_r(x,z)$$

In medium 2 ($z > 0$):

Known

$$\mathbf{E}_2(x,z) = 0$$

$$\mathbf{H}_2(x,z) = 0$$

for a perfect conductor ($\sigma = \infty$)

How many equations do we need to solve?

Oblique Incidence: Perpendicular Polarization (TE) (3)

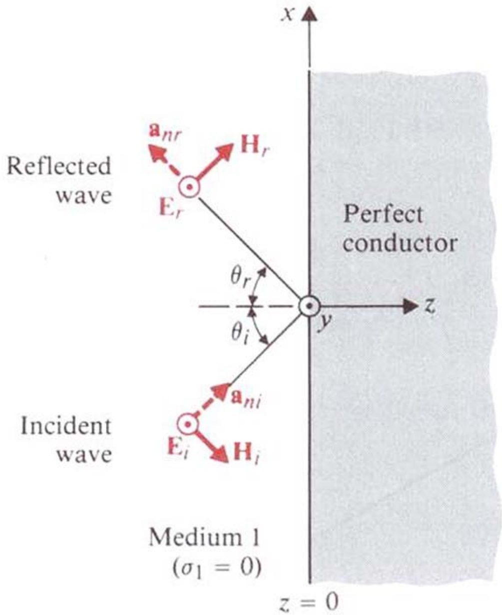
For tangential components of E-fields:

$$\mathbf{a}_{n2} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0 \qquad \rightarrow \mathbf{E}_{1t}(x,0) = \mathbf{E}_i(x,0) + \mathbf{E}_r(x,0) = \mathbf{E}_{2t}(x,0) = 0$$

$$\rightarrow \mathbf{E}_r(x,0) = -\mathbf{E}_i(x,0) = -\mathbf{a}_y E_{i0} e^{-j\beta_1 x \sin\theta_i}$$

$$\rightarrow \theta_r = \theta_i$$

$$\rightarrow \boldsymbol{\beta}_r = \beta_r \mathbf{a}_{nr} = \beta_1(\mathbf{a}_x \sin\theta_i - \mathbf{a}_z \cos\theta_i)$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$$\rightarrow \mathbf{E}_r(x,z) = -\mathbf{a}_y E_{i0} e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}$$

$$\rightarrow \mathbf{H}_r(x,z) = \frac{1}{\eta_1}\mathbf{a}_{nr} \times \mathbf{E}_r(x,z)$$

$$= -\frac{E_{i0}}{\eta_1}(\mathbf{a}_x \cos\theta_i + \mathbf{a}_z \sin\theta_i) e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}$$

What is the direction of the induced surface current density $\mathbf{J}_s$?

Poynting vector: $\mathbf{E}_1 \times \mathbf{H}_1 = ?$ $\mathbf{P}_{av} = \frac{1}{2}\mathrm{Re}[\mathbf{E}_1(x,z) \times \mathbf{H}_1^*(x,z)] = \mathbf{a}_x 2\frac{E_{i0}^2}{\eta_1}\sin\theta_i \sin^2(\beta_1 z\cos\theta_i)$

→ Propagating wave in x-direction *→ Standing wave in z-direction*

Oblique Incidence: Parallel Polarization (TM)

In medium 1 ($z < 0$):

$$\mathbf{E}_i(x,z) = E_{i0}(\mathbf{a}_x \cos\theta_i - \mathbf{a}_z \sin\theta_i)e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

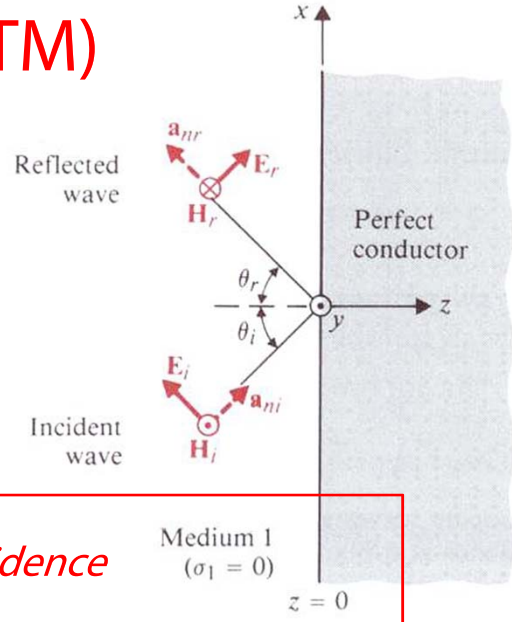
$$\mathbf{H}_i(x,z) = \frac{1}{\eta_1}[\mathbf{a}_{ni} \times \mathbf{E}_i(x,z)] = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1(x\sin\theta_i + z\cos\theta_i)}$$

$$\mathbf{E}_r(x,z) = \mathbf{a}_{er} E_{r0} e^{-j\boldsymbol{\beta}_r \cdot \mathbf{r}} = \mathbf{a}_{er} E_{r0} e^{-j\beta_1(x\sin\theta_r - z\cos\theta_r)}$$

$$\mathbf{H}_r(x,z) = \frac{1}{\eta_1}\mathbf{a}_{nr} \times \mathbf{E}_r(x,z)$$

→ $\boldsymbol{\beta}_r$: *Still on the plane of incidence*

Why?



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

For tangential components of E-fields:

$$\mathbf{a}_{n2} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0 \quad \rightarrow \mathbf{E}_{1t}(x,0) = \mathbf{E}_{it}(x,0) + \mathbf{E}_{rt}(x,0) = \mathbf{E}_{2t}(x,0) = 0$$

$$\rightarrow \mathbf{E}_{rt}(x,0) = -\mathbf{E}_{it}(x,0) = -\mathbf{a}_x E_{i0}\cos\theta_i e^{-j\beta_1 x\sin\theta_i}$$

$$\rightarrow \theta_r = \theta_i \quad \rightarrow \boldsymbol{\beta}_r = \beta_r \mathbf{a}_{nr} = \beta_1(\mathbf{a}_x \sin\theta_i - \mathbf{a}_z \cos\theta_i)$$

$$\rightarrow \mathbf{a}_{er} E_{r0} = -E_{i0}(\mathbf{a}_x \cos\theta_i + \mathbf{a}_z \sin\theta_i)$$ *Why?*

$$\rightarrow \mathbf{E}_r(x,z) = -E_{i0}(\mathbf{a}_x \cos\theta_i + \mathbf{a}_z \sin\theta_i)e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}$$

$$\rightarrow \mathbf{H}_r(x,z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1(x\sin\theta_i - z\cos\theta_i)}$$

Normal Incidence at a Plane Dielectric Boundary (1)

Lossless dielectric media: $\sigma_1 = \sigma_2 = 0$

In medium 1 ($z < 0$):

Given

$$\mathbf{E}_i(z) = \mathbf{a}_x E_{i0} e^{-j\beta_1 z}$$

$$\mathbf{H}_i(z) = \frac{1}{\eta_1}\mathbf{a}_{ni} \times \mathbf{E}_i(z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z}$$

Unknown

$$\mathbf{E}_r(z) = \mathbf{a}_x E_{r0} e^{+j\beta_1 z}$$

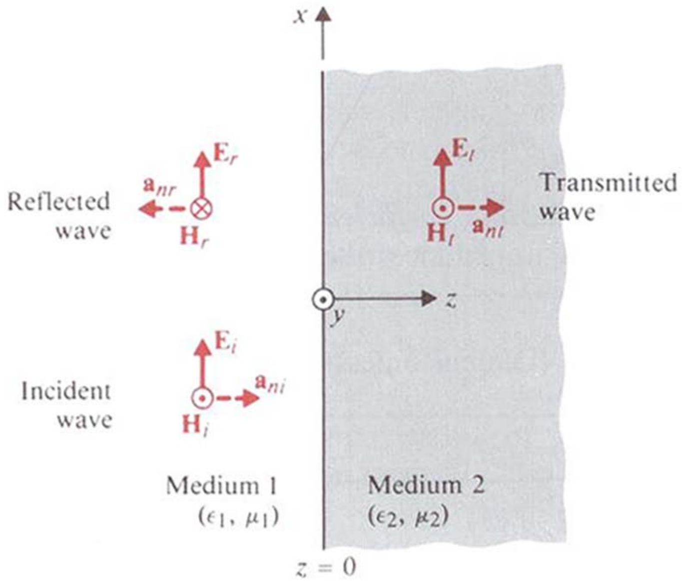
$$\mathbf{H}_r(z) = \frac{1}{\eta_1}\mathbf{a}_{nr} \times \mathbf{E}_r(z) = -\mathbf{a}_y \frac{E_{r0}}{\eta_1} e^{+j\beta_1 z}$$

In medium 2 ($z > 0$):

Unknown

$$\mathbf{E}_t(z) = \mathbf{a}_x E_{t0} e^{-j\beta_2 z}$$

$$\mathbf{H}_t(z) = \mathbf{a}_y \frac{E_{t0}}{\eta_2} e^{-j\beta_2 z}$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$$\boldsymbol{\beta}_i = \beta_i \mathbf{a}_{ni} = \beta_1 \mathbf{a}_z$$

$$\boldsymbol{\beta}_r = \beta_r \mathbf{a}_{nr} = \beta_1(-\mathbf{a}_z)$$

How many equations do we need to solve?

Normal Incidence at a Plane Dielectric Boundary (2)

For tangential components of E-fields:

$$\mathbf{a}_{n2} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0 \quad \rightarrow \mathbf{E}_i(0) + \mathbf{E}_r(0) = \mathbf{E}_t(0) \quad \rightarrow E_{i0} + E_{r0} = E_{t0}$$

For tangential components of H-fields:

$$\mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s = 0 \quad \rightarrow \mathbf{H}_i(0) + \mathbf{H}_r(0) = \mathbf{H}_t(0)$$

$$\rightarrow \frac{1}{\eta_1}(E_{i0} - E_{r0}) = \frac{1}{\eta_2} E_{t0}$$

Reflection and transmission coefficients:

$$E_{r0} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} E_{i0} \equiv \Gamma E_{i0}$$ → *Reflection coefficient*

$$E_{t0} = \frac{2\eta_2}{\eta_2 + \eta_1} E_{i0} \equiv \tau E_{i0}$$ → *Transmission coefficient*

$$\rightarrow 1 + \Gamma = \tau$$ ← *Continuity of E-field*

$$\rightarrow 1 - \Gamma^2 = \frac{\eta_1}{\eta_2} \tau^2$$ ← *Conservation of power density*