

Electromagnetics:

Oblique Incidence at a Plane Dielectric Boundary

(8-10)

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Boundary Conditions for a Plane Dielectric Interface

Lossless dielectric media: $\sigma_1 = \sigma_2 = 0$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \quad \rightarrow \quad \mathbf{a}_{n2} \times (\mathbf{E}_1 - \mathbf{E}_2) = 0$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \quad \rightarrow \quad \mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s$$

$$\nabla \cdot \mathbf{D} = \rho \quad \rightarrow \quad \mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$$

$$\nabla \cdot \mathbf{B} = 0 \quad \rightarrow \quad \mathbf{a}_{n2} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0$$



$$E_{1t} = E_{2t}$$

$$H_{1t} = H_{2t}$$

$$D_{1n} = D_{2n}$$

$$B_{1n} = B_{2n}$$

Incidence at a Plane Dielectric Boundary

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

For plane waves: $E, H \propto e^{-j\boldsymbol{\beta} \cdot \mathbf{r}}$
(Isotropic media)

In medium 1 ($z < 0$):

Incident wave:

$$\mathbf{E}_i = \mathbf{E}_{i0} e^{-j\boldsymbol{\beta}_i \cdot \mathbf{r}} \rightarrow \boldsymbol{\beta}_i = \beta_1 (\mathbf{a}_x \sin \theta_i + \mathbf{a}_z \cos \theta_i) = \beta_1 \mathbf{a}_{ni}$$

$$\mathbf{H}_i = \mathbf{H}_{i0} e^{-j\boldsymbol{\beta}_i \cdot \mathbf{r}} = \frac{1}{\eta_1} \mathbf{a}_{ni} \times \mathbf{E}_{i0} e^{-j\boldsymbol{\beta}_i \cdot \mathbf{r}}$$

Reflected wave:

$$\mathbf{E}_r = \mathbf{E}_{r0} e^{-j\boldsymbol{\beta}_r \cdot \mathbf{r}} \rightarrow \boldsymbol{\beta}_r = \beta_1 (\mathbf{a}_x \sin \theta_r - \mathbf{a}_z \cos \theta_r) = \beta_1 \mathbf{a}_{nr}$$

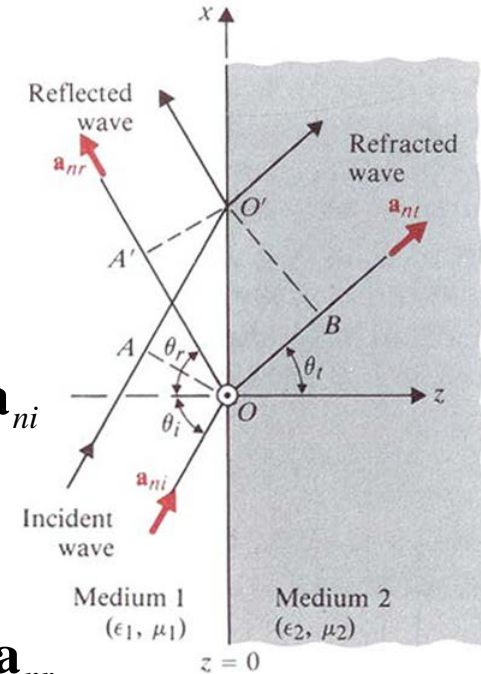
$$\mathbf{H}_r = \mathbf{H}_{r0} e^{-j\boldsymbol{\beta}_r \cdot \mathbf{r}} = \frac{1}{\eta_1} \mathbf{a}_{nr} \times \mathbf{E}_{r0} e^{-j\boldsymbol{\beta}_r \cdot \mathbf{r}}$$

In medium 2 ($z > 0$):

Transmitted wave:

$$\mathbf{E}_t = \mathbf{E}_{t0} e^{-j\boldsymbol{\beta}_t \cdot \mathbf{r}} \rightarrow \boldsymbol{\beta}_t = \beta_2 (\mathbf{a}_x \sin \theta_t + \mathbf{a}_z \cos \theta_t) = \beta_2 \mathbf{a}_{nt}$$

$$\mathbf{H}_t = \mathbf{H}_{t0} e^{-j\boldsymbol{\beta}_t \cdot \mathbf{r}} = \frac{1}{\eta_2} \mathbf{a}_{nt} \times \mathbf{E}_{t0} e^{-j\boldsymbol{\beta}_t \cdot \mathbf{r}}$$



Snell's Law & Total Reflection



Willebrord Snell
(1580-1626)

Boundary conditions at $z=0$: $\mathbf{E}_{1t} = \mathbf{E}_{2t}$, $\mathbf{H}_{1t} = \mathbf{H}_{2t}$

$$\mathbf{E}_{i0t} e^{-j\beta_1 \sin \theta_i x} + \mathbf{E}_{r0t} e^{-j\beta_1 \sin \theta_r x} = \mathbf{E}_{t0t} e^{-j\beta_2 \sin \theta_t x}$$

$$\mathbf{H}_{i0t} e^{-j\beta_1 \sin \theta_i x} + \mathbf{H}_{r0t} e^{-j\beta_1 \sin \theta_r x} = \mathbf{H}_{t0t} e^{-j\beta_2 \sin \theta_t x}$$

To hold for any x values: $\rightarrow e^{-j\beta_1 \sin \theta_i x} = e^{-j\beta_1 \sin \theta_r x} = e^{-j\beta_2 \sin \theta_t x}$

$$\rightarrow \beta_1 \sin \theta_i = \beta_1 \sin \theta_r = \beta_2 \sin \theta_t$$

$$\rightarrow \theta_i = \theta_r \rightarrow \text{Snell's law of reflection}$$

$$\rightarrow n_1 \sin \theta_i = n_2 \sin \theta_t \rightarrow \text{Snell's law of refraction}$$

For $n_1 > n_2$:

$$\sin \theta_t = \frac{n_1}{n_2} \sin \theta_i \rightarrow \sin \theta_t > 1 \text{ if } \theta_i > \theta_c \leftarrow \theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

Critical angle

\rightarrow Total internal reflection

Recall: $\boldsymbol{\beta}_t = \beta_2 (\mathbf{a}_x \sin \theta_t + \mathbf{a}_z \cos \theta_t)$

$$\rightarrow \cos \theta_t = \sqrt{1 - \sin^2 \theta_t} \rightarrow -j\sqrt{\sin^2 \theta_t - 1} \rightarrow \text{Evanescent wave in } z\text{-axis}$$

Continuity of E- and H-fields

Recall:

Boundary conditions at $z=0$: $\mathbf{E}_{1t} = \mathbf{E}_{2t}$, $\mathbf{H}_{1t} = \mathbf{H}_{2t}$

$$\mathbf{E}_{i0t} e^{-j\beta_1 \sin \theta_i x} + \mathbf{E}_{r0t} e^{-j\beta_1 \sin \theta_r x} = \mathbf{E}_{t0t} e^{-j\beta_2 \sin \theta_t x} \rightarrow \mathbf{E}_{i0t} + \mathbf{E}_{r0t} = \mathbf{E}_{t0t}$$

$$\mathbf{H}_{i0t} e^{-j\beta_1 \sin \theta_i x} + \mathbf{H}_{r0t} e^{-j\beta_1 \sin \theta_r x} = \mathbf{H}_{t0t} e^{-j\beta_2 \sin \theta_t x} \rightarrow \mathbf{H}_{i0t} + \mathbf{H}_{r0t} = \mathbf{H}_{t0t}$$

Consider two orthogonal polarizations: \perp & $//$

$$\mathbf{E}_{i0t} + \mathbf{E}_{r0t} = \mathbf{E}_{t0t} \rightarrow \mathbf{E}_{i0t}^{\perp} + \mathbf{E}_{i0t}^{//} + \mathbf{E}_{r0t}^{\perp} + \mathbf{E}_{r0t}^{//} = \mathbf{E}_{t0t}^{\perp} + \mathbf{E}_{t0t}^{//}$$

$$\mathbf{H}_{i0t} + \mathbf{H}_{r0t} = \mathbf{H}_{t0t} \rightarrow \mathbf{H}_{i0t}^{\perp} + \mathbf{H}_{i0t}^{//} + \mathbf{H}_{r0t}^{\perp} + \mathbf{H}_{r0t}^{//} = \mathbf{H}_{t0t}^{\perp} + \mathbf{H}_{t0t}^{//}$$

Perpendicular polarization (TE)

$$E_{i0t}^{\perp} + E_{r0t}^{\perp} = E_{t0t}^{\perp}$$

$$H_{i0t}^{\perp} + H_{r0t}^{\perp} = H_{t0t}^{\perp}$$

Parallel polarization (TM)

$$E_{i0t}^{//} + E_{r0t}^{//} = E_{t0t}^{//}$$

$$H_{i0t}^{//} + H_{r0t}^{//} = H_{t0t}^{//}$$

Perpendicular Polarization (TE) (1)

Perpendicular polarization (TE)

$$E_{i0t}^\perp + E_{r0t}^\perp = E_{t0t}^\perp$$

$$H_{i0t}^\perp + H_{r0t}^\perp = H_{t0t}^\perp$$

$$\mathbf{E}_{i0}^\perp = \mathbf{a}_y E_{i0}^\perp$$

$$\begin{aligned} \mathbf{H}_{i0}^\perp &= \frac{1}{\eta_1} \mathbf{a}_{ni} \times \mathbf{E}_{i0}^\perp \\ &= \frac{E_{i0}^\perp}{\eta_1} (-\mathbf{a}_x \cos \theta_i + \mathbf{a}_z \sin \theta_i) \end{aligned}$$

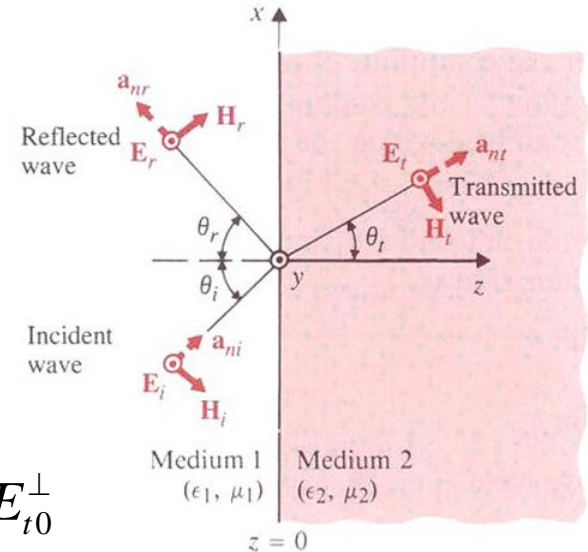
$$\mathbf{E}_{r0}^\perp = \mathbf{a}_y E_{r0}^\perp$$

$$\begin{aligned} \mathbf{H}_{r0}^\perp &= \frac{1}{\eta_1} \mathbf{a}_{nr} \times \mathbf{E}_{r0}^\perp \\ &= \frac{E_{r0}^\perp}{\eta_1} (\mathbf{a}_x \cos \theta_r + \mathbf{a}_z \sin \theta_r) \end{aligned}$$

$$\mathbf{E}_{t0}^\perp = \mathbf{a}_y E_{t0}^\perp$$

$$\begin{aligned} \mathbf{H}_{t0}^\perp &= \frac{1}{\eta_2} \mathbf{a}_{nt} \times \mathbf{E}_{t0}^\perp \\ &= \frac{E_{t0}^\perp}{\eta_2} (-\mathbf{a}_x \cos \theta_t + \mathbf{a}_z \sin \theta_t) \end{aligned}$$

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.



$$E_{i0}^\perp + E_{r0}^\perp = E_{t0}^\perp$$

$$\frac{1}{\eta_1} (E_{i0}^\perp - E_{r0}^\perp) \cos \theta_i = \frac{1}{\eta_2} E_{t0}^\perp \cos \theta_t$$

Perpendicular Polarization (TE) (2)

$$E_{i0}^{\perp} + E_{r0}^{\perp} = E_{t0}^{\perp}$$

$$\frac{1}{\eta_1} (E_{i0}^{\perp} - E_{r0}^{\perp}) \cos \theta_i = \frac{1}{\eta_2} E_{t0}^{\perp} \cos \theta_t$$

$$\Gamma_{\perp} = \frac{E_{r0}^{\perp}}{E_{i0}^{\perp}} = \frac{\eta_2 \cos \theta_i - \eta_1 \cos \theta_t}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t}$$

$$\tau_{\perp} = \frac{E_{t0}^{\perp}}{E_{i0}^{\perp}} = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_i + \eta_1 \cos \theta_t}$$

$$1 + \Gamma_{\perp} = \tau_{\perp}$$

$$\text{Note: } \eta_2 \cos \theta_i - \eta_1 \cos \theta_t = 0 \rightarrow \Gamma_{\perp} = 0$$

→ Brewster angle



Augustin Jean Fresnel
(1788-1827)

→ Fresnel's equations for S-pol.



David Brewster
(1781-1868)

Brewster Angle for S-Pol. (TE) Waves

In case: $\Gamma_{\perp} = 0 \quad \rightarrow \boxed{\eta_2 \cos \theta_{B\perp} = \eta_1 \cos \theta_t}$

$$\rightarrow \cos^2 \theta_{B\perp} = \frac{\eta_1^2}{\eta_2^2} \cos^2 \theta_t \rightarrow 1 - \sin^2 \theta_{B\perp} = \frac{\eta_1^2}{\eta_2^2} \left(1 - \frac{n_1^2}{n_2^2} \sin^2 \theta_{B\perp}\right)$$

$$\rightarrow 1 - \frac{\eta_1^2}{\eta_2^2} = \left(1 - \frac{\eta_1^2}{\eta_2^2} \frac{n_1^2}{n_2^2}\right) \sin^2 \theta_{B\perp}$$

$$\rightarrow \sin^2 \theta_{B\perp} = \frac{1 - \eta_1^2 / \eta_2^2}{1 - \eta_1^2 n_1^2 / \eta_2^2 n_2^2} = \frac{1 - \frac{\mu_1 \varepsilon_2}{\varepsilon_1 \mu_2}}{1 - \frac{\mu_1 \varepsilon_2}{\varepsilon_1 \mu_2} \frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2}} = \frac{1 - \mu_1 \varepsilon_2 / \mu_2 \varepsilon_1}{1 - (\mu_1 / \mu_2)^2}$$

*No Brewster angle
for $\mu_1 = \mu_2$*

In case: $\varepsilon_1 = \varepsilon_2, \mu_1 \neq \mu_2$

$$\rightarrow \sin \theta_{B\perp} = \frac{1}{\sqrt{1 + \mu_1 / \mu_2}}$$

→ For non-magnetic media, no Brewster angle exists!

Parallel Polarization (TM) (1)

Parallel polarization (TM)

$$E_{i0t}'' + E_{r0t}'' = E_{t0t}''$$

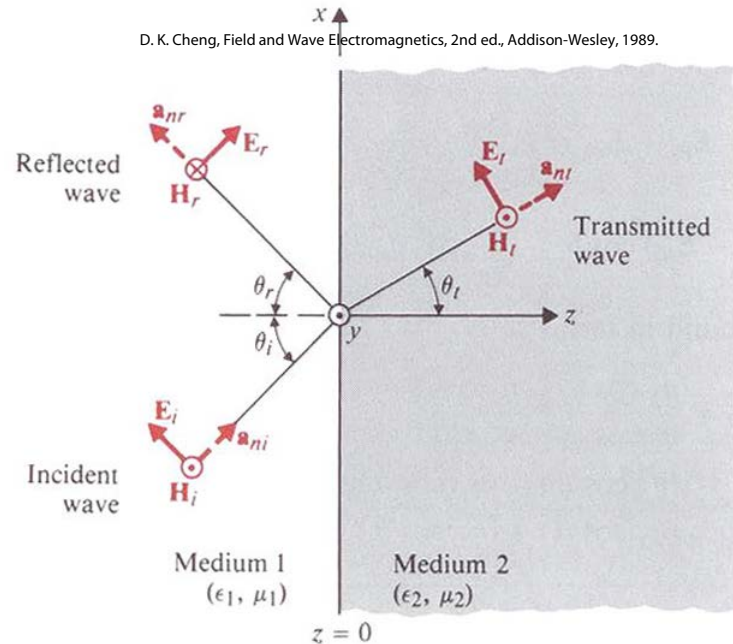
$$H_{i0t}'' + H_{r0t}'' = H_{t0t}''$$

$$\mathbf{E}_{i0}'' = E_{i0}'' (\mathbf{a}_x \cos \theta_i - \mathbf{a}_z \sin \theta_i)$$

$$\mathbf{H}_{i0}'' = \mathbf{a}_y \frac{1}{\eta_1} E_{i0}''$$

$$\mathbf{E}_{r0}'' = E_{r0}'' (\mathbf{a}_x \cos \theta_r + \mathbf{a}_z \sin \theta_r)$$

$$\mathbf{H}_{r0}'' = -\mathbf{a}_y \frac{1}{\eta_1} E_{r0}''$$



$$\mathbf{E}_{t0}'' = E_{t0}'' (\mathbf{a}_x \cos \theta_t - \mathbf{a}_z \sin \theta_t)$$

$$\mathbf{H}_{t0}'' = \mathbf{a}_y \frac{1}{\eta_2} E_{t0}''$$

$$(E_{i0}'' + E_{r0}'') \cos \theta_i = E_{t0}'' \cos \theta_t$$

$$\frac{1}{\eta_1} (E_{i0}'' - E_{r0}'') = \frac{1}{\eta_2} E_{t0}''$$

Parallel Polarization (TM) (2)

$$\begin{aligned}(E_{i0}'' + E_{r0}'') \cos \theta_i &= E_{t0}'' \cos \theta_t \\ \frac{1}{\eta_1} (E_{i0}'' - E_{r0}'') &= \frac{1}{\eta_2} E_{t0}''\end{aligned}$$

$$\Gamma_{//} = \frac{E_{r0}''}{E_{i0}''} = \frac{\eta_2 \cos \theta_t - \eta_1 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i}$$

$$\tau_{//} = \frac{E_{t0}''}{E_{i0}''} = \frac{2\eta_2 \cos \theta_i}{\eta_2 \cos \theta_t + \eta_1 \cos \theta_i}$$

→ Fresnel's equations for P-pol.

$$(1 + \Gamma_{//}) \cos \theta_i = \tau_{//} \cos \theta_t$$

$$\text{Note: } \eta_2 \cos \theta_t - \eta_1 \cos \theta_i = 0 \quad \rightarrow \Gamma_{//} = 0$$

→ Brewster angle

Brewster Angle for P-Pol. (TM) Waves

In case: $\Gamma_{//} = 0 \rightarrow \boxed{\eta_2 \cos \theta_t = \eta_1 \cos \theta_{B//}}$

$$\rightarrow \cos^2 \theta_{B//} = \frac{\eta_2^2}{\eta_1^2} \cos^2 \theta_t \rightarrow 1 - \sin^2 \theta_{B//} = \frac{\eta_2^2}{\eta_1^2} \left(1 - \frac{n_1^2}{n_2^2} \sin^2 \theta_{B//}\right)$$

$$\rightarrow 1 - \frac{\eta_2^2}{\eta_1^2} = \left(1 - \frac{\eta_2^2}{\eta_1^2} \frac{n_1^2}{n_2^2}\right) \sin^2 \theta_{B//}$$

$$\rightarrow \sin^2 \theta_{B//} = \frac{1 - \eta_2^2 / \eta_1^2}{1 - \eta_2^2 n_1^2 / \eta_1^2 n_2^2} = \frac{1 - \frac{\mu_2}{\varepsilon_2} \frac{\varepsilon_1}{\mu_1}}{1 - \frac{\mu_2}{\varepsilon_2} \frac{\varepsilon_1}{\mu_1} \frac{\mu_1 \varepsilon_1}{\mu_2 \varepsilon_2}} = \frac{1 - \mu_2 \varepsilon_1 / \mu_1 \varepsilon_2}{1 - (\varepsilon_1 / \varepsilon_2)^2}$$

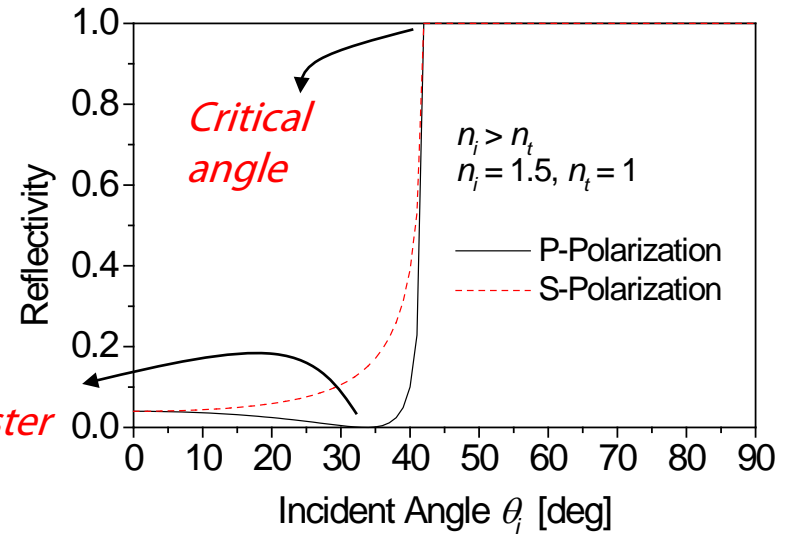
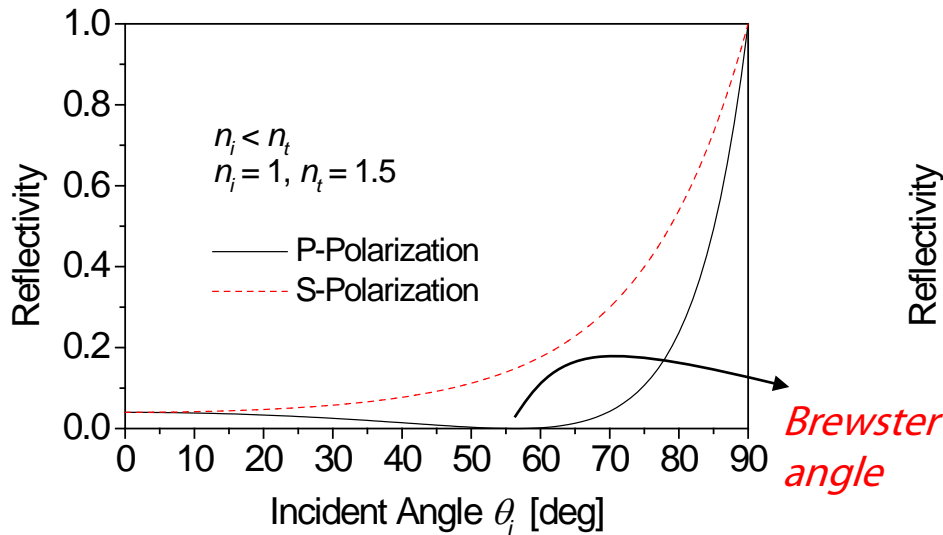
*No Brewster angle
for $\varepsilon_1 = \varepsilon_2$*

In case: $\varepsilon_1 \neq \varepsilon_2, \mu_1 = \mu_2 = \mu_0$

$$\rightarrow \sin \theta_{B//} = \frac{1}{\sqrt{1 + \varepsilon_1 / \varepsilon_2}} \rightarrow \tan \theta_{B//} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = \frac{n_2}{n_1} \rightarrow \theta_i + \theta_t = \frac{\pi}{2}$$

Brewster Angle & Critical Angle

Numerical examples:



Brewster angle (Non-magnetic):

Critical angle: Total internal reflection

For P-polarization

$$\theta_B = \tan^{-1}\left(\frac{n_t}{n_i}\right)$$

$$\rightarrow \theta_i + \theta_t = \frac{\pi}{2}$$

$$\theta_c = \sin^{-1}\left(\frac{n_t}{n_i}\right)$$

$$\rightarrow n_i > n_t, \theta_t = \frac{\pi}{2}$$