Electromagnetics:

Parallel-Plate Waveguides

(10-3)

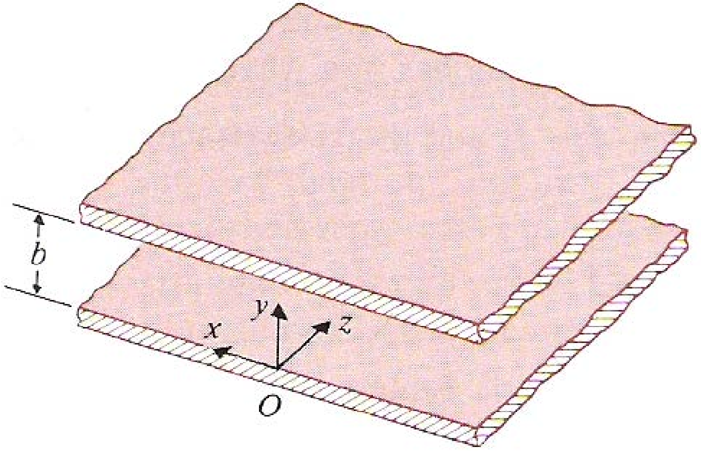
Yoonchan Jeong

School of Electrical Engineering, Seoul National University

Tel: +82 (0)2 880 1623, Fax: +82 (0)2 873 9953

Email: yoonchan@snu.ac.kr

Parallel-Plate Waveguides



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

An infinite parallel-plate waveguide

→ *TEM, TM, & TE modes possible*

Conditions given:

→ *Two perfectly conducting plates*

→ *Dielectric medium with ε, μ*

→ *Waves propagating in the z-direction*

→ *Infinitely wide in the x-direction: No variation in the x-direction*

TM Waves Between Parallel Plates (1)

Recall:

$$\gamma = \alpha + j\beta$$ → *Propagation constant*

$$\mathbf{E}(x, y, z; t) = \mathrm{Re}[\mathbf{E}^0(x, y) e^{(j\omega t - \gamma z)}]$$

$$H_z = 0 \quad \rightarrow \nabla_{xy}^2 E_z^0 + (\gamma^2 + k^2) E_z^0 = 0 \quad \rightarrow \nabla_{xy}^2 E_z^0 + h^2 E_z^0 = 0$$

$$\leftarrow h^2 = \gamma^2 + k^2$$

Characteristic equation:

$$\rightarrow \frac{\partial^2 E_z^0}{\partial y^2} + h^2 E_z^0 = 0 \quad (\because \frac{\partial}{\partial x} \rightarrow 0)$$

Recall:
BCs for a D-PEC Interface

$$E_{1t} = 0, \quad E_{2t} = 0$$
$$\mathbf{a}_{n2} \times \mathbf{H}_1 = \mathbf{J}_s, \quad H_{2t} = 0$$
$$\mathbf{a}_{2n} \cdot \mathbf{D}_1 = \rho_s, \quad D_{2n} = 0$$
$$B_{1n} = 0, \quad B_{2n} = 0$$

Boundary conditions:

$$E_z^0 = 0 \quad \leftarrow y = 0 \;\&\; y = b$$

Solution:

$$E_z^0 = A_n \sin hy$$

$$\leftarrow h = \frac{n\pi}{b}, \quad n = 0, 1, 2, \ldots$$

TM Waves Between Parallel Plates (2)

Solution:

$$E_z^0 = A_n \sin hy$$

$$\rightarrow H_x^0 = \frac{j\omega\varepsilon}{h} A_n \cos hy$$

$$\rightarrow E_y^0 = -\frac{\gamma}{h} A_n \cos hy$$

Recall:

$$\boxed{\begin{aligned}
H_x^0 &= -\frac{1}{h^2}\left(\gamma \cancel{\frac{\partial H_z^0}{\partial x}} - j\omega\varepsilon \frac{\partial E_z^0}{\partial y}\right) \\
H_y^0 &= -\frac{1}{h^2}\left(\gamma \cancel{\frac{\partial H_z^0}{\partial y}} + j\omega\varepsilon \cancel{\frac{\partial E_z^0}{\partial x}}\right) \\
E_x^0 &= -\frac{1}{h^2}\left(\gamma \cancel{\frac{\partial E_z^0}{\partial x}} + j\omega\mu \cancel{\frac{\partial H_z^0}{\partial y}}\right) \\
E_y^0 &= -\frac{1}{h^2}\left(\gamma \frac{\partial E_z^0}{\partial y} - j\omega\mu \cancel{\frac{\partial H_z^0}{\partial x}}\right)
\end{aligned}}$$

$$\rightarrow \gamma = \sqrt{\left(\frac{n\pi}{b}\right)^2 - \omega^2 \mu\varepsilon}$$

$$\rightarrow f_c = \frac{n}{2b\sqrt{\mu\varepsilon}}$$

If $n = 0$:

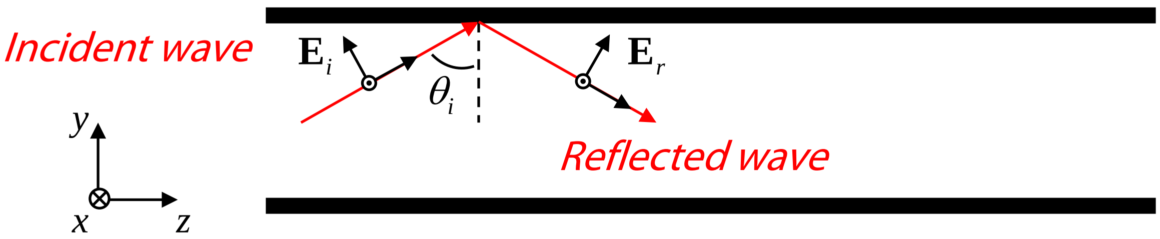
$$\rightarrow E_z^0 = 0$$

$TM_0 \rightarrow$ TEM mode

$$\rightarrow \gamma = ik$$

TM Waves: Alternative Approach (1)

An infinite parallel-plate waveguide



Plane-wave representation:

$$\rightarrow \mathbf{E}_i = E_i(\mathbf{a}_y \sin\theta_i - \mathbf{a}_z \cos\theta_i)e^{-jk(\cos\theta_i y + \sin\theta_i z)}$$

$$\rightarrow \mathbf{E}_r = E_r(\mathbf{a}_y \sin\theta_i + \mathbf{a}_z \cos\theta_i)e^{-jk(-\cos\theta_i y + \sin\theta_i z)}$$

$$\rightarrow \mathbf{E}_{total} = \mathbf{E}_i + \mathbf{E}_r \quad \rightarrow E_{total,t} = E_{i,z} + E_{r,z}$$

$$= (-E_i \cos\theta_i e^{-jk\cos\theta_i y} + E_r \cos\theta_i e^{+jk\cos\theta_i y})e^{-jk\sin\theta_i z}$$

$$\rightarrow E_{total,t}(y = 0\,\&\,b) = 0$$

$$\rightarrow E_i = E_r \quad \rightarrow \sin(k\cos\theta_i b) = 0 \quad \rightarrow k\cos\theta_i = \frac{n\pi}{b}$$

$$\rightarrow \mathbf{E}_{total} = E_i[\mathbf{a}_y 2\sin\theta_i \cos(k\cos\theta_i y) + \mathbf{a}_z 2j\cos\theta_i \sin(k\cos\theta_i y)]e^{-jk\sin\theta_i z}$$

TM Waves: Alternative Approach (2)

$$\rightarrow \mathbf{E}_{total} = E_i[\mathbf{a}_y 2\sin\theta_i \cos(\overbrace{k\cos\theta_i}^{\rightarrow h} y) + \mathbf{a}_z 2j\cos\theta_i \sin(k\cos\theta_i y)]e^{-\overbrace{jk\sin\theta_i}^{\rightarrow \gamma} z}$$

$$= A_n[\mathbf{a}_y \frac{2\sin\theta_i}{2j\cos\theta_i}\cos hy + \mathbf{a}_z \sin hy]e^{-\gamma z} \qquad \leftarrow h^2 = \gamma^2 + k^2$$

$$= A_n[\mathbf{a}_y \frac{-jk\sin\theta_i}{k\cos\theta_i}\cos hy + \mathbf{a}_z \sin hy]e^{-\gamma z}$$

$$= A_n[\mathbf{a}_y \frac{-\gamma}{h}\cos hy + \mathbf{a}_z \sin hy]e^{-\gamma z}$$

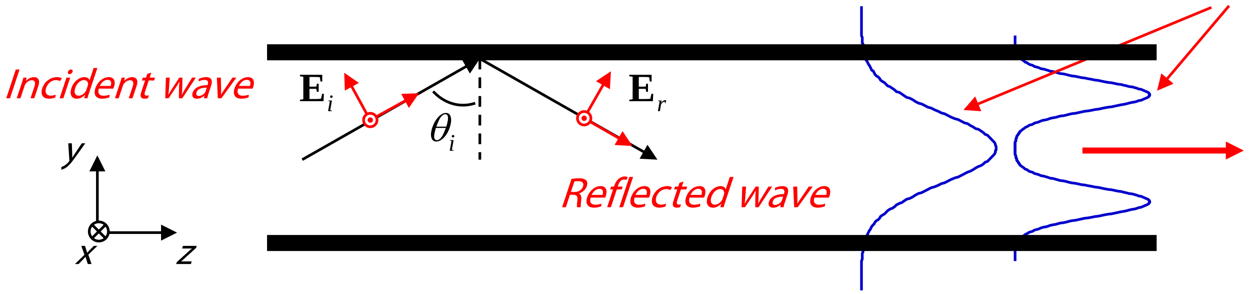
Exactly the same outcome!

Normal at the boundary

Tangential at the boundary

An infinite parallel-plate waveguide

Standing waves that satisfy the BCs



TE Waves Between Parallel Plates

Recall:

$$\gamma = \alpha + j\beta \qquad \rightarrow \textit{Propagation constant}$$

$$\mathbf{E}(x, y, z; t) = \mathrm{Re}[\mathbf{E}^0(x, y) e^{(j\omega t - \gamma z)}]$$

$$E_z = 0 \quad \rightarrow \nabla_{xy}^2 H_z^0 + (\gamma^2 + k^2) H_z^0 = 0 \quad \rightarrow \nabla_{xy}^2 H_z^0 + h^2 H_z^0 = 0$$

Characteristic equation:

$$\rightarrow \frac{\partial^2 H_z^0}{\partial y^2} + h^2 H_z^0 = 0 \qquad (\because \frac{\partial}{\partial x} \rightarrow 0)$$

Recall:

$$H_x^0 = -\frac{1}{h^2}\left(\gamma \cancel{\frac{\partial H_z^0}{\partial x}} - j\omega\varepsilon \cancel{\frac{\partial E_z^0}{\partial y}}\right)$$

$$H_y^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial H_z^0}{\partial y} + j\omega\varepsilon \cancel{\frac{\partial E_z^0}{\partial x}}\right)$$

$$E_x^0 = -\frac{1}{h^2}\left(\gamma \cancel{\frac{\partial E_z^0}{\partial x}} + j\omega\mu \frac{\partial H_z^0}{\partial y}\right)$$

$$E_y^0 = -\frac{1}{h^2}\left(\gamma \cancel{\frac{\partial E_z^0}{\partial y}} - j\omega\mu \cancel{\frac{\partial H_z^0}{\partial x}}\right)$$

Boundary conditions:

$$E_x^0 = 0 \quad \leftarrow y = 0 \;\&\; y = b$$

Solution:

$$H_z^0 = B_n \cos hy$$

$$\leftarrow h = \frac{n\pi}{b}, \quad n = 0, 1, 2, \ldots$$

If n = 0:

$TE_0 \rightarrow$ Null mode

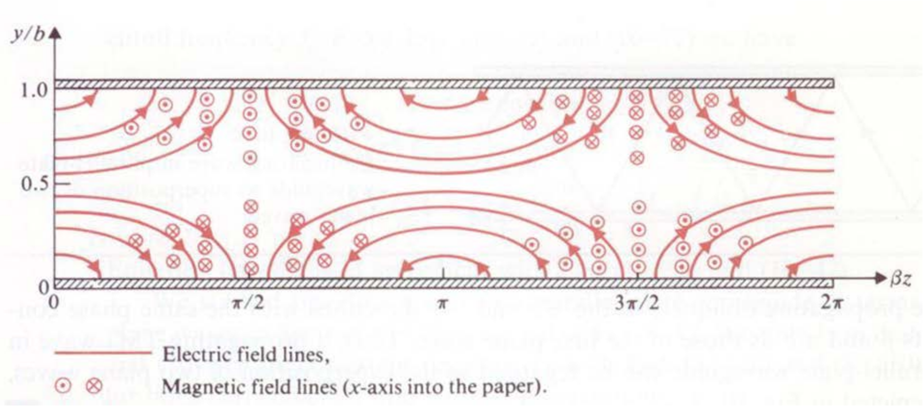
TM/TE Waves in Parallel-Plate Waveguides

TM$_1$ mode:

$$E_z^0 = A_n \sin\left(\frac{\pi}{b} y\right)$$

$$E_y^0 = -\frac{\gamma b}{\pi} A_n \cos\left(\frac{\pi}{b} y\right)$$

$$H_x^0 = \frac{j\omega\varepsilon b}{\pi} A_n \cos\left(\frac{\pi}{b} y\right)$$



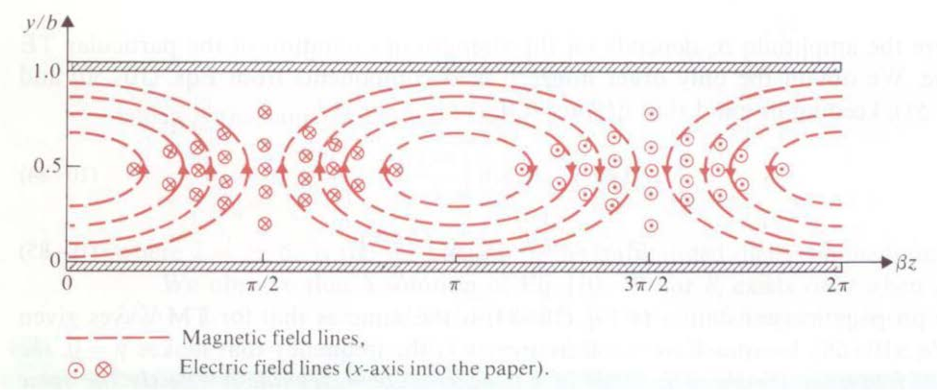
D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

TE$_1$ mode:

$$H_z^0 = B_n \cos\left(\frac{\pi}{b} y\right)$$

$$H_y^0 = \frac{\gamma b}{\pi} B_n \sin\left(\frac{\pi}{b} y\right)$$

$$E_x^0 = \frac{j\omega\mu b}{\pi} B_n \sin\left(\frac{\pi}{b} y\right)$$



Attenuation in Parallel-Plate Waveguides

Attenuation constant:

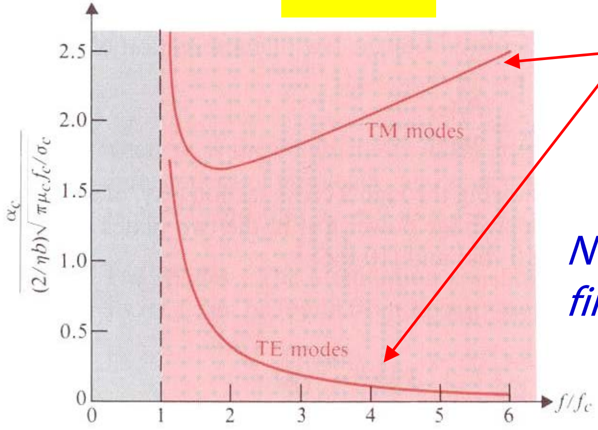
$$\rightarrow \alpha = \alpha_d + \alpha_c$$

Loss in the conducting wall

Loss in the dielectric

$$\rightarrow \alpha_c = \frac{P_L}{2P} = \frac{\textit{Time-averaged power lost in the conducting wall per unit length}}{\textit{Time-averaged power flow through the waveguide}}$$

→ *HW*



Why?

Normalized attenuation constant due to finite conductivity of the plates

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Energy-Transport Velocity (1)

Energy-transport velocity:

$$\rightarrow u_{en} = \frac{P_{z,av}}{W'_{av}} = \frac{\textit{Time-averaged propagated power}}{\textit{Time-averaged stored energy per unit guide length}}$$

$$\leftarrow P_{z,av} = \int_S \boldsymbol{P}_{av} \cdot d\mathbf{s} = \frac{1}{2}\mathrm{Re}\int_S \mathbf{E}\times\mathbf{H}^* \cdot d\mathbf{s}$$

$$\leftarrow W'_{av} = \int_S (w_{e,av} + w_{m,av})ds = \frac{1}{4}\mathrm{Re}\int_S (\mathbf{E}\cdot\varepsilon\mathbf{E}^* + \mathbf{H}\cdot\mu\mathbf{H}^*)ds$$

$$\rightarrow u_{en} = \frac{P_{z,av}}{W'_{av}} = u_g \quad \rightarrow \textit{Group velocity}$$

Derivation:

$$\nabla\times\mathbf{E} + \frac{\partial\mathbf{B}}{\partial t} = 0 \quad \rightarrow \nabla\times\mathbf{E} = -j\omega\mu\mathbf{H} \quad \rightarrow (\nabla_t + \nabla_z)\times\mathbf{E} = -j\omega\mu\mathbf{H}$$

$$\rightarrow \nabla_t\times\mathbf{E} - j\beta(\mathbf{a}_z\times\mathbf{E}) = -j\omega\mu\mathbf{H}$$

$$\nabla\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t} = \cancel{\mathbf{J}} \quad \rightarrow \nabla\times\mathbf{H} = j\omega\varepsilon\mathbf{E} \quad \rightarrow (\nabla_t + \nabla_z)\times\mathbf{H} = j\omega\varepsilon\mathbf{E}$$

$$\rightarrow \nabla_t\times\mathbf{H} - j\beta(\mathbf{a}_z\times\mathbf{H}) = j\omega\varepsilon\mathbf{E}$$

Energy-Transport Velocity (2)

First-order perturbation: $\rightarrow \delta\beta \;\; \rightarrow \delta\omega, \delta\mathbf{E}, \& \, \delta\mathbf{H}$

Recall:

$\rightarrow \nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B})$

$\rightarrow \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})=\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$

$$\rightarrow \boxed{\nabla_t\times\mathbf{E}-j\beta(\mathbf{a}_z\times\mathbf{E})=-j\omega\mu\mathbf{H}}$$

$$\rightarrow \nabla_t\times(\mathbf{E}+\delta\mathbf{E})-j(\beta+\delta\beta)[\mathbf{a}_z\times(\mathbf{E}+\delta\mathbf{E})]=-j(\omega+\delta\omega)\mu(\mathbf{H}+\delta\mathbf{H})$$

$$\rightarrow \nabla_t\times\delta\mathbf{E}-j\beta(\mathbf{a}_z\times\delta\mathbf{E})-j\delta\beta(\mathbf{a}_z\times\mathbf{E})=-j\omega\mu\delta\mathbf{H}-j\delta\omega\mu\mathbf{H} \qquad \leftarrow\cdot\mathbf{H}^*$$

$$\rightarrow \underline{\delta\mathbf{E}\cdot(\nabla_t\times\mathbf{H}^*)}+\nabla_t\cdot(\delta\mathbf{E}\times\mathbf{H}^*)+\underline{j\beta\delta\mathbf{E}\cdot(\mathbf{a}_z\times\mathbf{H}^*)}-j\delta\beta(\mathbf{E}\times\mathbf{H}^*)\cdot\mathbf{a}_z$$
$$=-j\omega\mathbf{H}^*\cdot\mu\delta\mathbf{H}-j\delta\omega\mathbf{H}^*\cdot\mu\mathbf{H}$$

$$\rightarrow \delta\mathbf{E}\cdot(\nabla\times\mathbf{H}^*)+\nabla_t\cdot(\delta\mathbf{E}\times\mathbf{H}^*)-j\delta\beta(\mathbf{E}\times\mathbf{H}^*)\cdot\mathbf{a}_z=-j\omega\mathbf{H}^*\cdot\mu\delta\mathbf{H}-j\delta\omega\mathbf{H}^*\cdot\mu\mathbf{H}$$

$$\rightarrow \boxed{\nabla_t\times\mathbf{H}-j\beta(\mathbf{a}_z\times\mathbf{H})=j\omega\varepsilon\mathbf{E}}$$

$$\rightarrow \delta\mathbf{H}\cdot(\nabla\times\mathbf{E}^*)+\nabla_t\cdot(\delta\mathbf{H}\times\mathbf{E}^*)+j\delta\beta(\mathbf{E}^*\times\mathbf{H})\cdot\mathbf{a}_z=j\omega\mathbf{E}^*\cdot\varepsilon\delta\mathbf{E}+j\omega\mathbf{E}^*\cdot\varepsilon\mathbf{E}$$

$$\rightarrow \delta\mathbf{E}\cdot(\nabla\times\bcancel{\mathbf{H}^*+j\omega\varepsilon\mathbf{E}^*})-\delta\mathbf{H}\cdot(\nabla\times\bcancel{\mathbf{E}^*-j\omega\mu\mathbf{H}^*})+\nabla_t\cdot(\delta\mathbf{E}\times\mathbf{H}^*-\delta\mathbf{H}\times\mathbf{E}^*)-j\delta\beta\cdot2\,\mathrm{Re}(\mathbf{E}\times\mathbf{H}^*)\cdot\mathbf{a}_z$$
$$=-j\delta\omega(\mathbf{E}^*\cdot\varepsilon\mathbf{E}+\mathbf{H}^*\cdot\mu\mathbf{H})$$

$$\rightarrow \boxed{\nabla_t\cdot\mathbf{F}-j\delta\beta\cdot4\,\mathrm{Re}(\mathbf{E}\times\mathbf{H}^*)\cdot\mathbf{a}_z=-j\delta\omega\cdot2\,\mathrm{Re}(\mathbf{E}\cdot\varepsilon\mathbf{E}^*+\mathbf{H}\cdot\mu\mathbf{H}^*)}$$

$$\leftarrow \mathbf{F}=\delta\mathbf{E}\times\mathbf{H}^*-\delta\mathbf{H}\times\mathbf{E}^*-\delta\mathbf{E}^*\times\mathbf{H}+\delta\mathbf{H}^*\times\mathbf{E}$$

Energy-Transport Velocity (3)

Continued:

$$\rightarrow \nabla_t \cdot \mathbf{F} - j\delta\beta \cdot 4\,\mathrm{Re}(\mathbf{E}\times\mathbf{H}^*)\cdot\mathbf{a}_z = -j\delta\omega \cdot 2\,\mathrm{Re}(\mathbf{E}\cdot\varepsilon\mathbf{E}^* + \mathbf{H}\cdot\mu\mathbf{H}^*)$$

$$\leftarrow \mathbf{F} = \delta\mathbf{E}\times\mathbf{H}^* - \delta\mathbf{H}\times\mathbf{E}^* - \delta\mathbf{E}^*\times\mathbf{H} + \delta\mathbf{H}^*\times\mathbf{E}$$

$$\rightarrow \int_S \nabla_t \cdot \mathbf{F}\,ds = \oint_C \mathbf{F}\cdot\mathbf{n}\,dl \rightarrow 0$$

$$\leftarrow \frac{1}{2}\mathrm{Re}\int_S \mathbf{E}\times\mathbf{H}^*\cdot\mathbf{a}_z\,ds = P_{z,av}$$

$$\leftarrow \frac{1}{4}\mathrm{Re}\int_S (\mathbf{E}\cdot\varepsilon\mathbf{E}^* + \mathbf{H}\cdot\mu\mathbf{H}^*)\,ds = W'_{av}$$

$$\rightarrow \delta\beta\, P_{z,av} = \delta\omega W'_{av}$$

$$\rightarrow \boxed{\frac{P_{z,av}}{W'_{av}} = \frac{\delta\omega}{\delta\beta}} \quad \rightarrow u_{en} = u_g$$

→ *Energy-transport velocity = Group velocity*