Electromagnetics:
Rectangular Waveguides
(10-4)

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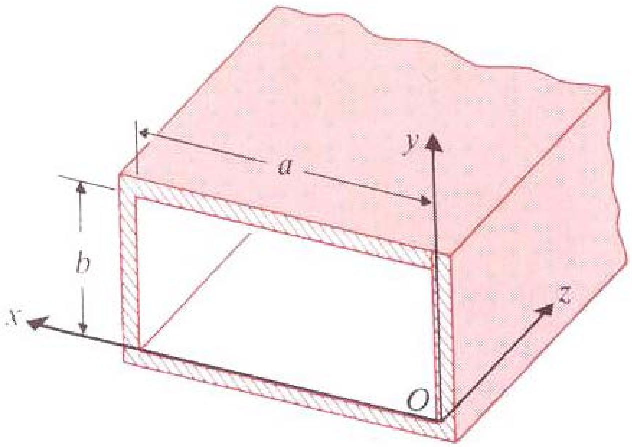
Rectangular Waveguides



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Very similar to parallel-plate waveguides

Just recall:

$$\gamma = \alpha + j\beta$$ $\rightarrow$ *Propagation constant*

$$\mathbf{E}(x, y, z; t) = \mathrm{Re}[\mathbf{E}^0(x, y)e^{(j\omega t - \gamma z)}]$$

$$\leftarrow h^2 = \gamma^2 + k^2$$

$$H_z = 0 \quad \rightarrow \nabla_{xy}^2 E_z^0 + (\gamma^2 + k^2)E_z^0 = 0 \quad \rightarrow \nabla_{xy}^2 E_z^0 + h^2 E_z^0 = 0$$

$$E_z = 0 \quad \rightarrow \nabla_{xy}^2 H_z^0 + (\gamma^2 + k^2)H_z^0 = 0 \quad \rightarrow \nabla_{xy}^2 H_z^0 + h^2 H_z^0 = 0$$

TM Waves in Rectangular Waveguides (1)

Characteristic equation:

$$H_z = 0 \quad \rightarrow \nabla_{xy}^2 E_z^0 + (\gamma^2 + k^2) E_z^0 = 0 \quad \rightarrow \nabla_{xy}^2 E_z^0 + h^2 E_z^0 = 0$$

$$\leftarrow h^2 = \gamma^2 + k^2$$

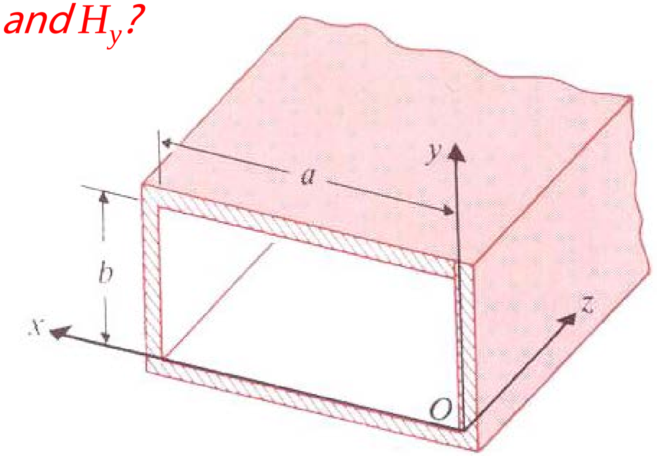
$$\rightarrow \frac{\partial^2 E_z^0}{\partial x^2} + \frac{\partial^2 E_z^0}{\partial y^2} + h^2 E_z^0 = 0$$

Boundary conditions:

What about E_x and E_y or H_x and H_y?

$$E_z^0 = 0 \qquad \leftarrow x = 0 \;\&\; x = a$$

$$E_z^0 = 0 \qquad \leftarrow y = 0 \;\&\; y = b$$



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Solution:

$$E_z^0 = E_0 \sin k_x x \sin k_y y$$

$$\rightarrow k_x^2 + k_y^2 = h^2$$

$$\rightarrow k_x = \frac{m\pi}{a}, \; k_y = \frac{n\pi}{b}$$ *→ m & n: Integer numbers*

TM Waves in Rectangular Waveguides (2)

With: $E_z^0 = E_0 \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$

Recall:

$$H_x^0 = -\frac{1}{h^2}\left(\gamma \cancel{\frac{\partial H_z^0}{\partial x}} - j\omega\varepsilon\frac{\partial E_z^0}{\partial y}\right)$$

$$H_y^0 = -\frac{1}{h^2}\left(\gamma \cancel{\frac{\partial H_z^0}{\partial y}} + j\omega\varepsilon\frac{\partial E_z^0}{\partial x}\right)$$

$$E_x^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial E_z^0}{\partial x} + j\omega\mu\cancel{\frac{\partial H_z^0}{\partial y}}\right)$$

$$E_y^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial E_z^0}{\partial y} - j\omega\mu\cancel{\frac{\partial H_z^0}{\partial x}}\right)$$

$$E_x^0 = -\frac{\gamma}{h^2}\left(\frac{m\pi}{a}\right)E_0\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$$E_y^0 = -\frac{\gamma}{h^2}\left(\frac{n\pi}{b}\right)E_0\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

$$H_x^0 = \frac{j\omega\varepsilon}{h^2}\left(\frac{n\pi}{b}\right)E_0\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

$$H_y^0 = -\frac{j\omega\varepsilon}{h^2}\left(\frac{m\pi}{a}\right)E_0\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

TM_{11} → Lowest-order mode

$$\leftarrow \gamma = j\beta = j\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Cutoff frequency and wavelength:

→ Neither m nor n can be zero!

$$\rightarrow f_{c,mn} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$\rightarrow \lambda_{c,mn} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

TE Waves in Rectangular Waveguides (1)

Characteristic equation:

$$E_z = 0 \quad \to \nabla_{xy}^2 H_z^0 + (\gamma^2 + k^2) H_z^0 = 0 \quad \to \nabla_{xy}^2 H_z^0 + h^2 H_z^0 = 0$$

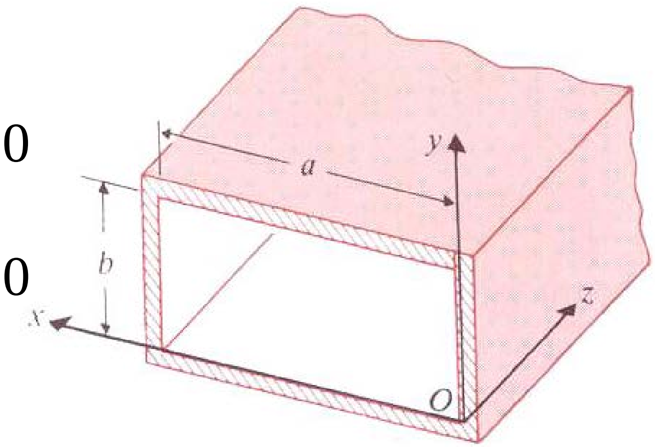
$$\leftarrow h^2 = \gamma^2 + k^2$$

$$\to \frac{\partial^2 H_z^0}{\partial x^2} + \frac{\partial^2 H_z^0}{\partial y^2} + h^2 H_z^0 = 0$$

Boundary conditions:

$$E_y^0 = 0 \quad \leftarrow x = 0 \;\&\; x = a \quad \to \frac{\partial H_z^0}{\partial x} = 0$$

$$E_x^0 = 0 \quad \leftarrow y = 0 \;\&\; y = b \quad \to \frac{\partial H_z^0}{\partial y} = 0$$



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Solution:

$$H_z^0 = H_0 \cos k_x x \cos k_y y$$

$$\leftarrow k_x^2 + k_y^2 = h^2$$

$$\to k_x = \frac{m\pi}{a}, \; k_y = \frac{n\pi}{b}$$ *$\to$ m & n: Integer numbers*

TE Waves in Rectangular Waveguides (2)

With: $H_z^0 = H_0 \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$

Recall:

$$H_x^0 = -\frac{1}{h^2}\left(\gamma\frac{\partial H_z^0}{\partial x} - \cancel{j\omega\varepsilon\frac{\partial E_z^0}{\partial y}}\right)$$

$$H_y^0 = -\frac{1}{h^2}\left(\gamma\frac{\partial H_z^0}{\partial y} + \cancel{j\omega\varepsilon\frac{\partial E_z^0}{\partial x}}\right)$$

$$E_x^0 = -\frac{1}{h^2}\left(\cancel{\gamma\frac{\partial E_z^0}{\partial x}} + j\omega\mu\frac{\partial H_z^0}{\partial y}\right)$$

$$E_y^0 = -\frac{1}{h^2}\left(\cancel{\gamma\frac{\partial E_z^0}{\partial y}} - j\omega\mu\frac{\partial H_z^0}{\partial x}\right)$$

$$E_x^0 = \frac{j\omega\mu}{h^2}\left(\frac{n\pi}{b}\right)H_0\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$$E_y^0 = -\frac{j\omega\mu}{h^2}\left(\frac{m\pi}{a}\right)H_0\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

$$H_x^0 = \frac{\gamma}{h^2}\left(\frac{m\pi}{a}\right)H_0\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

$$H_y^0 = \frac{\gamma}{h^2}\left(\frac{n\pi}{b}\right)H_0\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

TE_{10} or TE_{01} → Lowest-order mode

$$\leftarrow \gamma = j\beta = j\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

Cutoff frequency or wavelength:

→ Either m or n can be zero but not both!

$$f_{c,mn} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$$

$$\lambda_{c,mn} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}}$$

TM/TE Waves in Rectangular Waveguides (1)

TM modes: $\rightarrow$ *m ≠ 0 and n ≠ 0*

$$E_x^0 = -\frac{\gamma}{h^2}\left(\frac{m\pi}{a}\right)E_0\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$$E_y^0 = -\frac{\gamma}{h^2}\left(\frac{n\pi}{b}\right)E_0\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

$$H_x^0 = \frac{j\omega\varepsilon}{h^2}\left(\frac{n\pi}{b}\right)E_0\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

$$H_y^0 = -\frac{j\omega\varepsilon}{h^2}\left(\frac{m\pi}{a}\right)E_0\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

Lowest-order mode: TM_{11}

$$f_{c,11} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}$$

$$\lambda_{c,mn} = 2\Big/\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2}$$

TE modes: $\rightarrow$ *m ≠ 0 or n ≠ 0*

$$E_x^0 = \frac{j\omega\mu}{h^2}\left(\frac{n\pi}{b}\right)H_0\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

$$E_y^0 = -\frac{j\omega\mu}{h^2}\left(\frac{m\pi}{a}\right)H_0\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

$$H_x^0 = \frac{\gamma}{h^2}\left(\frac{m\pi}{a}\right)H_0\sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)$$

$$H_y^0 = \frac{\gamma}{h^2}\left(\frac{n\pi}{b}\right)H_0\cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)$$

Lowest-order mode: TE_{10} *(a > b)*

$$f_{c,10} = \frac{1}{2\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{0}{b}\right)^2} = \frac{1}{2a\sqrt{\mu\varepsilon}}$$

$\lambda_{c,mn} = 2a$ $\rightarrow$ *Dominant mode*

What if: $f_{c.10} < f < f_{c,01}$ $\rightarrow$ *Single-mode excitation!*

TM/TE Waves in Rectangular Waveguides (2)

TM$_{11}$ mode:

$$E_x^0 = -\frac{\gamma}{h^2}\left(\frac{\pi}{a}\right)E_0 \cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{b}y\right)$$

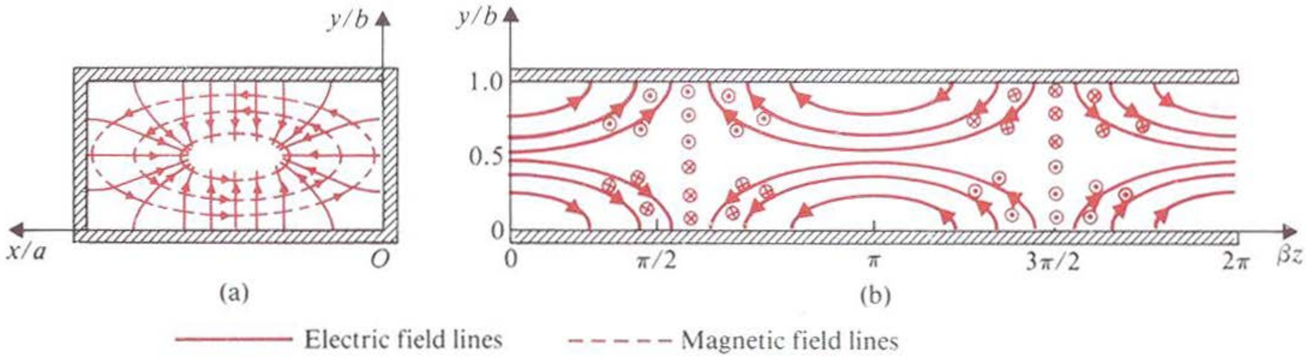
$$E_y^0 = -\frac{\gamma}{h^2}\left(\frac{\pi}{b}\right)E_0 \sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{b}y\right)$$

$$E_z^0 = E_0 \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{b}y\right)$$

$$H_x^0 = \frac{j\omega\varepsilon}{h^2}\left(\frac{\pi}{b}\right)E_0 \sin\left(\frac{\pi}{a}x\right)\cos\left(\frac{\pi}{b}y\right)$$

$$H_y^0 = -\frac{j\omega\varepsilon}{h^2}\left(\frac{\pi}{a}\right)E_0 \cos\left(\frac{\pi}{a}x\right)\sin\left(\frac{\pi}{b}y\right)$$

$$H_z^0 = 0$$



TE$_{10}$ mode:

$$E_x^0 = 0$$

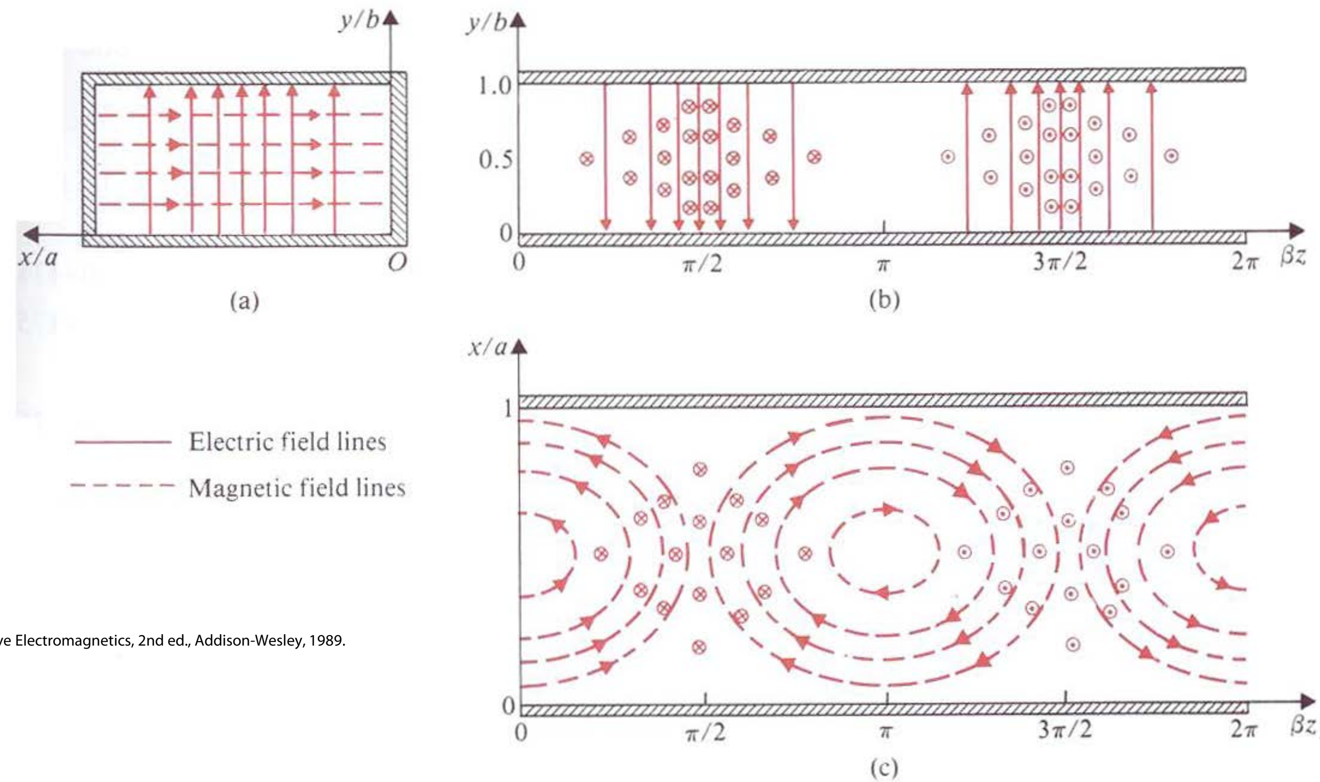
$$E_y^0 = -\frac{j\omega\mu}{h^2}\left(\frac{\pi}{a}\right)H_0 \sin\left(\frac{\pi}{a}x\right)$$

$$E_z^0 = 0$$

$$H_x^0 = \frac{\gamma}{h^2}\left(\frac{\pi}{a}\right)H_0 \sin\left(\frac{\pi}{a}x\right)$$

$$H_y^0 = 0$$

$$H_z^0 = H_0 \cos\left(\frac{\pi}{a}x\right)$$



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Attenuation in Rectangular Waveguides

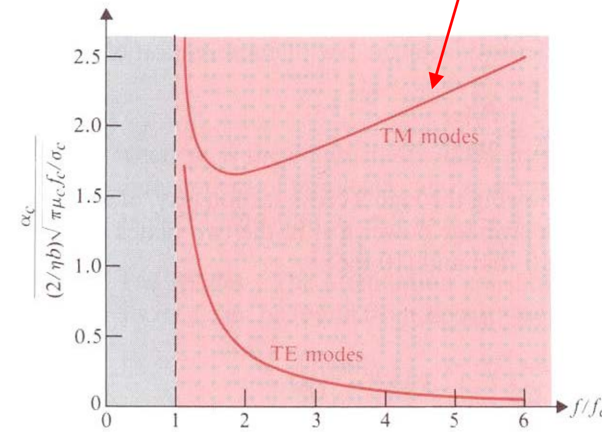
Attenuation constant:

$$\rightarrow \alpha = \alpha_d + \alpha_c$$

Loss in the conducting wall

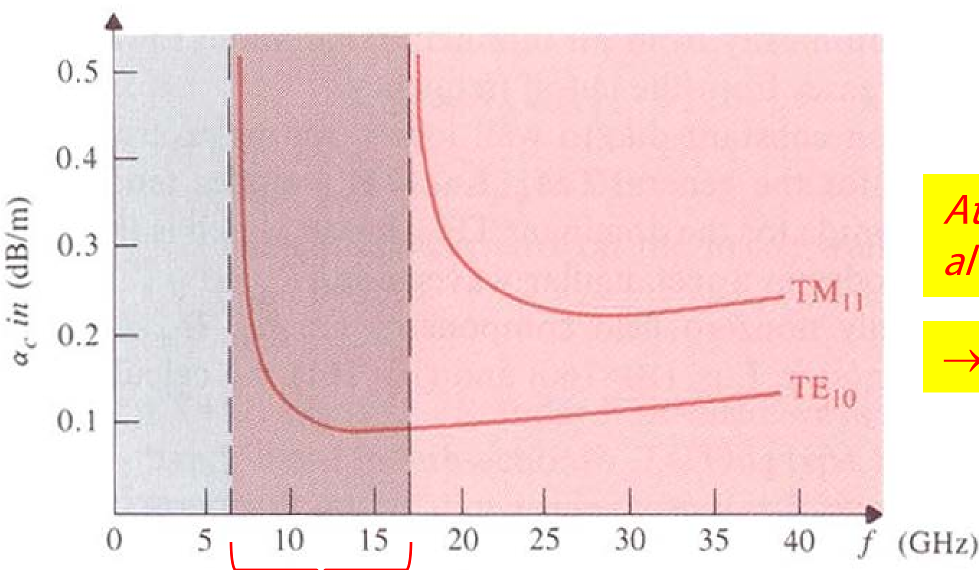
Loss in the dielectric

Recall: $\rightarrow \eta_c = \sqrt{\dfrac{\mu}{\varepsilon_c}} \cong \dfrac{1+j}{\sqrt{2}}\sqrt{\dfrac{\omega\mu}{\sigma}}$



PP waveguide

Attenuation due to wall losses:



Single-mode operation

Attenuation of TE_{10} mode is always lower than that of TM_{11}.

→ HW