Electromagnetics:
Circular Waveguides
(10-5)

Yoonchan Jeong

School of Electrical Engineering, Seoul National University

Tel: +82 (0)2 880 1623, Fax: +82 (0)2 873 9953

Email: yoonchan@snu.ac.kr

Circular Waveguides

Just recall:

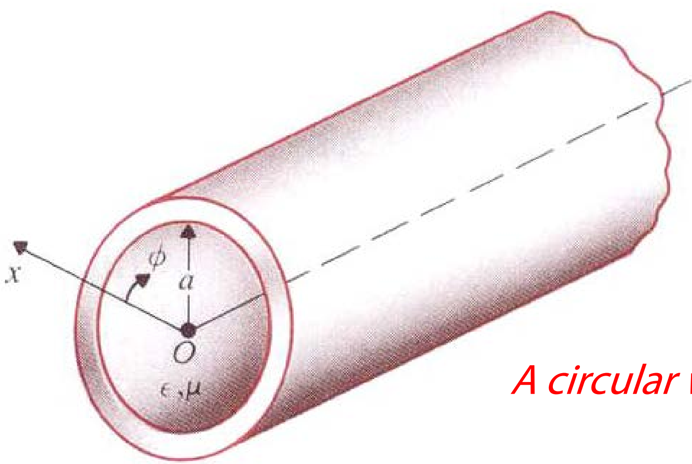
$$\gamma = \alpha + j\beta \qquad \rightarrow \textit{Propagation constant}$$

$$\mathbf{E}(x, y, z; t) = \mathrm{Re}[\mathbf{E}^0(x, y)e^{(j\omega t - \gamma z)}]$$

TM: $H_z = 0 \quad \rightarrow \nabla_{xy}^2 E_z^0 + (\gamma^2 + k^2)E_z^0 = 0 \quad \rightarrow \nabla_{xy}^2 E_z^0 + h^2 E_z^0 = 0$

TE: $E_z = 0 \quad \rightarrow \nabla_{xy}^2 H_z^0 + (\gamma^2 + k^2)H_z^0 = 0 \quad \rightarrow \nabla_{xy}^2 H_z^0 + h^2 H_z^0 = 0$

$$\leftarrow h^2 = \gamma^2 + k^2$$



A circular waveguide

→ *Choice of the coordinate system:*

$$\rightarrow \nabla_{r\phi}^2 E_z^0 + h^2 E_z^0 = 0$$

$$\rightarrow \nabla_{r\phi}^2 H_z^0 + h^2 H_z^0 = 0$$

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Bessel's Differential Equation

Consider:

$$\nabla_{r\phi}^2 E_z^0 + h^2 E_z^0 = 0$$

$$\rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z^0}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 E_z^0}{\partial \phi^2} + h^2 E_z^0 = 0$$

Suppose: $E_z^0 = R(r)\Phi(\phi)$

$$\rightarrow \frac{d^2\Phi(\phi)}{d\phi^2} + n^2\Phi(\phi) = 0$$

$$\rightarrow \frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + (h^2 - \frac{n^2}{r^2})R(r) = 0$$

$\rightarrow$ *Bessel's differential equation*

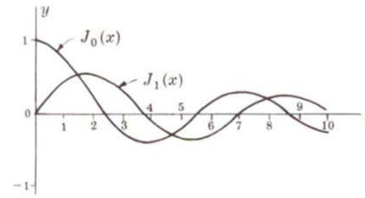


Fig. 24-1

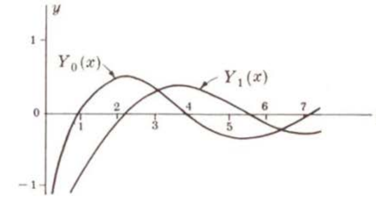


Fig. 24-2

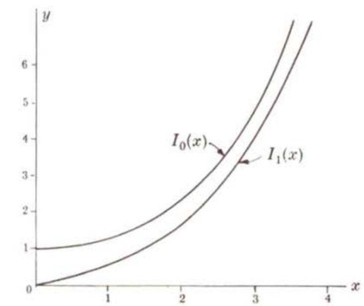


Fig. 24-3

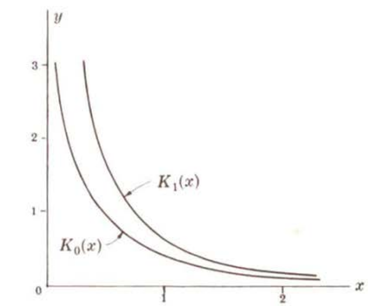


Fig. 24-4

M. R. Spiegel, Mathematical Handbook of Formulas and Tables, McGraw-Hill, 1990.

$$R(r) = \begin{cases} CJ_n(hr) + DN_n(hr) & for \;\; h^2 > 0 \\ CI_n(hr) + DK_n(hr) & for \;\; h^2 < 0 \end{cases}$$

cf.

$$H_n^{(1)}(hr) = J_n(hr) + jN_n(hr)$$

$$H_n^{(2)}(hr) = J_n(hr) - jN_n(hr)$$

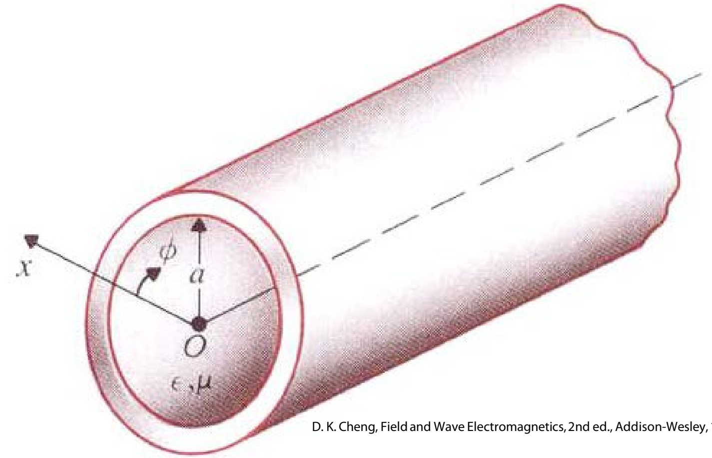
TM Waves in Circular Waveguides (1)

Let: $E_z(r,\phi,z) = E_z^0(r,\phi)e^{-\gamma z}$

Recall:

$$R(r) = \begin{cases} CJ_n(hr) + DN_n(hr) & \text{for } h^2 > 0 \\ CI_n(hr) + DK_n(hr) & \text{for } h^2 < 0 \end{cases}$$

$$\rightarrow E_z^0 = C_n J_n(hr)\cos n\phi$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

A circular waveguide with a conducting wall

Also recall: $\mathbf{E}_T^0 = -\frac{\gamma}{h^2}\nabla_T E_z^0$ $\rightarrow \nabla_T E_z^0 = \left(\mathbf{a}_r \frac{\partial}{\partial r} + \mathbf{a}_\phi \frac{\partial}{r\partial\phi}\right)E_z^0$

Solution:

$$E_r^0 = -\frac{j\beta}{h}C_n J_n'(hr)\cos n\phi$$

$$E_\phi^0 = \frac{j\beta n}{h^2 r}C_n J_n(hr)\sin n\phi$$

$$H_r^0 = -\frac{j\omega\varepsilon n}{h^2 r}C_n J_n(hr)\sin n\phi$$

$$H_\phi^0 = -\frac{j\omega\varepsilon}{h}C_n J_n'(hr)\cos n\phi$$

$$\rightarrow \mathbf{H} = \frac{1}{Z_{TM}}\mathbf{a}_z \times \mathbf{E}$$

$$\leftarrow \gamma = j\beta$$

TM Waves in Circular Waveguides (2)

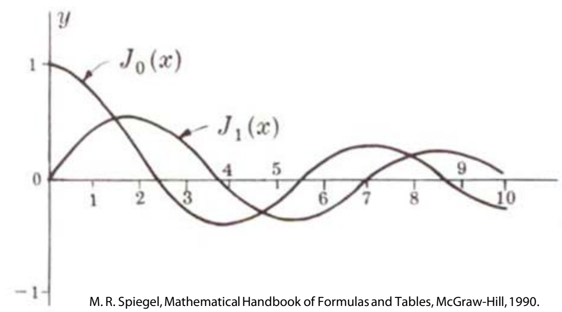
Solution:

$$E_r^0 = -\frac{j\beta}{h} C_n J_n'(hr)\cos n\phi \qquad H_r^0 = -\frac{j\omega\varepsilon n}{h^2 r} C_n J_n(hr)\sin n\phi$$

$$E_\phi^0 = \frac{j\beta n}{h^2 r} C_n J_n(hr)\sin n\phi \qquad H_\phi^0 = -\frac{j\omega\varepsilon}{h} C_n J_n'(hr)\cos n\phi$$

$$E_z^0 = C_n J_n(hr)\cos n\phi \qquad H_z^0 = 0$$

Recall:



M. R. Spiegel, Mathematical Handbook of Formulas and Tables, McGraw-Hill, 1990.

Boundary condition:

TABLE 10–2
Zeros of $J_n(x)$, x_{np}

p \ n	n = 0	n = 1	n = 2
1	2.405	3.832	5.136
2	5.520	7.016	8.417

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$$E_z^0 = E_\phi^0 = 0 \quad at \quad r = a$$

$$\rightarrow J_n(ha) = 0 \quad at \quad r = a$$

$$\rightarrow ha = x_{nm}$$

where x_{nm} represents the m^{th} zero of $J_n(x)$.

For TM_{01} mode:

$$h_{TM_{01}} = \frac{2.405}{a} \qquad \rightarrow f_{c,TM_{01}} = \frac{h_{TM_{01}}}{2\pi\sqrt{\mu\varepsilon}} = \frac{0.383}{a\sqrt{\mu\varepsilon}}$$

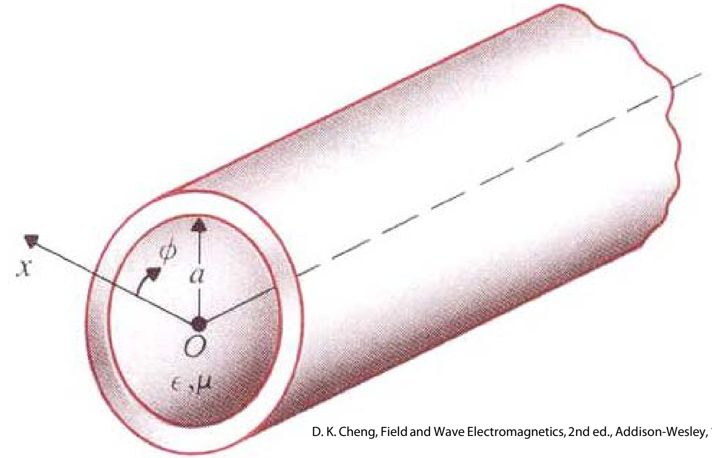
TE Waves in Circular Waveguides (1)

Let: $H_z(r,\phi,z) = H_z^0(r,\phi)e^{-\gamma z}$

Recall again:

$$R(r) = \begin{cases} CJ_n(hr) + DN_n(hr) & for \quad h^2 > 0 \\ CI_n(hr) + DK_n(hr) & for \quad h^2 < 0 \end{cases}$$

$$\rightarrow H_z^0 = C_n' J_n(hr)\cos n\phi$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

A circular waveguide with a conducting wall

Also recall: $\mathbf{H}_T^0 = -\frac{\gamma}{h^2}\nabla_T H_z^0$ & $\mathbf{E} = -Z_{TE}(\mathbf{a}_z \times \mathbf{H})$

Solution:

$$H_r^0 = -\frac{j\beta}{h} C_n' J_n'(hr)\cos n\phi$$

$$H_\phi^0 = \frac{j\beta n}{h^2 r} C_n' J_n(hr)\sin n\phi$$

$$E_r^0 = \frac{j\omega\mu n}{h^2 r} C_n' J_n(hr)\sin n\phi$$

$$E_\phi^0 = \frac{j\omega\mu}{h} C_n' J_n'(hr)\cos n\phi$$

$\leftarrow \gamma = j\beta$

TE Waves in Circular Waveguides (2)

Solution:

$$H_r^0 = -\frac{j\beta}{h} C_n' J_n'(hr)\cos n\phi$$
$$H_\phi^0 = \frac{j\beta n}{h^2 r} C_n' J_n(hr)\sin n\phi$$
$$H_z^0 = C_n' J_n(hr)\cos n\phi$$

$$E_r^0 = \frac{j\omega\mu n}{h^2 r} C_n' J_n(hr)\sin n\phi$$
$$E_\phi^0 = \frac{j\omega\mu}{h} C_n' J_n'(hr)\cos n\phi$$
$$E_z^0 = 0$$

Boundary condition:

TABLE 10–3
Zeros of $J_n'(x)$, x_{np}'

p \ n	n = 0	n = 1	n = 2
1	3.832	1.841	3.054
2	7.016	5.331	6.706

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

$$E_z^0 = E_\phi^0 = 0 \quad at \quad r = a$$

$$\rightarrow J_n'(ha) = 0 \quad at \quad r = a \quad \rightarrow J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$$

$$\rightarrow ha = x_{nm}'$$

where x_{nm}' represents the m^{th} zero of $J_n'(x)$.

For TE_{11} mode:

$$h_{TE_{11}} = \frac{1.841}{a} \qquad \rightarrow f_{c,TE_{11}} = \frac{h_{TE_{11}}}{2\pi\sqrt{\mu\varepsilon}} = \frac{0.293}{a\sqrt{\mu\varepsilon}}$$

→ Dominant mode!

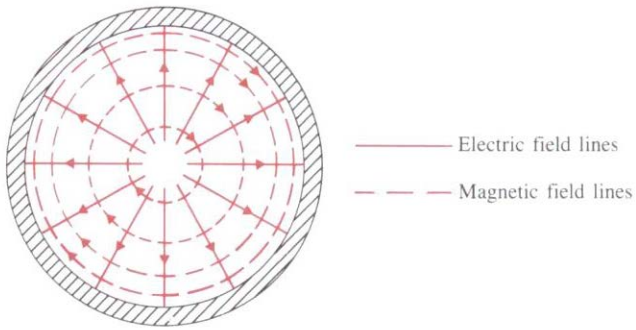
TM/TE Waves in Circular Waveguides

TM_{01} mode:

$$E_r^0 = -\frac{j\beta}{h} C_0 J_0'(hr) \qquad H_r^0 = 0$$

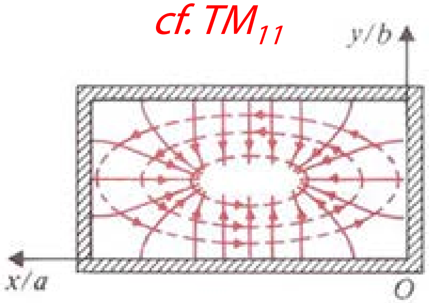
$$E_\phi^0 = 0 \qquad H_\phi^0 = -\frac{j\omega\varepsilon}{h} C_0 J_0'(hr)$$

$$E_z^0 = C_0 J_0(hr) \qquad H_z^0 = 0$$



$$h_{TM_{01}} = \frac{2.405}{a} \rightarrow f_{c,TM_{01}} = \frac{0.383}{a\sqrt{\mu\varepsilon}}$$

cf. TM_{11}

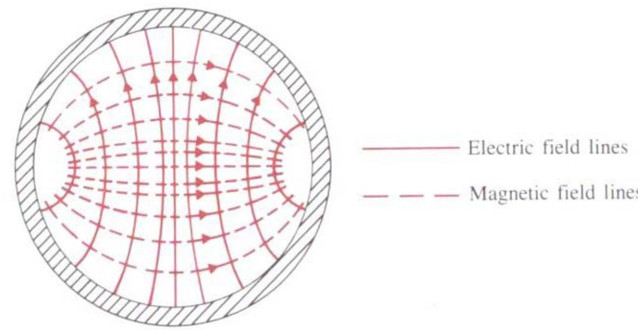


TE_{11} mode:

$$E_r^0 = \frac{j\omega\mu}{h^2 r} C_1' J_1(hr)\sin\phi \qquad H_r^0 = -\frac{j\beta}{h} C_1' J_1'(hr)\cos\phi$$

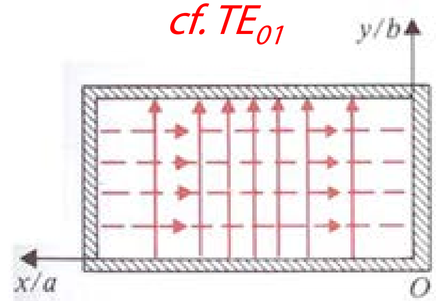
$$E_\phi^0 = \frac{j\omega\mu}{h} C_1' J_1'(hr)\cos\phi \qquad H_\phi^0 = \frac{j\beta}{h^2 r} C_1' J_1(hr)\sin\phi$$

$$E_z^0 = 0 \qquad H_z^0 = C_1' J_1(hr)\cos\phi$$



$$h_{TE_{11}} = \frac{1.841}{a} \rightarrow f_{c,TE_{11}} = \frac{0.293}{a\sqrt{\mu\varepsilon}}$$

cf. TE_{01}



→ Dominant mode!