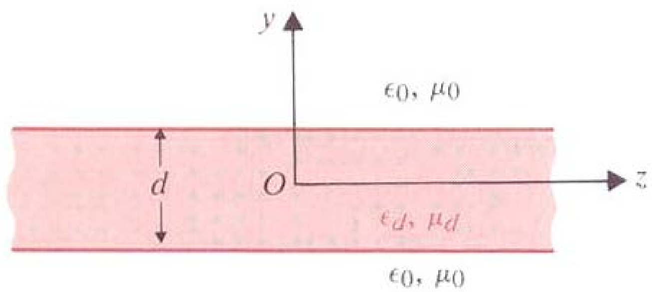
Electromagnetics:
Dielectric Waveguides
(10-6)

Yoonchan Jeong
School of Electrical Engineering, Seoul National University
Tel: +82 (0)2 880 1623, Fax: +82 (0)2 873 9953
Email: yoonchan@snu.ac.kr

Dielectric Waveguides



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

A dielectric slab waveguide: Lossless

$\rightarrow$ *No variation in the x-direction* $\quad (\frac{\partial}{\partial x} \rightarrow 0)$

Just recall:

$\gamma = j\beta$ $\quad \rightarrow$ *Propagation constant*

$$\mathbf{E}(y, z; t) = \mathrm{Re}[\mathbf{E}^0(y) e^{(j\omega t - \gamma z)}]$$

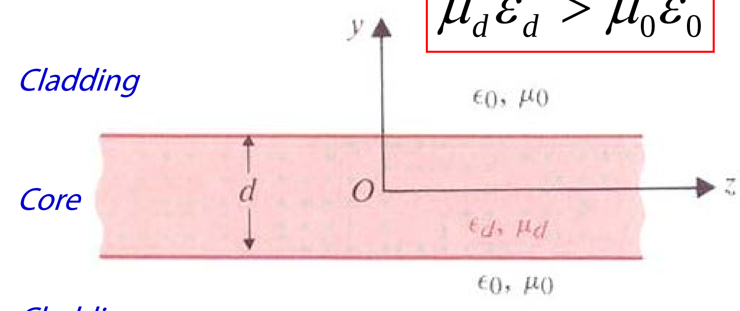
TM mode: $H_z = 0 \quad \rightarrow \nabla_y^2 E_z^0 + (\gamma^2 + k^2) E_z^0 = 0 \quad \rightarrow \nabla_y^2 E_z^0 + h^2 E_z^0 = 0$

TE mode: $E_z = 0 \quad \rightarrow \nabla_y^2 H_z^0 + (\gamma^2 + k^2) H_z^0 = 0 \quad \rightarrow \nabla_y^2 H_z^0 + h^2 H_z^0 = 0$

$$\leftarrow h^2 = \gamma^2 + k^2$$

TM Modes along a Dielectric Slab

$$\mu_d \varepsilon_d > \mu_0 \varepsilon_0$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Characteristic equation:

$$H_z = 0 \;\rightarrow \frac{\partial^2 E_z^0}{\partial y^2} + h^2 E_z^0 = 0$$

$$\leftarrow h^2 = \gamma^2 + k^2$$

$$\rightarrow \gamma = j\beta \;\rightarrow h^2 = k^2 - \beta^2 \qquad \leftarrow k^2 = \omega^2 \mu\varepsilon = \omega^2 \begin{bmatrix} \mu_d \varepsilon_d, & |y| \le d/2 \\ \mu_0 \varepsilon_0, & |y| > d/2 \end{bmatrix}$$

Boundary condition:

Tangential components of E and H fields: Continuous across the boundary

$\rightarrow \beta$ *must be the same in the core and cladding regions.*

$\rightarrow h^2 \ge 0$ *in the core: Propagating modes*

$\rightarrow h^2 < 0$ *in the cladding: Evanescent modes*

Solution:

$$E_z^0(y) = E_o \sin k_y y + E_e \cos k_y y, \;\; |y| \le d/2 \qquad \leftarrow k_y^2 = \omega^2 \mu_d \varepsilon_d - \beta^2 = h_d^2$$

$$E_z^0(y) = \begin{cases} C_u e^{-\alpha(y-d/2)}, & y > d/2 \\ C_l e^{\alpha(y+d/2)}, & y < -d/2 \end{cases} \qquad \leftarrow \alpha^2 = \beta^2 - \omega^2 \mu_0 \varepsilon_0 = -h_0^2$$

Odd TM Modes (1)

Odd-mode solution:

$$E_z^0(y) = E_o \sin k_y y + E_e \cos k_y y, \quad |y| \le d/2$$

$$E_z^0(y) = \begin{cases} C_u e^{-\alpha(y-d/2)}, & y > d/2 \\ C_l e^{\alpha(y+d/2)}, & y < -d/2 \end{cases}$$

In the core:

$$E_z^0(y) = E_o \sin k_y y$$

$$E_y^0(y) = -\frac{j\beta}{k_y} E_o \cos k_y y$$

$$H_x^0(y) = \frac{j\omega\varepsilon_d}{k_y} E_o \cos k_y y$$

Recall:

$$H_x^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial H_z^0}{\partial x} - j\omega\varepsilon \frac{\partial E_z^0}{\partial y}\right)$$

$$H_y^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial H_z^0}{\partial y} + j\omega\varepsilon \frac{\partial E_z^0}{\partial x}\right)$$

$$E_x^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial E_z^0}{\partial x} + j\omega\mu \frac{\partial H_z^0}{\partial y}\right)$$

$$E_y^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial E_z^0}{\partial y} - j\omega\mu \frac{\partial H_z^0}{\partial x}\right)$$

In the cladding:

Upper cladding:

$$E_z^0(y) = C_u e^{-\alpha(y-d/2)}$$

$$E_y^0(y) = -\frac{j\beta}{\alpha} C_u e^{-\alpha(y-d/2)}$$

$$H_x^0(y) = \frac{j\omega\varepsilon_0}{\alpha} C_u e^{-\alpha(y-d/2)}$$

Lower cladding:

$$E_z^0(y) = C_l e^{\alpha(y+d/2)}$$

$$E_y^0(y) = \frac{j\beta}{\alpha} C_l e^{\alpha(y+d/2)}$$

$$H_x^0(y) = -\frac{j\omega\varepsilon_0}{\alpha} C_l e^{\alpha(y+d/2)}$$

Odd TM Modes (2)

E_z^0 & H_x^0 *must be continuous across the boundary!*

Upper cl.:	$E_z^0(y) = C_u e^{-\alpha(y-d/2)}$	$E_y^0(y) = -\dfrac{j\beta}{\alpha} C_u e^{-\alpha(y-d/2)}$	$H_x^0(y) = \dfrac{j\omega\varepsilon_0}{\alpha} C_u e^{-\alpha(y-d/2)}$
Core:	$E_z^0(y) = E_o \sin k_y y$	$E_y^0(y) = -\dfrac{j\beta}{k_y} E_o \cos k_y y$	$H_x^0(y) = \dfrac{j\omega\varepsilon_d}{k_y} E_o \cos k_y y$
Lower cl.:	$E_z^0(y) = C_l e^{\alpha(y+d/2)}$	$E_y^0(y) = \dfrac{j\beta}{\alpha} C_l e^{\alpha(y+d/2)}$	$H_x^0(y) = -\dfrac{j\omega\varepsilon_0}{\alpha} C_l e^{\alpha(y+d/2)}$

$$\rightarrow C_u = E_o \sin\frac{k_y d}{2}$$

$$\rightarrow C_l = -E_o \sin\frac{k_y d}{2}$$

$$\rightarrow \frac{j\omega\varepsilon_0}{\alpha}\left(E_o \sin\frac{k_y d}{2}\right) = \frac{j\omega\varepsilon_d}{k_y} E_o \cos\frac{k_y d}{2}$$

$$\rightarrow \alpha = \frac{\varepsilon_0}{\varepsilon_d} k_y \tan\frac{k_y d}{2}$$

Recall: $\alpha^2 = \beta^2 - \omega^2\mu_0\varepsilon_0,\ \beta^2 = \omega^2\mu_d\varepsilon_d - k_y^2$

$$\rightarrow \alpha = [\omega^2(\mu_d\varepsilon_d - \mu_0\varepsilon_0) - k_y^2]^{1/2}$$

Transcendental equation:

$$[\omega^2(\mu_d\varepsilon_d - \mu_0\varepsilon_0) - k_y^2]^{1/2} = \frac{\varepsilon_0}{\varepsilon_d} k_y \tan\frac{k_y d}{2}$$

To be solved numerically

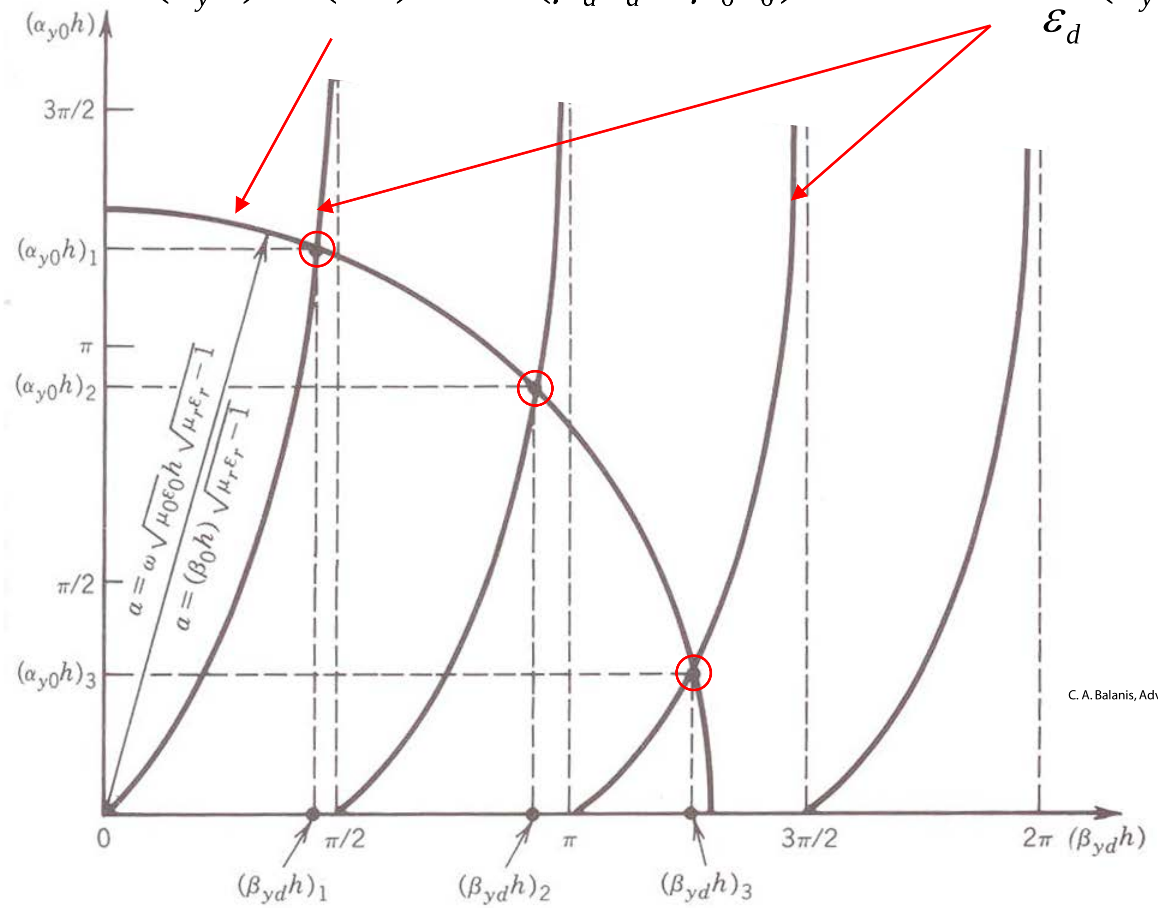
Odd TM Modes (3)

Graphical method to determine: k_y

Transcendental equation:

$$[\omega^2(\mu_d\varepsilon_d-\mu_0\varepsilon_0)-k_y^2]^{1/2}=\frac{\varepsilon_0}{\varepsilon_d}k_y\tan\frac{k_y d}{2}$$

$$(k_y d)^2+(\alpha d)^2=\omega^2(\mu_d\varepsilon_d-\mu_0\varepsilon_0)d^2 \qquad \alpha d=\frac{\varepsilon_0}{\varepsilon_d}(k_y d)\tan\frac{k_y d}{2}$$



C. A. Balanis, Advanced Engineering Electromagnetics, John-Wiley, 1989.

Note:

$$\beta_{yd}h\equiv\frac{k_y d}{2}$$

Even TM Modes

E_z^0 & H_x^0 *must be continuous across the boundary!*

Upper cl.:

$$E_z^0(y) = C_u e^{-\alpha(y-d/2)} \qquad E_y^0(y) = -\frac{j\beta}{\alpha} C_u e^{-\alpha(y-d/2)} \qquad H_x^0(y) = \frac{j\omega\varepsilon_0}{\alpha} C_u e^{-\alpha(y-d/2)}$$

Core:

$$E_z^0(y) = E_e \cos k_y y \qquad E_y^0(y) = \frac{j\beta}{k_y} E_e \sin k_y y \qquad H_x^0(y) = -\frac{j\omega\varepsilon_d}{k_y} E_e \sin k_y y$$

Lower cl.:

$$E_z^0(y) = C_l e^{\alpha(y+d/2)} \qquad E_y^0(y) = \frac{j\beta}{\alpha} C_l e^{\alpha(y+d/2)} \qquad H_x^0(y) = -\frac{j\omega\varepsilon_0}{\alpha} C_l e^{\alpha(y+d/2)}$$

$$\rightarrow C_u = E_e \cos\frac{k_y d}{2}$$

$$\rightarrow C_l = E_e \cos\frac{k_y d}{2}$$

$$\rightarrow \frac{j\omega\varepsilon_0}{\alpha}\left(E_e \cos\frac{k_y d}{2}\right) = -\frac{j\omega\varepsilon_d}{k_y} E_e \sin\frac{k_y d}{2}$$

$$\rightarrow \alpha = -\frac{\varepsilon_0}{\varepsilon_d} k_y \cot\frac{k_y d}{2}$$

Recall: $\alpha^2 = \beta^2 - \omega^2\mu_0\varepsilon_0,\ \beta^2 = \omega^2\mu_d\varepsilon_d - k_y^2$

$$\rightarrow \alpha = [\omega^2(\mu_d\varepsilon_d - \mu_0\varepsilon_0) - k_y^2]^{1/2}$$

Transcendental equation:

$$[\omega^2(\mu_d\varepsilon_d - \mu_0\varepsilon_0) - k_y^2]^{1/2} = -\frac{\varepsilon_0}{\varepsilon_d} k_y \cot\frac{k_y d}{2}$$

To be solved numerically

Cutoff Frequencies of TM modes

Lowest-order mode?

Odd TM_1

Guiding condition:

$$k_y^2 = \omega^2 \mu_d \varepsilon_d - \beta^2 \quad \rightarrow \beta^2 < \omega^2 \mu_d \varepsilon_d$$

$$\alpha^2 = \beta^2 - \omega^2 \mu_0 \varepsilon_0 \quad \rightarrow \beta^2 > \omega^2 \mu_0 \varepsilon_0$$

$$\rightarrow \omega^2 \mu_0 \varepsilon_0 < \beta^2 < \omega^2 \mu_d \varepsilon_d$$

Cutoff condition: $\alpha = 0 \quad \rightarrow k_y = \omega_{co} \sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0}$

Odd TM modes:

$$\alpha = [\omega^2 (\mu_d \varepsilon_d - \mu_0 \varepsilon_0) - k_y^2]^{1/2}$$

$$= \frac{\varepsilon_0}{\varepsilon_d} k_y \tan \frac{k_y d}{2} = 0$$

$$\rightarrow \pi f_{co} d \sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0} = (n-1)\pi$$

$$\rightarrow f_{co} = \frac{(n-1)}{d\sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0}}$$

Even TM modes:

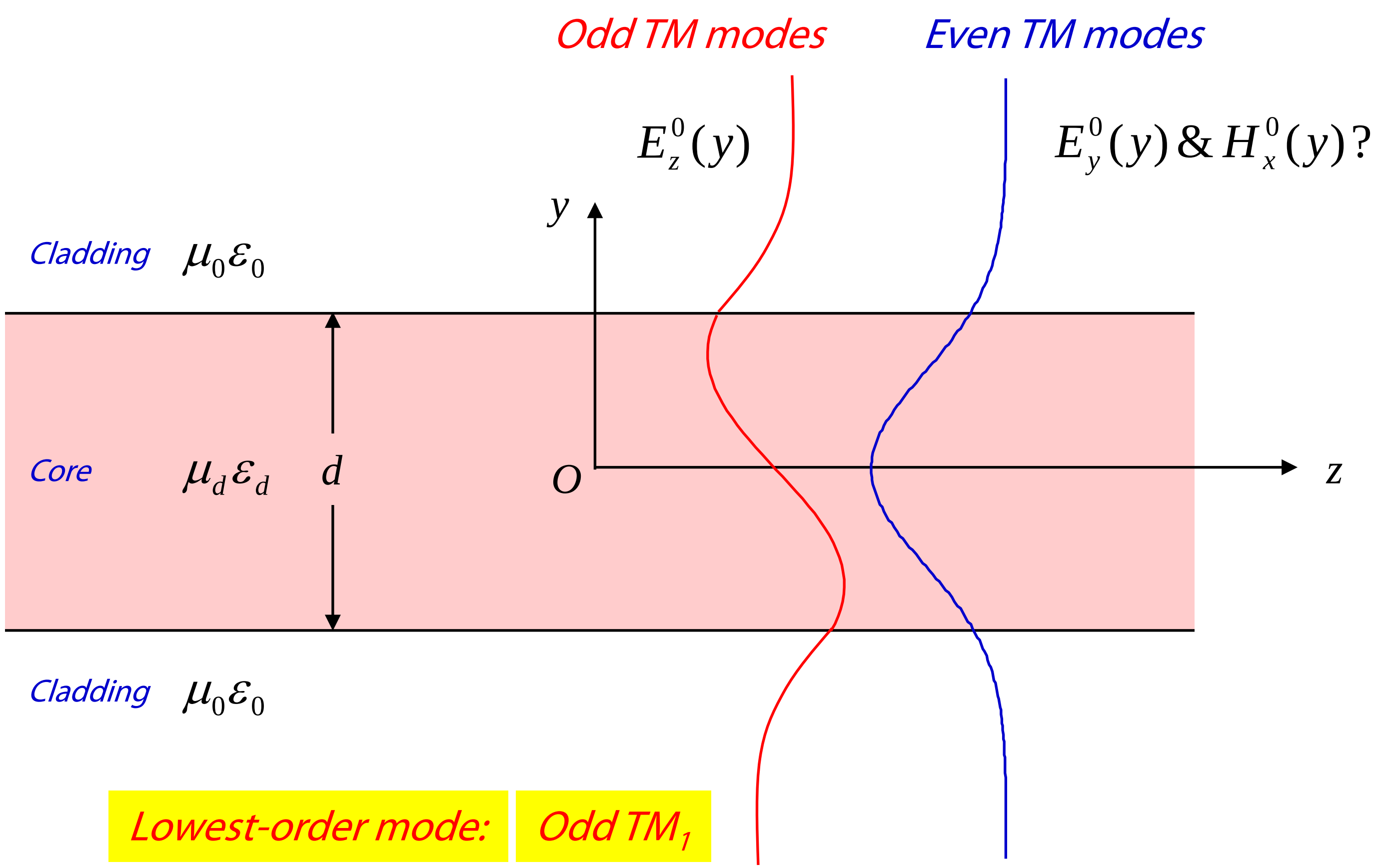
$$\alpha = [\omega^2 (\mu_d \varepsilon_d - \mu_0 \varepsilon_0) - k_y^2]^{1/2}$$

$$= -\frac{\varepsilon_0}{\varepsilon_d} k_y \cot \frac{k_y d}{2} = 0$$

$$\rightarrow \pi f_{co} d \sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0} = (n - \frac{1}{2})\pi$$

$$\rightarrow f_{co} = \frac{(n - \frac{1}{2})}{d\sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0}}$$

TM Modes along a Dielectric Slab



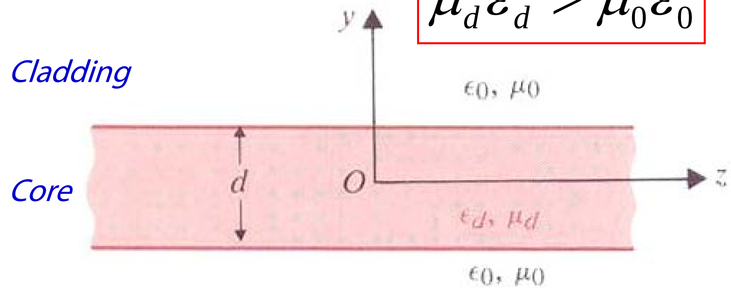
Lowest-order mode: *Odd TM$_1$*

$$\rightarrow f_{co} = \frac{(n-1)}{d\sqrt{\mu_d\varepsilon_d - \mu_0\varepsilon_0}} = 0$$

Why?

TE Modes along a Dielectric Slab

$\mu_d \varepsilon_d > \mu_0 \varepsilon_0$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Characteristic equation:

$$E_z = 0 \rightarrow \frac{\partial^2 H_z^0}{\partial y^2} + h^2 H_z^0 = 0$$

$$\leftarrow h^2 = \gamma^2 + k^2$$

$$\rightarrow \gamma = j\beta \;\; \rightarrow h^2 = k^2 - \beta^2 \qquad \leftarrow k^2 = \omega^2 \mu\varepsilon = \omega^2 \begin{bmatrix} \mu_d \varepsilon_d, & |y| \le d/2 \\ \mu_0 \varepsilon_0, & |y| > d/2 \end{bmatrix}$$

Boundary condition:

Tangential components of E and H fields: Continuous across the boundary

$\rightarrow \beta$ *must be the same in the core and cladding regions.*

$\rightarrow h^2 \ge 0$ *in the core: Propagating modes*

$\rightarrow h^2 < 0$ *in the cladding: Evanescent modes*

Solution:

$$H_z^0(y) = H_o \sin k_y y + H_e \cos k_y y, \;\; |y| \le d/2 \quad \leftarrow k_y^2 = \omega^2 \mu_d \varepsilon_d - \beta^2 = h_d^2$$

$$H_z^0(y) = \begin{cases} C_u e^{-\alpha(y-d/2)}, & y > d/2 \\ C_l e^{\alpha(y+d/2)}, & y < -d/2 \end{cases} \quad \leftarrow \alpha^2 = \beta^2 - \omega^2 \mu_0 \varepsilon_0 = -h_0^2$$

Odd TE Modes (1)

Odd-mode solution:

$$H_z^0(y) = H_o \sin k_y y + H_e \cos k_y y, \quad |y| \le d/2$$

$$H_z^0(y) = \begin{cases} C_u e^{-\alpha(y-d/2)}, & y > d/2 \\ C_l e^{\alpha(y+d/2)}, & y < -d/2 \end{cases}$$

Recall:

$$H_x^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial H_z^0}{\partial x} - j\omega\varepsilon \frac{\partial E_z^0}{\partial y}\right)$$

$$H_y^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial H_z^0}{\partial y} + j\omega\varepsilon \frac{\partial E_z^0}{\partial x}\right)$$

$$E_x^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial E_z^0}{\partial x} + j\omega\mu \frac{\partial H_z^0}{\partial y}\right)$$

$$E_y^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial E_z^0}{\partial y} - j\omega\mu \frac{\partial H_z^0}{\partial x}\right)$$

In the core:

$$H_z^0(y) = H_o \sin k_y y$$

$$H_y^0(y) = -\frac{j\beta}{k_y} H_o \cos k_y y$$

$$E_x^0(y) = -\frac{j\omega\mu_d}{k_y} H_o \cos k_y y$$

In the cladding:

Upper cladding:

$$H_z^0(y) = C_u e^{-\alpha(y-d/2)}$$

$$H_y^0(y) = -\frac{j\beta}{\alpha} C_u e^{-\alpha(y-d/2)}$$

$$E_x^0(y) = -\frac{j\omega\mu_0}{\alpha} C_u E_o e^{-\alpha(y-d/2)}$$

Lower cladding:

$$H_z^0(y) = C_l e^{\alpha(y+d/2)}$$

$$H_y^0(y) = \frac{j\beta}{\alpha} C_l e^{\alpha(y+d/2)}$$

$$E_x^0(y) = \frac{j\omega\mu_0}{\alpha} C_l E_o e^{\alpha(y+d/2)}$$

Odd TE Modes (2)

H_z^0 & E_x^0 *must be continuous across the boundary!*

Upper cl.: $H_z^0(y) = C_u e^{-\alpha(y-d/2)}$ $\quad H_y^0(y) = -\frac{j\beta}{\alpha} C_u e^{-\alpha(y-d/2)}$ $\quad E_x^0(y) = -\frac{j\omega\mu_0}{\alpha} C_u H_o e^{-\alpha(y-d/2)}$

Core: $H_z^0(y) = H_o \sin k_y y$ $\quad H_y^0(y) = -\frac{j\beta}{k_y} H_o \cos k_y y$ $\quad E_x^0(y) = -\frac{j\omega\mu_d}{k_y} H_o \cos k_y y$

Lower cl.: $H_z^0(y) = C_l e^{\alpha(y+d/2)}$ $\quad H_y^0(y) = \frac{j\beta}{\alpha} C_l e^{\alpha(y+d/2)}$ $\quad E_x^0(y) = \frac{j\omega\mu_0}{\alpha} C_l H_o e^{\alpha(y+d/2)}$

$$\rightarrow C_u = H_o \sin \frac{k_y d}{2}$$

$$\rightarrow C_l = -H_o \sin \frac{k_y d}{2}$$

$$\rightarrow -\frac{j\omega\mu_0}{\alpha}\left(H_o \sin \frac{k_y d}{2}\right) = -\frac{j\omega\mu_d}{k_y} H_o \cos \frac{k_y d}{2}$$

$$\rightarrow \alpha = \frac{\mu_0}{\mu_d} k_y \tan \frac{k_y d}{2}$$

Recall: $\alpha^2 = \beta^2 - \omega^2 \mu_0 \varepsilon_0,\ \beta^2 = \omega^2 \mu_d \varepsilon_d - k_y^2$

$$\rightarrow \alpha = [\omega^2(\mu_d\varepsilon_d - \mu_0\varepsilon_0) - k_y^2]^{1/2}$$

Transcendental equation:

$$[\omega^2(\mu_d\varepsilon_d - \mu_0\varepsilon_0) - k_y^2]^{1/2} = \frac{\mu_0}{\mu_d} k_y \tan \frac{k_y d}{2}$$

To be solved numerically.

Even TE Modes

H_z^0 & E_x^0 *must be continuous across the boundary!*

Upper cl.: $H_z^0(y) = C_u e^{-\alpha(y-d/2)}$ $\quad H_y^0(y) = -\dfrac{j\beta}{\alpha} C_u e^{-\alpha(y-d/2)}$ $\quad E_x^0(y) = -\dfrac{j\omega\mu_0}{\alpha} C_u H_e e^{-\alpha(y-d/2)}$

Core: $H_z^0(y) = H_e \cos k_y y$ $\quad H_y^0(y) = \dfrac{j\beta}{k_y} H_e \sin k_y y$ $\quad E_x^0(y) = \dfrac{j\omega\mu_d}{k_y} H_e \sin k_y y$

Lower cl.: $H_z^0(y) = C_l e^{\alpha(y+d/2)}$ $\quad H_y^0(y) = \dfrac{j\beta}{\alpha} C_l e^{\alpha(y+/2)}$ $\quad E_x^0(y) = \dfrac{j\omega\mu_0}{\alpha} C_l H_e e^{\alpha(y+d/2)}$

$$\rightarrow C_u = H_e \cos\frac{k_y d}{2}$$

$$\rightarrow C_l = H_e \cos\frac{k_y d}{2}$$

$$\rightarrow -\frac{j\omega\mu_0}{\alpha}\left(H_e \cos\frac{k_y d}{2}\right) = \frac{j\omega\mu_d}{k_y} H_e \sin\frac{k_y d}{2}$$

$$\rightarrow \alpha = -\frac{\mu_0}{\mu_d} k_y \cot\frac{k_y d}{2}$$

Recall: $\alpha^2 = \beta^2 - \omega^2\mu_0\varepsilon_0,\ \beta^2 = \omega^2\mu_d\varepsilon_d - k_y^2$

$$\rightarrow \alpha = [\omega^2(\mu_d\varepsilon_d - \mu_0\varepsilon_0) - k_y^2]^{1/2}$$

Transcendental equation:

$$[\omega^2(\mu_d\varepsilon_d - \mu_0\varepsilon_0) - k_y^2]^{1/2} = -\frac{\mu_0}{\mu_d} k_y \cot\frac{k_y d}{2}$$

To be solved numerically

Cutoff Frequencies of TE modes

Guiding condition:

$$k_y^2 = \omega^2 \mu_d \varepsilon_d - \beta^2 \quad \rightarrow \beta^2 < \omega^2 \mu_d \varepsilon_d$$

$$\alpha^2 = \beta^2 - \omega^2 \mu_0 \varepsilon_0 \quad \rightarrow \beta^2 > \omega^2 \mu_0 \varepsilon_0$$

$$\rightarrow \omega^2 \mu_0 \varepsilon_0 < \beta^2 < \omega^2 \mu_d \varepsilon_d$$

Cutoff condition: $\alpha = 0 \quad \rightarrow k_y = \omega_{co} \sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0}$

Odd TE modes:

$$\alpha = [\omega^2 (\mu_d \varepsilon_d - \mu_0 \varepsilon_0) - k_y^2]^{1/2}$$

$$= \frac{\mu_0}{\mu_d} k_y \tan \frac{k_y d}{2} = 0$$

$$\rightarrow \pi f_{co} d \sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0} = (n-1)\pi$$

$$\rightarrow f_{co} = \frac{(n-1)}{d\sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0}}$$

Even TE modes:

$$\alpha = [\omega^2 (\mu_d \varepsilon_d - \mu_0 \varepsilon_0) - k_y^2]^{1/2}$$

$$= -\frac{\mu_0}{\mu_d} k_y \cot \frac{k_y d}{2} = 0$$

$$\rightarrow \pi f_{co} d \sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0} = (n - \frac{1}{2})\pi$$

$$\rightarrow f_{co} = \frac{(n - \frac{1}{2})}{d\sqrt{\mu_d \varepsilon_d - \mu_0 \varepsilon_0}}$$