Electromagnetics:
The Smith Chart
(9-6)

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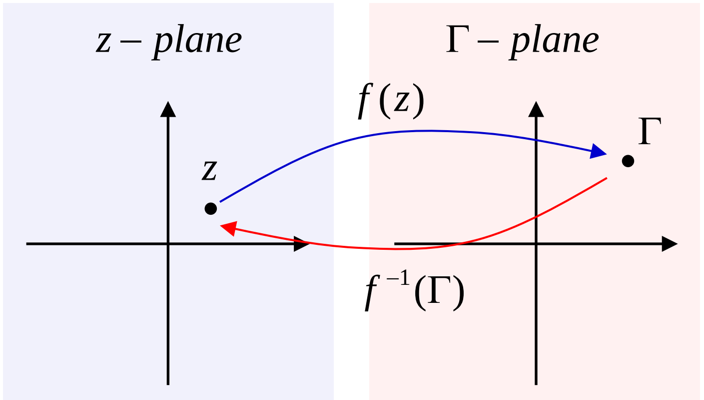
A Conformal Mapping (1)



Mapping between complex-valued variables: z & Γ

$$\Gamma = f(z) \quad \leftarrow f: \textit{ Mapping function}$$

$$\rightarrow z = f^{-1}(\Gamma)$$

Suppose a mapping function: $f(z) = \dfrac{z-1}{z+1}$

$$z = z_r + jz_i = r + jx$$

$$\Gamma = \Gamma_r + j\Gamma_i = |\Gamma| e^{j\phi}$$

Polar coordinates

Cartesian coordinates

$$\rightarrow \Gamma = \frac{z-1}{z+1} \quad \rightarrow z = \frac{1+\Gamma}{1-\Gamma}$$

Consider:

$$z = f^{-1}(\Gamma) \quad \rightarrow z = \frac{1+\Gamma}{1-\Gamma} \quad \rightarrow r + jx = \frac{1+\Gamma_r + j\Gamma_i}{1-\Gamma_r - j\Gamma_i}$$

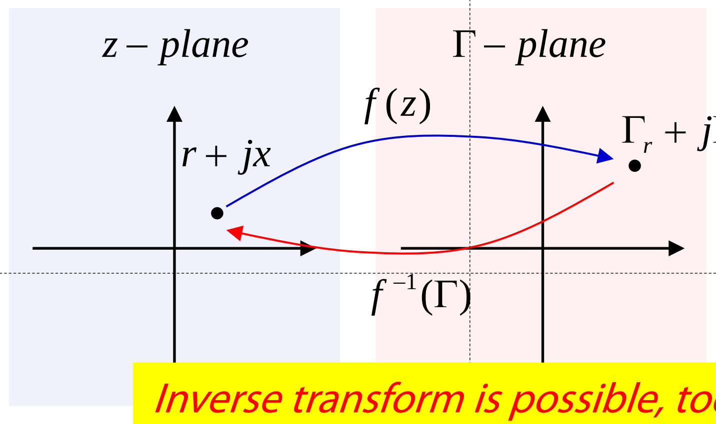
$$\rightarrow r = \frac{1-\Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2} \qquad \rightarrow \left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r}\right)^2 \quad \leftarrow r\textit{-circle}$$

$$\rightarrow x = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2} \qquad \rightarrow (\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2 \quad \leftarrow x\textit{-circle}$$

A Conformal Mapping (2)

r-circle: $\left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r}\right)^2$

x-circle: $(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2$



Inverse transform is possible, too!

Transformation: $r + jx \xrightarrow{f(z)} \Gamma_r + j\Gamma_i$

e.g. $r + jx = 1 + j0.5$

r-circle:

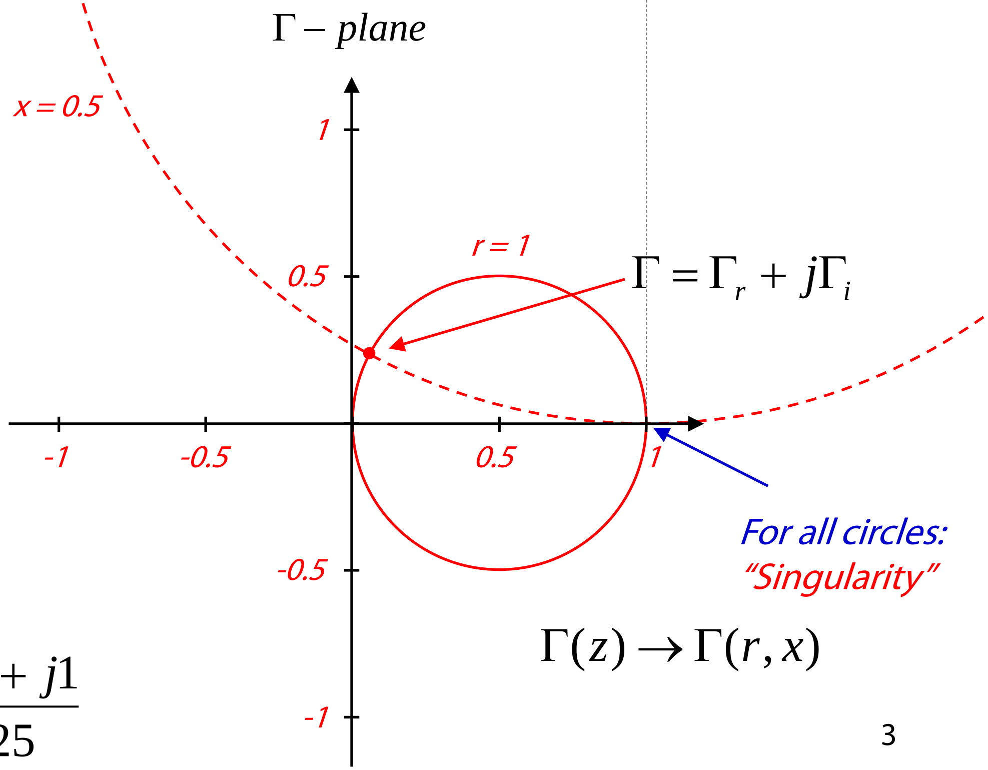
$$(\Gamma_r - 0.5)^2 + \Gamma_i^2 = 0.5^2$$

x-circle:

$$(\Gamma_r - 1)^2 + (\Gamma_i - 2)^2 = 2^2$$

Recall:

$$\Gamma = \frac{z-1}{z+1} = \frac{1 + j0.5 - 1}{1 + j0.5 + 1} = \frac{0.25 + j1}{4.25}$$



$\Gamma(z) \rightarrow \Gamma(r, x)$

The Smith Chart: A Conformal Mapping (1)

Voltage reflection coefficient at the load for a lossless transmission line:

$$\Gamma = \frac{Z_L - R_0}{Z_L + R_0} = \Gamma_r + j\Gamma_i = |\Gamma| e^{j\theta_\Gamma}$$

Polar coordinates

Cartesian coordinates

Normalized load impedance:

$$z_L = \frac{Z_L}{R_0} = \frac{R_L}{R_0} + j\frac{X_L}{R_0} = r + jx$$

$$\rightarrow \Gamma = \frac{Z_L - R_0}{Z_L + R_0} = \frac{z_L - 1}{z_L + 1}$$

$$\rightarrow z_L = \frac{1+\Gamma}{1-\Gamma} = \frac{1+|\Gamma|e^{j\theta_\Gamma}}{1-|\Gamma|e^{j\theta_\Gamma}}$$

$$\rightarrow r + jx = \frac{(1+\Gamma_r) + j\Gamma_i}{(1-\Gamma_r) - j\Gamma_i}$$

$$\rightarrow r = \frac{1-\Gamma_r^2 - \Gamma_i^2}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

$$\rightarrow x = \frac{2\Gamma_i}{(1-\Gamma_r)^2 + \Gamma_i^2}$$

r-circle: $$\left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r}\right)^2$$

x-circle: $$(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2$$

The Smith Chart: A Conformal Mapping (2)

r-circle: $\left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r}\right)^2$

x-circle: $(\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2$

$\rightarrow |\Gamma|^2 \leq 1$ ← *Reflection coefficient*

In the Smith chart, there are many circles already drawn for various values of "r" and "x"!

"in solid lines" *"in dashed lines"*

e.g.

1. Let us assume that the voltage reflection coefficient is given by: $\Gamma_1 = \Gamma_{1r} + j\Gamma_{1i}$

2. Find r- and x-circles that pass through Γ_1

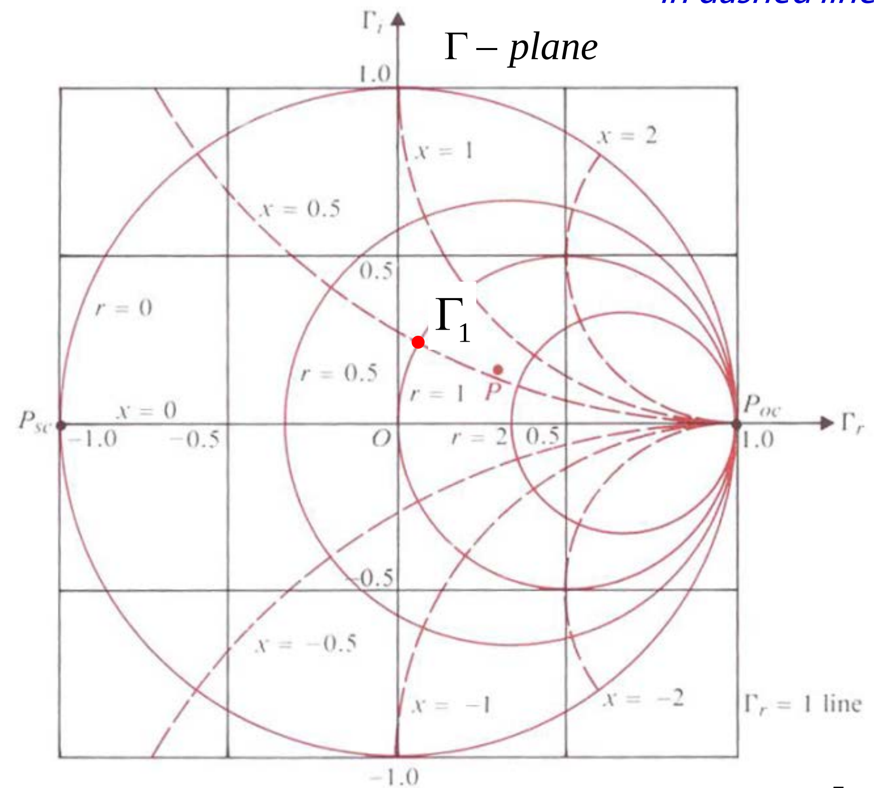
3. Read the r- and x-values for the circles:

$$\rightarrow r = 1,\ x = 0.5$$

In result, when the voltage reflection coefficient is Γ_1*, the normalized load impedance is given by:* $z_L = r + jx = 1 + j0.5$

This can also be verified:

$$\Gamma_1 = \frac{z_L - 1}{z_L + 1} = \frac{1 + j0.5 - 1}{1 + j0.5 + 1} = \frac{0.25 + j1}{4.25}$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

The Smith Chart: A Conformal Mapping (3)

$$\Gamma = \frac{z_L - 1}{z_L + 1} \qquad \left(\Gamma_r - \frac{r}{1+r}\right)^2 + \Gamma_i^2 = \left(\frac{1}{1+r}\right)^2$$

$$z_L = \frac{1+\Gamma}{1-\Gamma} \qquad (\Gamma_r - 1)^2 + \left(\Gamma_i - \frac{1}{x}\right)^2 = \left(\frac{1}{x}\right)^2$$

The r-circles:

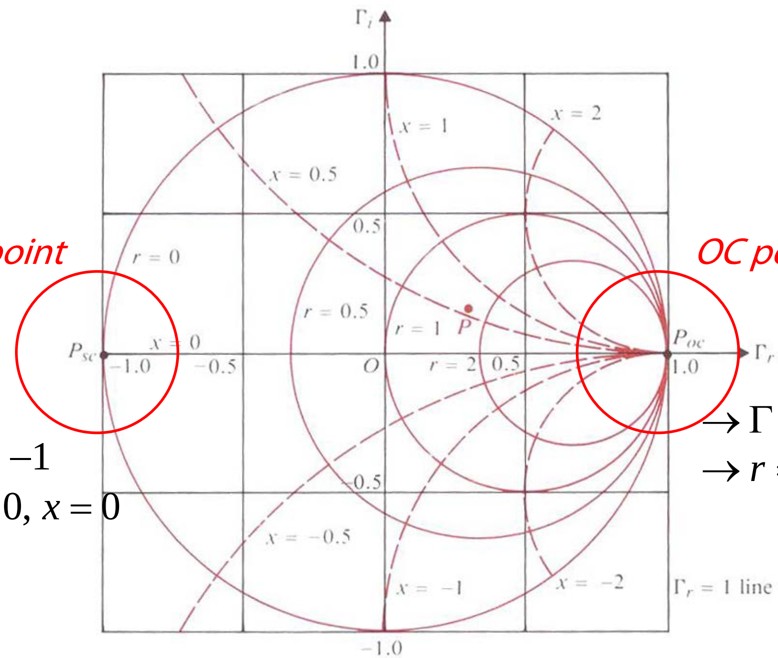
1. *Centred at ($\Gamma_r = r/(r+1)$, $\Gamma_i = 0$): On the Γ_r-axis*
2. *$r = 0$: Centred at the origin with a unity radius*
3. *$r \to \infty$: Centred at ($\Gamma_r = 1$, $\Gamma_i = 0$)*
4. *All r-circles: Passing through ($\Gamma_r = 1$, $\Gamma_i = 0$).*

The x-circles:

1. *Centred at ($\Gamma_r = 1$, $\Gamma_i = 1/x$): On the $\Gamma_r = 1$ line*
2. *$x = 0$: $\Gamma_i = 0$ line, i.e., Γ_r-axis*
3. *$x \to \infty$: Centred at ($\Gamma_r = 1$, $\Gamma_i = 0$)*
4. *All x-circles: Passing through ($\Gamma_r = 1$, $\Gamma_i = 0$).*

SC point

OC point



$\to \Gamma = -1$

$\to r = 0,\ x = 0$

$\to \Gamma = 1$

$\to r = \infty,\ x = \infty$

D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

The r- and x-circles are everywhere orthogonal to one another!

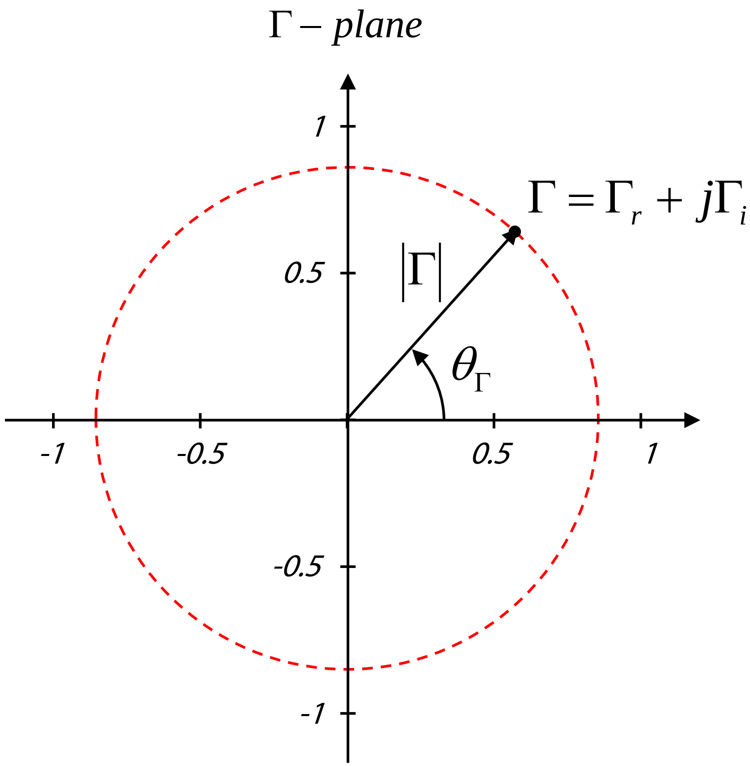
Smith Chart with Polar Coordinates (1)

Voltage reflection coefficient at the load for a lossless transmission line:

$$\Gamma = \frac{Z_L - R_0}{Z_L + R_0} = \frac{z_L - 1}{z_L + 1} = \Gamma_r + j\Gamma_i = |\Gamma| e^{j\theta_\Gamma}$$

Polar coordinates

Cartesian coordinates

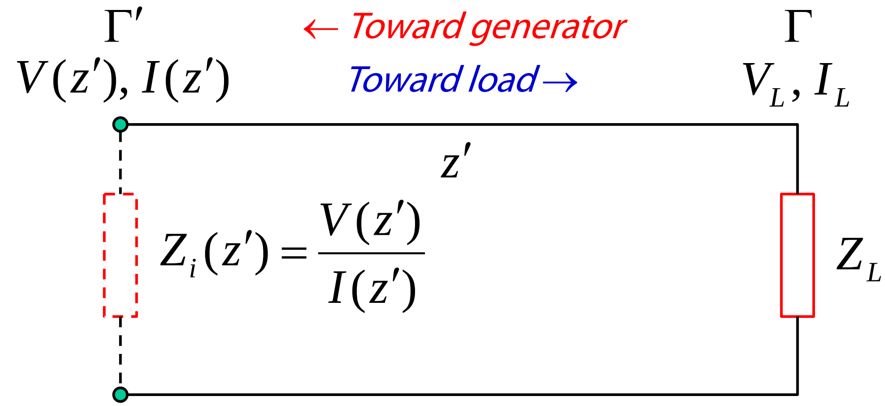


Properties to note:

1. All $|\Gamma|$ -circles: Radii from 0 to 1

2. θ_Γ: Measured from the positive real axis

Smith Chart with Polar Coordinates (2)



Input impedance:

$$Z_i(z') = \frac{V(z')}{I(z')} = Z_0\left[\frac{1+\Gamma e^{-j2\beta z'}}{1-\Gamma e^{-j2\beta z'}}\right]$$

Normalized "input" impedance:

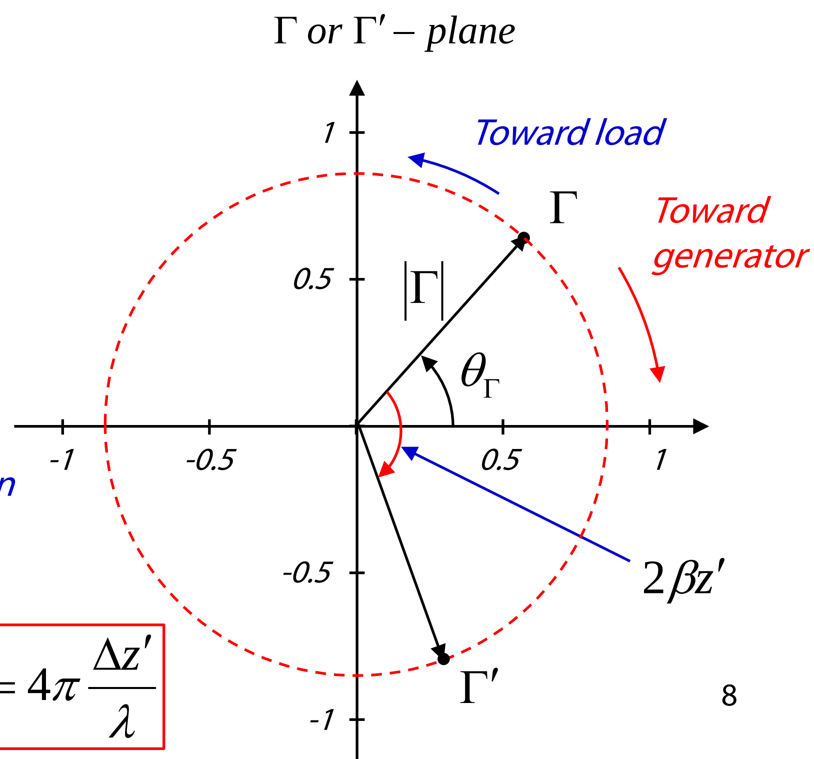
$$z_i = \frac{Z_i}{Z_0} = \frac{1+\Gamma e^{-j2\beta z'}}{1-\Gamma e^{-j2\beta z'}} = \frac{1+|\Gamma|e^{j\phi}}{1-|\Gamma|e^{j\phi}}$$

$$\leftarrow \phi = \theta_\Gamma - 2\beta z'$$

Of exactly same form!!

Normalized "load" impedance:

$$\rightarrow z_L = \frac{1+\Gamma}{1-\Gamma} = \frac{1+|\Gamma|e^{j\theta_\Gamma}}{1-|\Gamma|e^{j\theta_\Gamma}}$$



Transformation: $\Gamma(z_L) \rightarrow \Gamma'(z_i)$

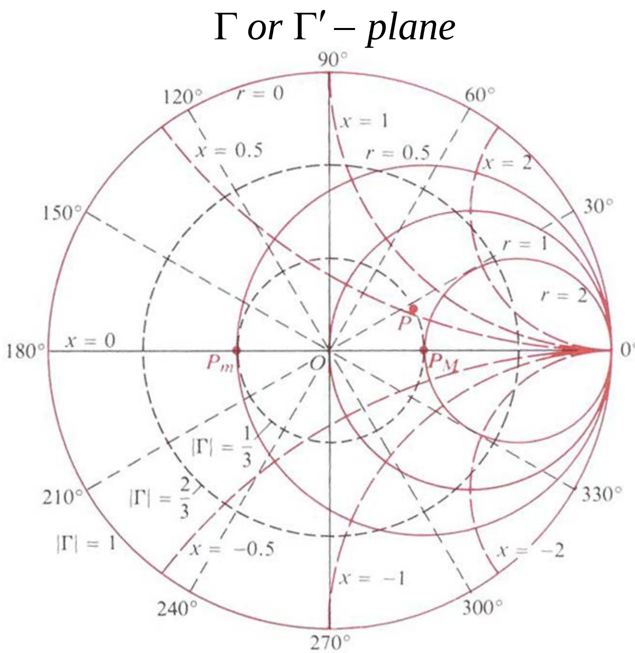
$$z_i = \frac{1+|\Gamma|e^{j\phi}}{1-|\Gamma|e^{j\phi}} = \frac{1+\Gamma'}{1-\Gamma'}$$

→ *"Effective" reflection coefficient at z'*

$$\leftarrow |\Gamma'| = |\Gamma|$$

$$\phi = \theta_\Gamma - 2\beta z' \rightarrow \boxed{\Delta\phi = 2\beta\Delta z' = 4\pi\frac{\Delta z'}{\lambda}}$$

Smith Chart with Polar Coordinates (3)



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Normalized load impedance:

$$z_L = \frac{1+\Gamma}{1-\Gamma} = \frac{1+|\Gamma|e^{j\theta_\Gamma}}{1-|\Gamma|e^{j\theta_\Gamma}}$$

Normalized input impedance:

$$z_i = \frac{1+\Gamma'}{1-\Gamma'} = \frac{1+|\Gamma|e^{j\phi}}{1-|\Gamma|e^{j\phi}} \quad \leftarrow \phi = \theta_\Gamma - 2\beta z'$$

Additional properties to note:

1. P_M: The r-circle value → S (SWR)

2. P_m: The r-circle value → 1/S

Recall:

$$S = \left|\frac{V_{\max}}{V_{\min}}\right| = \left|\frac{I_{\max}}{I_{\min}}\right| = \frac{1+|\Gamma|}{1-|\Gamma|} \rightarrow |\Gamma| = \frac{S-1}{S+1}$$

$$\rightarrow z_i = \frac{1+|\Gamma|e^{j\phi}}{1-|\Gamma|e^{j\phi}} \rightarrow r_M = \frac{1+|\Gamma|}{1-|\Gamma|} \rightarrow r_m = \frac{1-|\Gamma|}{1+|\Gamma|}$$

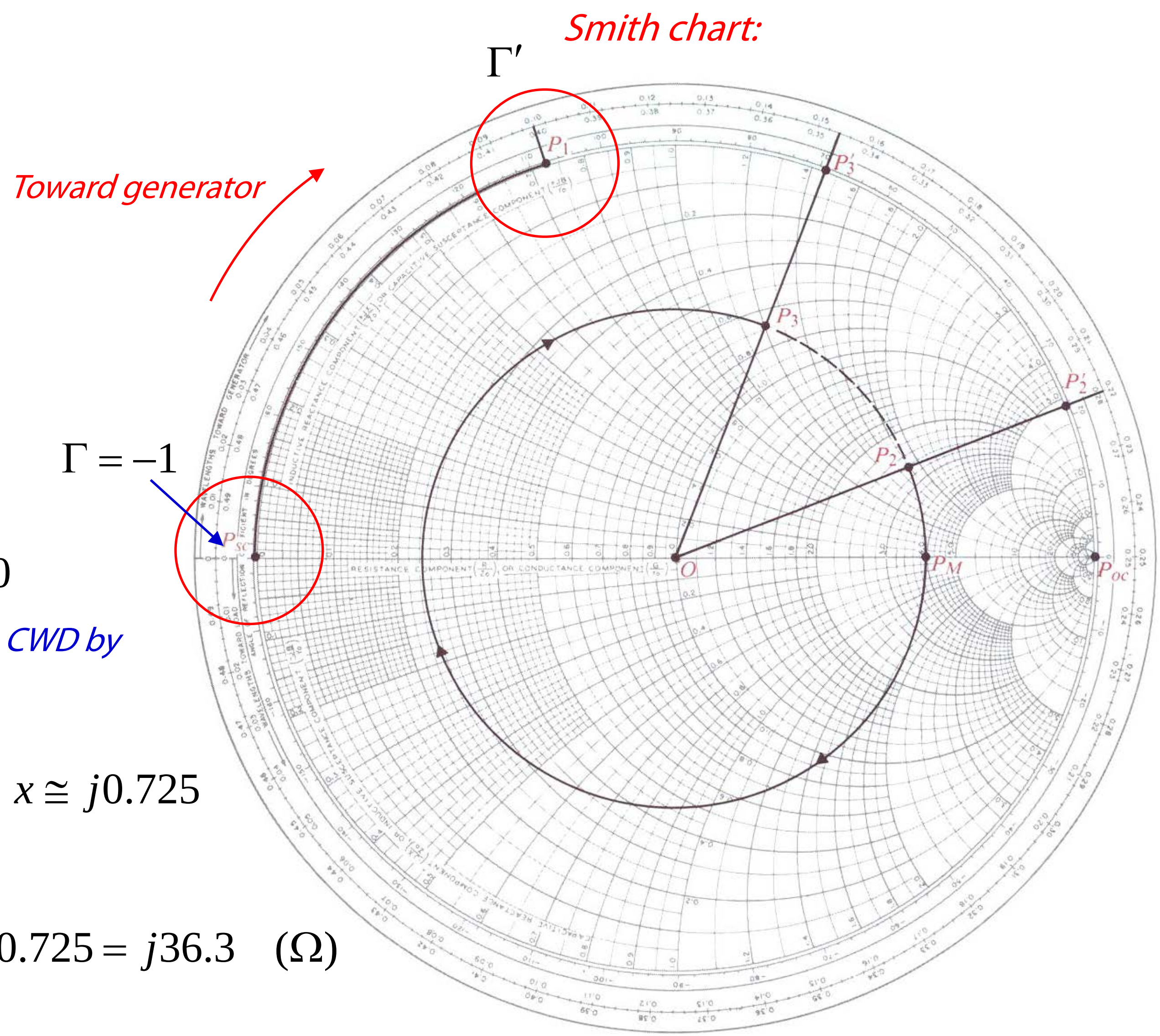
Example 9-13:

Given:

$$R_0 = 50 \ (\Omega)$$

$$z_L = 0$$

$$z' = 0.1\lambda$$



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Solution:

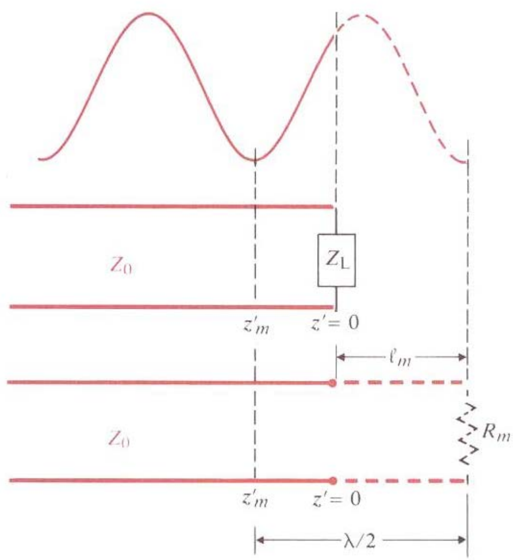
1. Point P_{sc}: $r = x = 0$
2. Azimuthal shift in the CWD by 0.1 from P_{sc} to P_1
3. At P_1, read r & x

$$r = 0, \quad x \cong j0.725$$

In result:

$$Z_i = R_0 z_i = 50 \times j0.725 = j36.3 \quad (\Omega)$$

Example 9-15:



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Given:

$$R_0 = 50 \ (\Omega)$$

$$S = 3.0$$

$$\lambda = 2 \times 0.2 = 0.4 \ (m)$$

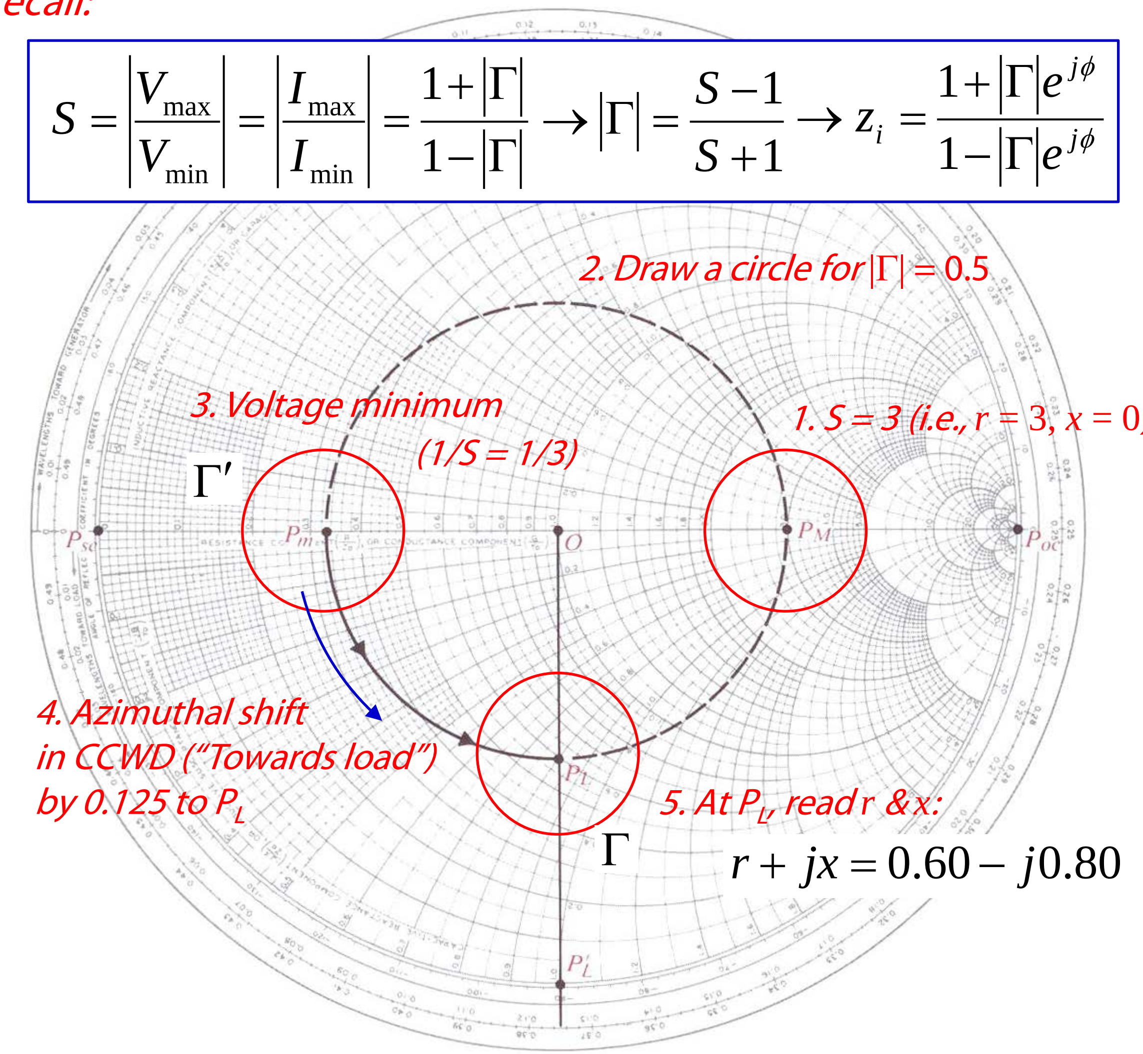
$$z'_m = 0.05 \ (m)$$

(First voltage minimum)

Smith chart:

Recall:

$$S = \left|\frac{V_{\max}}{V_{\min}}\right| = \left|\frac{I_{\max}}{I_{\min}}\right| = \frac{1+|\Gamma|}{1-|\Gamma|} \rightarrow |\Gamma| = \frac{S-1}{S+1} \rightarrow z_i = \frac{1+|\Gamma|e^{j\phi}}{1-|\Gamma|e^{j\phi}}$$



D. K. Cheng, Field and Wave Electromagnetics, 2nd ed., Addison-Wesley, 1989.

Note: $z'_m / \lambda = 0.05 / 0.4 = 0.125$

$$Z_L = 50(0.60 - j0.80) = 30 - j40 \quad (\Omega)$$

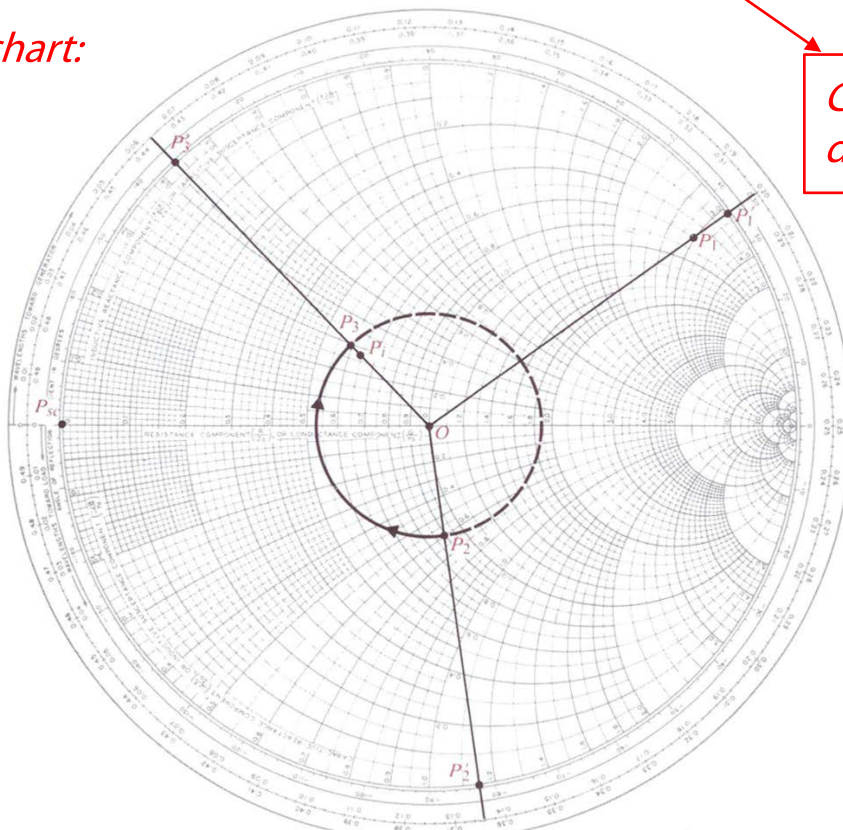
$$\Gamma = |\Gamma| e^{-j\pi/2} = -j0.5$$

Smith-Chart Calculations for Lossy Lines

Normalized input impedance for a lossy line:

$$z_i = \frac{1+\Gamma e^{-2\alpha z'} e^{-j2\beta z'}}{1-\Gamma e^{-2\alpha z'} e^{-j2\beta z'}} = \frac{1+|\Gamma| e^{-2\alpha z} e^{j\phi}}{1-|\Gamma| e^{-2\alpha z} e^{j\phi}} \qquad \leftarrow \phi = \theta_\Gamma - 2\beta z'$$

Smith chart:



Consider shrinkage due to attenuation: $e^{-2\alpha z'}$

$$|\Gamma'| = |\Gamma| e^{-2\alpha z'}$$

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