

Electro-Optics:

Electromagnetic Fields (2)

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Complex-Function Formalism

Field component:

$$a(t) = |A| \cos(\omega t + \alpha)$$

Amplitude *Angular frequency*
Phase

Consider a complex amplitude:

$$A = |A|e^{i\alpha}$$
$$\rightarrow a(t) = \operatorname{Re}[Ae^{i\omega t}]$$

Product of two sinusoidal functions:

$$a(t) = |A| \cos(\omega t + \alpha) \quad b(t) = |B| \cos(\omega t + \beta)$$
$$= \operatorname{Re}[Ae^{i\omega t}] \quad = \operatorname{Re}[Be^{i\omega t}]$$
$$\rightarrow a(t)b(t) = \frac{1}{2}|AB|[\cos(2\omega t + \alpha + \beta) + \cos(\alpha - \beta)]$$
$$\rightarrow Ae^{i\omega t} Be^{i\omega t} = |AB|e^{i(2\omega t + \alpha + \beta)}$$
$$\rightarrow \boxed{\operatorname{Re}[x]\operatorname{Re}[y] \neq \operatorname{Re}[xy]}$$

Time-Averaging Sinusoidal Products

Time average of the product of two sinusoidal functions of the same freq.:

$$\begin{aligned}\langle a(t)b(t) \rangle &= \frac{1}{T} \int_0^T |A| \cos(\omega t + \alpha) |B| \cos(\omega t + \beta) dt \\ &= \frac{1}{T} \int_0^T \frac{1}{2} |AB| [\cos(2\omega t + \alpha + \beta) + \cos(\alpha - \beta)] dt \\ &= \frac{1}{2} |AB| \cos(\alpha - \beta)\end{aligned}$$

$$\rightarrow \langle a(t)b(t) \rangle = \frac{1}{2} \operatorname{Re}[AB^*]$$

$$\rightarrow \langle \operatorname{Re}[X(t)] \operatorname{Re}[Y(t)] \rangle = \frac{1}{2} \operatorname{Re}[X(t)Y(t)^*]$$

Time-averaged Poynting vector and energy density:

$$\mathbf{S} = \frac{1}{2} \operatorname{Re}[\mathbf{E} \times \mathbf{H}^*]$$

$$U = \frac{1}{4} \operatorname{Re}[\mathbf{E} \cdot \mathbf{D}^* + \mathbf{B} \cdot \mathbf{H}^*]$$

Wave Equations & Monochromatic Plane Waves

For an isotropic medium:

No charge density & no current density:

$$\begin{aligned} \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 & \quad \rightarrow \quad \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{E} \right) + \frac{\partial}{\partial t} \nabla \times \mathbf{H} = 0 \quad \leftarrow \quad \boxed{\rho = 0, \mathbf{J} = 0} \\ & \rightarrow \quad \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{E} \right) + \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \\ & \leftarrow \quad \nabla \times \left(\frac{1}{\mu} \nabla \times \mathbf{E} \right) = \frac{1}{\mu} \nabla \times (\nabla \times \mathbf{E}) + \left(\nabla \frac{1}{\mu} \right) \times \nabla \times \mathbf{E} \\ & \leftarrow \quad \nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \\ & \rightarrow \quad \nabla^2 \mathbf{E} - \mu \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + (\nabla \log \mu) \times (\nabla \times \mathbf{E}) - \nabla (\nabla \cdot \mathbf{E}) = 0 \\ & \rightarrow \quad \nabla \cdot (\varepsilon \mathbf{E}) = \varepsilon (\nabla \cdot \mathbf{E}) + \mathbf{E} \cdot \nabla \varepsilon = \rho = 0 \end{aligned}$$

Wave equation:

$$\begin{aligned} & \rightarrow \quad \nabla^2 \mathbf{E} - \mu \varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + (\nabla \log \mu) \times (\nabla \times \mathbf{E}) + \nabla (\mathbf{E} \cdot \nabla \log \varepsilon) = 0 \\ & \rightarrow \quad \nabla^2 \mathbf{H} - \mu \varepsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} + (\nabla \log \varepsilon) \times (\nabla \times \mathbf{H}) + \nabla (\mathbf{H} \cdot \nabla \log \mu) = 0 \end{aligned}$$

Monochromatic Plane Waves

Wave equations: *Homogeneous, isotropic & no source*

$$\nabla^2 \mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0, \quad \nabla^2 \mathbf{H} - \mu\epsilon \frac{\partial^2 \mathbf{H}}{\partial t^2} = 0$$

Plane waves:

$$\psi = e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})}, \quad |\mathbf{k}| = \omega \sqrt{\mu\epsilon}$$

Phase velocity:

$$\omega t - \mathbf{k} \cdot \mathbf{r} = \text{constant}, \quad v_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}},$$

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 2.997930 \times 10^8 \text{ (m/s)}$$

Time-averaged power flow & energy density:

$$\mathbf{S} = \frac{|E_0|^2 \mathbf{u}_3}{2\eta} = \frac{\mathbf{k}}{2\omega\mu} |E_0|^2$$

$$U = \frac{1}{2} \epsilon |E_0|^2$$

Field vectors:

$$\mathbf{E} = \mathbf{u}_1 E_0 e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})}$$

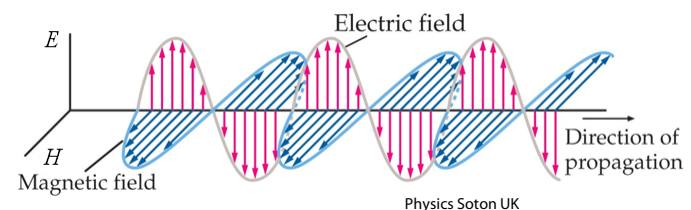
$$\mathbf{H} = \mathbf{u}_2 H_0 e^{i(\omega t - \mathbf{k} \cdot \mathbf{r})}$$

$$\leftarrow \mathbf{u}_1 \cdot \mathbf{k} = \mathbf{u}_2 \cdot \mathbf{k} = 0$$

$$\leftarrow \mathbf{u}_2 = \frac{\mathbf{k} \times \mathbf{u}_1}{|\mathbf{k}|}$$

Impedance of space:

$$\eta = \frac{E_0}{H_0} = \sqrt{\frac{\mu}{\epsilon}}$$



Propagation of a Laser Pulse (1)

Wave packet:

$$\psi(z, t) = \int_{-\infty}^{\infty} A(k) e^{i[\omega(k)t - kz]} dk$$

Dispersion relation: $\omega = f(k)$

→ Taylor series expansion

$$\omega(k) = \omega_0 + \left(\frac{d\omega}{dk} \right)_0 (k - k_0) + \dots$$

$$\rightarrow \psi(z, t) = \int_{-\infty}^{\infty} A(k) e^{i[\omega(k)t - kz]} dk$$

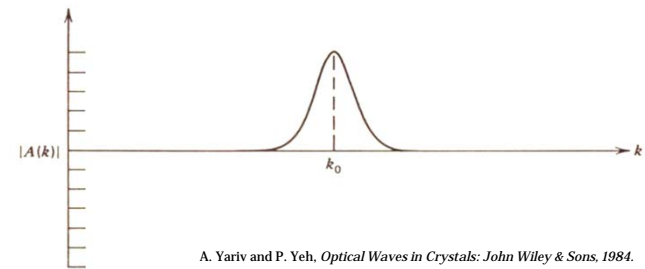
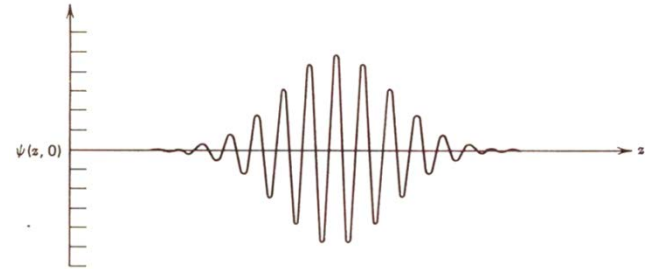
$$\cong \int_{-\infty}^{\infty} A(k) e^{i\left\{ \left[\omega_0 + \left(\frac{d\omega}{dk} \right)_0 (k - k_0) \right] t - kz \right\}} dk$$

$$= e^{i(\omega_0 t - k_0 z)} \int_{-\infty}^{\infty} A(k) \exp \left\{ i \left[\left(\frac{d\omega}{dk} \right)_0 t - z \right] (k - k_0) \right\} dk$$

Carrier

$$= e^{i(\omega_0 t - k_0 z)} F \left[z - \left(\frac{d\omega}{dk} \right)_0 t \right]$$

Envelope



A. Yariv and P. Yeh, *Optical Waves in Crystals: John Wiley & Sons, 1984.*

← First-order perturbation

Propagation of a Laser Pulse (2)

Wave packet:

$$\psi(z, t) = \int_{-\infty}^{\infty} A(k) e^{i[\omega(k)t - kz]} dk \cong e^{i(\omega_0 t - k_0 z)} F \left[z - \left(\frac{d\omega}{dk} \right)_0 t \right]$$

Group velocity: **Velocity of the envelope**

$$v_g = \left(\frac{d\omega}{dk} \right)_0 = \frac{c}{n + \omega(dn/d\omega)}$$

Dispersion relation:

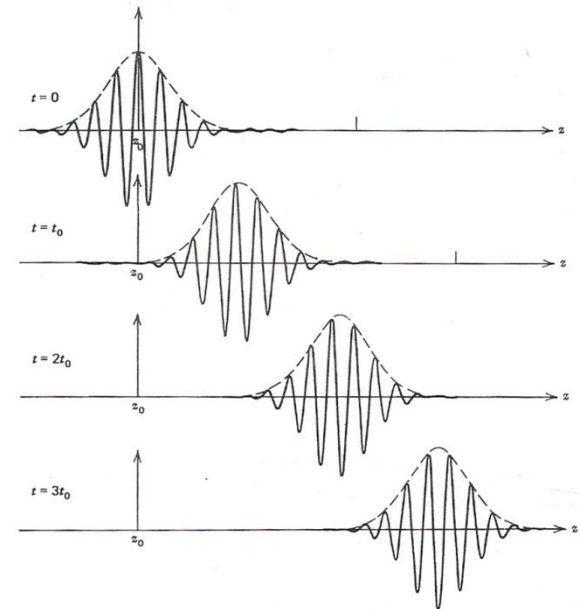
$$k = \frac{\omega}{c} n(\omega)$$

Phase velocity:

$$v_p = \frac{c}{n(\omega)}$$

Group velocity dispersion:

$$\Delta v_g \sim \left(\frac{d^2 \omega}{dk^2} \right)_0 \Delta k$$



A. Yariv and P. Yeh, *Optical Waves in Crystals: John Wiley & Sons, 1984.*