Electro-Optics:

Propagation of Laser Beams (1)

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Wave Equations in Quadratic Index Media (1)

Maxwell's equations:

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad \nabla \cdot \mathbf{D} = \rho$$

$$\nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{J} \qquad \nabla \cdot \mathbf{B} = 0$$

Isotropic, non-magnetic, charge-free, and current-free medium:

$$\nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + (\nabla \log \mu) \times (\nabla \times \mathbf{E}) + \nabla(\mathbf{E} \cdot \nabla \log \varepsilon) = 0$$

Wave equation:

$$\nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = -\nabla\left(\mathbf{E} \cdot \nabla \log \varepsilon\right) \cong 0$$

$$\rightarrow \nabla^2 \mathbf{E} + K^2(\mathbf{r})\mathbf{E} = 0 \qquad \leftarrow K^2(\mathbf{r}) = \omega^2 \mu\varepsilon(r)$$

Quadratic index media:

$$K^2(r) = k^2 - kk_2 r^2$$

$$\leftarrow k^2 = K^2(0) = \omega^2 \mu\varepsilon(0)$$

Wave Equations in Quadratic Index Media (2)

Scalar-wave approximation:

$$\nabla^2 \mathbf{E} + K^2(r)\mathbf{E} = 0$$

No azimuthal variation:

$$\rightarrow \nabla^2 = \nabla_t^2 + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \quad \leftarrow \quad \frac{\partial}{\partial \phi} = 0$$

$$\rightarrow E = \psi(r,\phi,z)e^{-ikz}$$

Paraxial approximation

$$\rightarrow \nabla_t^2\psi + \frac{\partial^2\psi}{\partial z^2} - 2ik\frac{\partial\psi}{\partial z} - k^2\psi + (k^2 - kk_2 r^2)\psi = 0$$

$$\rightarrow \nabla_t^2\psi - 2ik\psi' - kk_2 r^2\psi = 0$$

Trial solution:

$$\psi = \exp\left[-i\left(P(z) + \frac{k}{2q(z)}r^2\right)\right]$$

$$\rightarrow -\left(\frac{k}{q}\right)^2 r^2 - 2i\left(\frac{k}{q}\right) - k^2 r^2\left(\frac{1}{q}\right)' - 2kP' - kk_2 r^2 = 0$$

$$\rightarrow \left(\frac{1}{q}\right)^2 + \left(\frac{1}{q}\right)' + \frac{k_2}{k} = 0$$ $\rightarrow$ For all r $\rightarrow P' = -\frac{i}{q}$

Gaussian Beams in a Homogeneous Medium (1)

Trial solution:

$$\rightarrow \left(\frac{1}{q}\right)^2 + \left(\frac{1}{q}\right)' + \frac{k_2}{k} = 0, \quad P' = -\frac{i}{q}$$

For a homogeneous medium: $\rightarrow k_2 = 0$

$$\rightarrow \left(\frac{1}{q}\right)^2 + \left(\frac{1}{q}\right)' = 0, \quad P' = -\frac{i}{q}$$

$$\rightarrow \frac{1}{q} = \frac{1}{u}\frac{du}{dz} \rightarrow \frac{d^2u}{dz^2} = 0 \rightarrow u = az + b$$

$$\rightarrow \frac{1}{q} = \frac{a}{az+b} = \frac{1}{z+q_0} \rightarrow q = z + q_0$$

$$\rightarrow P' = -\frac{i}{q} = -\frac{i}{z+q_0}$$

$$\rightarrow P = -i\ln\left(1+\frac{z}{q_0}\right)$$

Scalar-wave solution:

$$\rightarrow \psi = \exp\left[-i\left(P(z) + \frac{k}{2q(z)}r^2\right)\right] = \exp\left\{-i\left[-i\ln\left(1+\frac{z}{q_0}\right) + \frac{k}{2(z+q_0)}r^2\right]\right\}$$

$$\rightarrow q_0 \equiv i\frac{\pi\omega_0^2 n}{\lambda} \equiv iz_0 \rightarrow \lambda = \frac{2\pi n}{k}$$

$\leftarrow$ Choice of a pure imaginary number?

Gaussian Beams in a Homogeneous Medium (2)

Scalar-wave solution:

$$\psi = \exp\left\{-i\left[-i\ln\left(1+\frac{z}{q_0}\right)+\frac{k}{2(z+q_0)}r^2\right]\right\} \quad \leftarrow \quad q_0 = iz_0$$

$$\rightarrow \quad \exp\left[-\ln\left(1-i\frac{z}{z_0}\right)\right] = \frac{1}{\sqrt{1+(z/z_0)^2}}\exp\left[i\tan^{-1}\left(\frac{z}{z_0}\right)\right]$$

$$\rightarrow \quad \exp\left[\frac{-ikr^2}{2(z+iz_0)}\right] = \exp\left\{\frac{-r^2}{\omega_0^2[1+(z/z_0)^2]} - \frac{ikr^2}{2z[1+(z_0/z)^2]}\right\}$$

$$\rightarrow \quad \omega^2(z) = \omega_0^2\left(1+\frac{z^2}{z_0^2}\right), \qquad R(z) = z\left(1+\frac{z_0^2}{z^2}\right), \qquad \eta(z) = \tan^{-1}\left(\frac{z}{z_0}\right)$$

$$\rightarrow \quad \frac{1}{q(z)} = \frac{1}{z+iz_0} = \frac{1}{R(z)} - i\frac{2}{k\omega^2(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi n\omega^2(z)}$$

Gaussian beam solution:

$$E(r,\phi,z) = \psi(r,\phi,z)e^{-ikz} = E_0\frac{\omega_0}{\omega(z)}\exp\left\{-i[kz-\eta(z)] - i\frac{kr^2}{2q(z)}\right\}$$

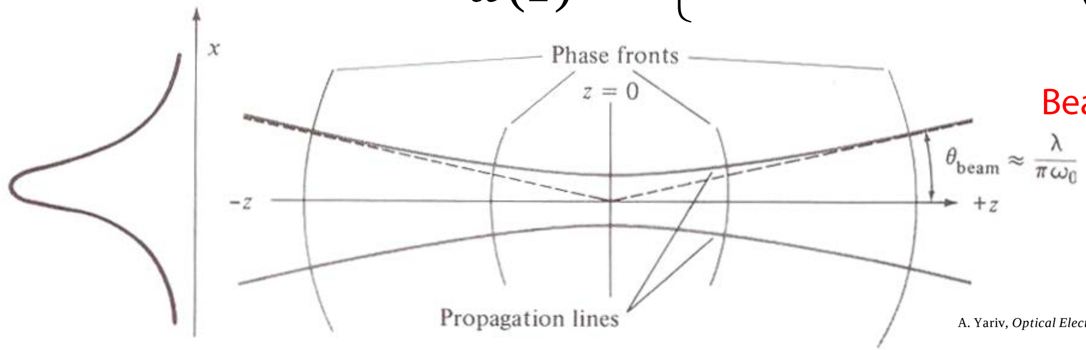
$$= E_0\frac{\omega_0}{\omega(z)}\exp\left\{-i[kz-\eta(z)] - r^2\left(\frac{1}{\omega^2(z)}+\frac{ik}{2R(z)}\right)\right\}$$

Gaussian Beams in a Homogeneous Medium (3)

Gaussian beam:

$$E(r,\phi,z) = E_0 \frac{\omega_0}{\omega(z)} \exp\left\{-i[kz-\eta(z)] - r^2\left(\frac{1}{\omega^2(z)} + \frac{ik}{2R(z)}\right)\right\}$$

Beam radius



A. Yariv, *Optical Electronics*, 4th ed. Saunders, 1991.

Similarity

Spherical wave:

$$E \propto \frac{1}{R}e^{-ikR} = \frac{1}{R}\exp\left(-ik\sqrt{x^2+y^2+z^2}\right)$$

$$\approx \frac{1}{R}\exp\left[-ikz - ik\left(\frac{x^2+y^2}{2R}\right)\right] \leftarrow x^2+y^2 << z^2$$

Radius of curvature of the wavefront

Energy propagation and angular beam spread:

$$r^2 = x^2+y^2 = \omega^2(z) \rightarrow r \approx \frac{\lambda}{\pi\omega_0 n} z$$ → For large z

$$\rightarrow \theta_{beam} = \tan^{-1}\left(\frac{\lambda}{\pi\omega_0 n}\right) \approx \frac{\lambda}{\pi\omega_0 n} \rightarrow \theta_{beam}\omega_0 \approx \frac{\lambda}{\pi n} = const.$$

Invariant

Fundamental Gaussian Beam in a LL Medium

For a lenslike medium: $\rightarrow\ k_2 \neq 0$

$$\left(\frac{1}{q}\right)^2 + \left(\frac{1}{q}\right)' + \frac{k_2}{k} = 0, \quad P' = -\frac{i}{q}$$

$$\rightarrow\ \frac{1}{q} = \frac{u'}{u} \quad \rightarrow\ u'' + u\left(\frac{k_2}{k}\right) = 0$$

$$\rightarrow u(z) = a\sin\sqrt{\frac{k_2}{k}}z + b\cos\sqrt{\frac{k_2}{k}}z$$

$$\rightarrow u'(z) = a\sqrt{\frac{k_2}{k}}\cos\sqrt{\frac{k_2}{k}}z - b\sqrt{\frac{k_2}{k}}\sin\sqrt{\frac{k_2}{k}}z$$

Complex beam radius:

$$q(z) = \frac{u}{u'} = \frac{q_0\cos\sqrt{\frac{k_2}{k}}z + \sqrt{k/k_2}\sin\sqrt{\frac{k_2}{k}}z}{-q_0\sqrt{k_2/k}\sin\sqrt{\frac{k_2}{k}}z + \cos\sqrt{\frac{k_2}{k}}z}$$

$$\rightarrow\ \frac{1}{q(z)} = \frac{1}{R(z)} - i\frac{\lambda}{\pi n\omega^2(z)}$$

High-Order Gaussian Beam Modes

Scalar-wave approximation:

$$\nabla^2 \mathbf{E} + K^2(r)\mathbf{E} = 0$$

$$\rightarrow \nabla^2 = \nabla_t^2 + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\rightarrow \nabla_t^2 \psi - 2ik\psi' - kk_2 r^2 \psi = 0$$

For a homogeneous medium: $\rightarrow k_2 = 0$

$$E_{l,m}(x,y,z) = E_0 \frac{\omega_0}{\omega(z)} H_l\left(\sqrt{2}\frac{x}{\omega(z)}\right) H_m\left(\sqrt{2}\frac{y}{\omega(z)}\right) \exp\left\{-i\frac{k(x^2+y^2)}{2q(z)} - ikz + i(l+m+1)\eta\right\}$$

$$= E_0 \frac{\omega_0}{\omega(z)} H_l\left(\sqrt{2}\frac{x}{\omega(z)}\right) H_m\left(\sqrt{2}\frac{y}{\omega(z)}\right) \exp\left\{-\frac{x^2+y^2}{\omega^2(z)} - \frac{ik(x^2+y^2)}{2R(z)} - ikz + i(l+m+1)\eta\right\}$$

For a lenslike medium: $\rightarrow k_2 \neq 0$

$$E_{l,m}(x,y,z) = E_0 H_l\left(\sqrt{2}\frac{x}{\omega}\right) H_m\left(\sqrt{2}\frac{y}{\omega}\right) \exp\left(-\frac{x^2+y^2}{\omega^2}\right) \exp(-i\beta_{l,m} z)$$

H.W.

$$\rightarrow \beta_{l,m} = k\left[1 - \frac{2}{k}\sqrt{\frac{n_2}{n_0}}(l+m+1)\right]^{1/2}$$