Electro-Optics:

Electromagnetic Propagation in Periodic Media (1)

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Periodic Media

Translational symmetry:

$$\varepsilon(\mathbf{x}) = \varepsilon(\mathbf{x}+\mathbf{a}), \quad \mu(\mathbf{x}) = \mu(\mathbf{x}+\mathbf{a})$$

Maxwell's equations:

$$\nabla \times \mathbf{H} = i\omega\varepsilon\mathbf{E}$$

$$\nabla \times \mathbf{E} = -i\omega\mu\mathbf{H}$$

Normal modes:

$$\rightarrow \ \mathbf{E} = \mathbf{E}_{\mathbf{K}}(\mathbf{x})e^{-i\mathbf{K}\cdot\mathbf{x}} \qquad \leftarrow \ \mathbf{E}_{\mathbf{K}}(\mathbf{x}) = \mathbf{E}_{\mathbf{K}}(\mathbf{x}+\mathbf{a})$$

$$\rightarrow \ \mathbf{H} = \mathbf{H}_{\mathbf{K}}(\mathbf{x})e^{-i\mathbf{K}\cdot\mathbf{x}} \qquad \leftarrow \ \mathbf{H}_{\mathbf{K}}(\mathbf{x}) = \mathbf{H}_{\mathbf{K}}(\mathbf{x}+\mathbf{a})$$

$\leftarrow$ Floquet (or Bloch) theorem (to be proved)

$\leftarrow$ $\mathbf{K}$: Bloch wave vector

Dispersion relation:

$$\rightarrow \ \omega = \omega(\mathbf{K})$$

One-Dimensional Periodic Media (1)

Permittivity:

$$\varepsilon(z) = \varepsilon(z + l\Lambda)$$

Wave equation:

$$\nabla \times (\nabla \times \mathbf{E}) - \omega^2 \mu \varepsilon \mathbf{E} = 0$$

Dielectric tensor in a Fourier series:

$$\rightarrow \ \varepsilon(\mathbf{x}) = \sum_{\mathbf{G}} \varepsilon_{\mathbf{G}} e^{-i\mathbf{G}\cdot\mathbf{x}} \quad \leftarrow \ \mathbf{G} = l\mathbf{g} = l\frac{2\pi}{\Lambda}\hat{\mathbf{z}}$$

← Reciprocal-lattice vector

$$\rightarrow \ \varepsilon(z) = \sum_{l} \varepsilon_l e^{-il(2\pi/\Lambda)z}$$

Electric-field vector in a Fourier integral:

$$\mathbf{E} = \int d^3k \mathbf{A}(\mathbf{k}) e^{-i\mathbf{k}\cdot\mathbf{x}}$$

$$\rightarrow \int d^3k \, \mathbf{k} \times [\mathbf{k} \times \mathbf{A}(\mathbf{k})] e^{-i\mathbf{k}\cdot\mathbf{x}} + \omega^2 \mu \sum_{\mathbf{G}} \int d^3k \, \varepsilon_{\mathbf{G}} \mathbf{A}(\mathbf{k} - \mathbf{G}) e^{-i\mathbf{k}\cdot\mathbf{x}} = 0$$

$$\rightarrow \ \mathbf{k} \times [\mathbf{k} \times \mathbf{A}(\mathbf{k})] + \omega^2 \mu \sum_{\mathbf{G}} \varepsilon_{\mathbf{G}} \mathbf{A}(\mathbf{k} - \mathbf{G}) = 0 \quad \textit{for all } \mathbf{k}$$

One-Dimensional Periodic Media (2)

Electric-field vector:

$$\rightarrow\ \mathbf{k}\times[\mathbf{k}\times\mathbf{A}(\mathbf{k})]+\omega^2\mu\sum_{\mathbf{G}}\varepsilon_{\mathbf{G}}\mathbf{A}(\mathbf{k}-\mathbf{G})=0 \quad \textit{for all } \mathbf{k}$$

Solution of a subset labeled by K (normal mode):

$$\rightarrow\ \mathbf{E}_{(\mathbf{K})}=\sum_{\mathbf{G}}\mathbf{A}(\mathbf{K}-\mathbf{G})\,e^{-i(\mathbf{K}-\mathbf{G})\cdot\mathbf{x}}$$

$$=e^{-i\mathbf{K}\cdot\mathbf{x}}\sum_{\mathbf{G}}\mathbf{A}(\mathbf{K}-\mathbf{G})e^{i\mathbf{G}\cdot\mathbf{x}}$$

$$=e^{-i\mathbf{K}\cdot\mathbf{x}}\mathbf{E}_{\mathbf{K}}(\mathbf{x}) \quad \leftarrow\ \mathbf{E}_{\mathbf{K}}(\mathbf{x})=\sum_{l}\mathbf{A}(\mathbf{K}-l\mathbf{g})e^{il(2\pi/\Lambda)z}$$

$$\rightarrow\ \mathbf{E}_{\mathbf{K}}(\mathbf{x})=\mathbf{E}_{\mathbf{K}}(\mathbf{x}+\mathbf{a})$$

→ Floquet (or Bloch) theorem proved

Bloch mode or wave:

$$\rightarrow \mathbf{E}=e^{-i(K_x x+K_y y)}e^{-iK_z z}\mathbf{E}_{\mathbf{K}}(z)$$

One-Dimensional Periodic Media (3)

Wave propagating in the *z* direction (isotropic):

$$\rightarrow\ k^2 A(k) - \omega^2 \mu \sum_l \varepsilon_l A(k - lg) = 0$$

$$\leftarrow\ k = K,\ K \pm g,\ K \pm 2g,\ \cdots$$

$$\rightarrow\ K^2 A(K) - \omega^2 \mu\varepsilon_0 A(K) - \omega^2 \mu\varepsilon_1 A(K-g) - \omega^2 \mu\varepsilon_{-1} A(K+g) - \cdots = 0$$

$$\rightarrow\ A(K) = \frac{1}{K^2 - \omega^2\mu\varepsilon_0}\left[\omega^2\mu\varepsilon_1 A(K-g) + \omega^2\mu\varepsilon_{-1} A(K+g) + \cdots\right]$$

$$\rightarrow\ A(K-g) = \frac{1}{(K-g)^2 - \omega^2\mu\varepsilon_0}\left[\omega^2\mu\varepsilon_1 A(K-2g) + \omega^2\mu\varepsilon_{-1} A(K) + \cdots\right]$$

$$\rightarrow\ A(K+g) = \frac{1}{(K+g)^2 - \omega^2\mu\varepsilon_0}\left[\omega^2\mu\varepsilon_1 A(K) + \omega^2\mu\varepsilon_{-1} A(K+2g) + \cdots\right]$$

$$\vdots$$

If: $\left|K - g\right| \cong K \ \&\ K^2 \cong \omega^2 \mu\varepsilon_0$

$$\rightarrow\ (K^2 - \omega^2\mu\varepsilon_0) A(K) - \omega^2\mu\varepsilon_1 A(K-g) = 0$$

$$\rightarrow\ -\omega^2\mu\varepsilon_{-1} A(K) + [(K-g)^2 - \omega^2\mu\varepsilon_0] A(K-g) = 0$$

Bragg condition

One-Dimensional Periodic Media (4)

Nontrivial solution:

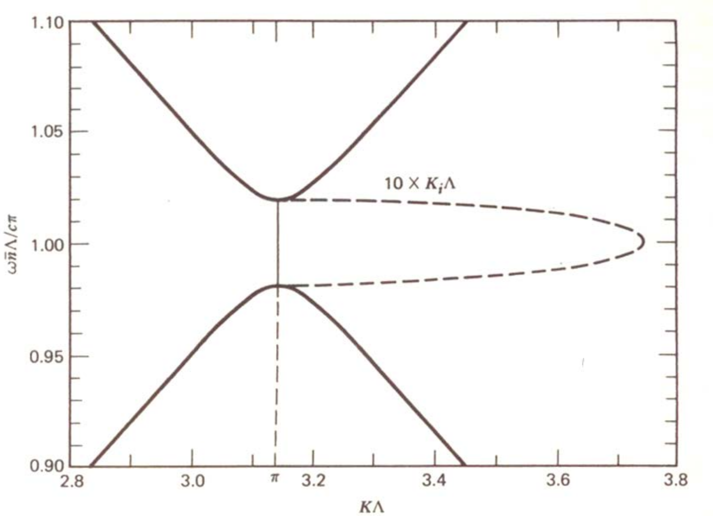
$$\rightarrow \begin{vmatrix} K^2-\omega^2\mu\varepsilon_0 & -\omega^2\mu\varepsilon_1 \\ -\omega^2\mu\varepsilon_{-1} & (K-g)^2-\omega^2\mu\varepsilon_0 \end{vmatrix} = 0 \qquad \leftarrow \ \varepsilon_{-1} = \varepsilon_1^*$$

$$\rightarrow (K^2-\omega^2\mu\varepsilon_0)\left[(K-g)^2-\omega^2\mu\varepsilon_0\right]-\left(\omega^2\mu|\varepsilon_1|\right)^2 = 0$$

Bragg condition:

$$\rightarrow K = \frac{1}{2}g = \frac{\pi}{\Lambda}$$

$$\rightarrow \omega_\pm^2 = \frac{K^2}{\mu(\varepsilon_0 \pm |\varepsilon_1|)}$$



Optical Waves in Crystals, A. Yariv and P. Yeh, 1984.

$\rightarrow$ Forbidden band between ω_+ and ω_-

One-Dimensional Periodic Media (5)

Forbidden band:

$$\rightarrow \omega^2 = \left(\frac{1}{2}g\right)^2 / \mu\varepsilon_0$$

$$\rightarrow K = \frac{1}{2}g + x, \quad |x| << \frac{1}{2}g$$

$$\rightarrow (K^2 - \omega^2\mu\varepsilon_0)\left[(K-g)^2 - \omega^2\mu\varepsilon_0\right] - \left(\omega^2\mu|\varepsilon_1|\right)^2 = 0$$

$$\rightarrow g^2x^2 + \left(\frac{|\varepsilon_1|}{\varepsilon_0}\right)^2\left(\frac{1}{4}g^2\right)^2 = 0$$

$$\rightarrow K = \frac{1}{2}g\left(1 \pm i\frac{|\varepsilon_1|}{2\varepsilon_0}\right) \quad \leftarrow \quad \Delta\omega_{gap} = |\omega_+ - \omega_-| = \omega\frac{|\varepsilon_1|}{\varepsilon_0}$$

$$\rightarrow K_i = \frac{1}{4}g\frac{\Delta\omega_{gap}}{\omega}$$

One-Dimensional Periodic Media (6)

Higher-order forbidden band:

$$|K - lg| \cong K, \quad l = \pm 1, \pm 2, \cdots$$

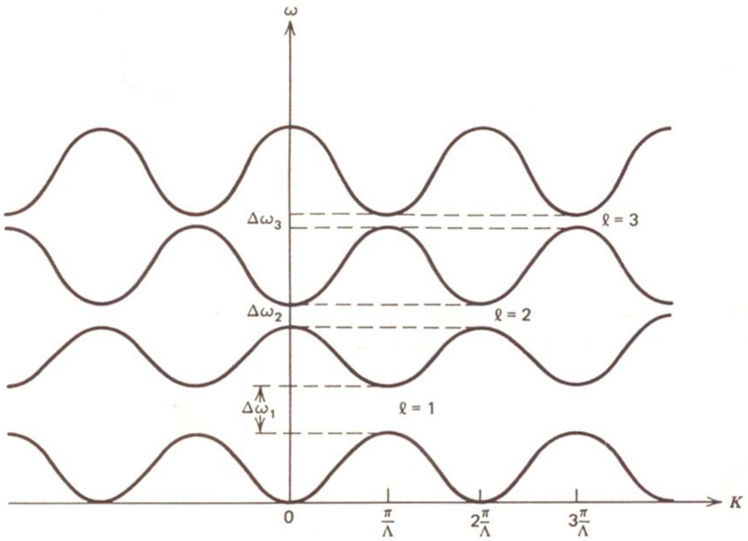
$$\& \; K^2 \cong \omega^2 \mu \varepsilon_0$$

$$\rightarrow \; (K^2 - \omega^2 \mu \varepsilon_0) A(K) - \omega^2 \mu \varepsilon_l A(K - lg) = 0$$

$$\rightarrow \; -\omega^2 \mu \varepsilon_{-l} A(K) + [(K - lg)^2 - \omega^2 \mu \varepsilon_0] A(K - lg) = 0$$

$$\rightarrow \; K = l\frac{g}{2} = l\frac{\pi}{\Lambda}$$

$$\rightarrow \; (\Delta\omega_{gap})_l = \omega \frac{|\varepsilon_l|}{\varepsilon_0}$$



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