

Electro-Optics:

Electromagnetic Propagation in Periodic Media (3)

Yoonchan Jeong

School of Electrical Engineering, Seoul National University

Tel: +82 (0)2 880 1623, Fax: +82 (0)2 873 9953

Email: yunchan@snu.ac.kr

Coupled-Mode Theory (1)

Dielectric tensor:

$$\varepsilon(x, y, z) = \varepsilon_0(x, y) + \Delta\varepsilon(x, y, z)$$

Unperturbed part

Perturbed (periodic) part

Normal modes:

$$\mathbf{E}_m(x, y)e^{i(\omega t - \beta_m z)}$$

$$\rightarrow \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \omega^2 \mu \varepsilon_0(x, y) - \beta_m^2 \right] \mathbf{E}_m(x, y) = 0$$

Electric-field for the “unperturbed” medium:

$$\mathbf{E} = \sum_m A_m \mathbf{E}_m(x, y)e^{i(\omega t - \beta_m z)}$$

Orthogonal relation of the normal modes:

TM modes? → H.W.

$$\frac{1}{2} \int (\mathbf{E}_k \times \mathbf{H}_l^*)_z dx dy = \delta_{lk} \quad \leftarrow \mathbf{a}_z \cdot \mathbf{E} = 0 \quad (\text{TE modes})$$

$$\rightarrow \int \mathbf{E}_k^*(x, y) \cdot \mathbf{E}_l(x, y) dx dy = \frac{2\omega\mu}{|\beta_k|} \delta_{lk}$$

Coupled-Mode Theory (2)

In the presence of the dielectric perturbation:

$$\Delta \mathbf{P} = \Delta \varepsilon(x, y, z) \mathbf{E}_l(x, y) e^{i(\omega t - \beta_l z)}$$

$$\rightarrow \mathbf{E} = \sum_m A_m(z) \mathbf{E}_m(x, y) e^{i(\omega t - \beta_m z)} \quad \leftarrow \text{Variation of constants}$$

Wave equation:

$$\left\{ \nabla^2 + \omega^2 \mu [\varepsilon_0(x, y) + \Delta \varepsilon(x, y, z)] \right\} \mathbf{E} = 0$$

$$\rightarrow \sum_k \left[\frac{d^2}{dz^2} A_k - 2i\beta_k \frac{d}{dz} A_k \right] \mathbf{E}_k(x, y) e^{-i\beta_k z} = -\omega^2 \mu \sum_l \Delta \varepsilon(x, y, z) A_l \mathbf{E}_l(x, y) e^{-i\beta_l z}$$

“Slowly varying amplitude”
 $\leftarrow \left| \frac{d^2}{dz^2} A_k \right| \ll \left| \beta_k \frac{d}{dz} A_k \right|$

$$\rightarrow -2i \sum_k \beta_k \left(\frac{d}{dz} A_k \right) \mathbf{E}_k(x, y) e^{-i\beta_k z} = -\omega^2 \mu \sum_l \Delta \varepsilon(x, y, z) A_l \mathbf{E}_l(x, y) e^{-i\beta_l z}$$

Coupled-Mode Theory (3)

Wave equation with the orthogonal relation:

$$\begin{aligned} \rightarrow -2i \sum_k \beta_k \left(\frac{d}{dz} A_k \right) \mathbf{E}_k(x, y) e^{-i\beta_k z} \\ = -\omega^2 \mu \sum_l \Delta \varepsilon(x, y, z) A_l \mathbf{E}_l(x, y) e^{-i\beta_l z} \end{aligned}$$

$$\rightarrow \langle k | k \rangle \frac{d}{dz} A_k(z) = \frac{\omega^2 \mu}{2i\beta_k} \sum_l \langle k | \Delta \varepsilon | l \rangle A_l(z) e^{i(\beta_k - \beta_l)z}$$

$$\leftarrow \langle k | k \rangle = \int \mathbf{E}_k^*(x, y) \cdot \mathbf{E}_k(x, y) dx dy = \frac{2\omega\mu}{|\beta_k|} \quad (\text{TE modes})$$

$$\leftarrow \langle k | \Delta \varepsilon | l \rangle = \int \mathbf{E}_k^*(x, y) \cdot \Delta \varepsilon(x, y, z) \mathbf{E}_l(x, y) dx dy$$

Dielectric perturbation in a Fourier series:

$$\Delta \varepsilon(x, y, z) = \sum_{m \neq 0} \varepsilon_m(x, y) \exp \left[-im \frac{2\pi}{\Lambda} z \right]$$

$$\begin{aligned} \rightarrow \frac{d}{dz} A_k(z) = -i \frac{\beta_k}{|\beta_k|} \sum_l \sum_m C_{kl}^{(m)} A_l(z) e^{i(\beta_k - \beta_l - m2\pi/\Lambda)z} \\ \leftarrow C_{kl}^{(m)} = \frac{\omega}{4} \langle k | \varepsilon_m | l \rangle \quad 4 \end{aligned}$$

Coupled-Mode Theory (4)

Resonant coupling:

$$\frac{d}{dz} A_k(z) = -i \frac{\beta_k}{|\beta_k|} \sum_l \sum_m C_{kl}^{(m)} A_l(z) e^{i(\beta_k - \beta_l - m2\pi/\Lambda)z}$$
$$\rightarrow \beta_k - \beta_l - m \frac{2\pi}{\Lambda} = 0$$

Coupled-mode equations:

$$\frac{d}{dz} A_1 = -i \frac{\beta_1}{|\beta_1|} C_{12}^{(m)} A_2 e^{i\Delta\beta z}$$

$$\frac{d}{dz} A_2 = -i \frac{\beta_2}{|\beta_2|} C_{21}^{(-m)} A_1 e^{-i\Delta\beta z} \quad \leftarrow \Delta\beta = \beta_1 - \beta_2 - m \frac{2\pi}{\Lambda}$$

$$\leftarrow C_{12}^{(m)} = [C_{21}^{(-m)}]^*$$

For plane waves:

$$\rightarrow C_{kl}^{(m)} = \frac{\omega^2 \mu}{2\sqrt{|\beta_k \beta_l|}} \mathbf{p}_k^* \cdot \boldsymbol{\varepsilon}_m \mathbf{p}_l \quad \leftarrow \mathbf{p}_k, \mathbf{p}_l: \text{Unit pol. vectors}$$

Codirectional Coupling

Coupled-mode equations:

$$\frac{d}{dz} A_1 = -i\kappa A_2 e^{i\Delta\beta z}$$

$$\frac{d}{dz} A_2 = -i\kappa^* A_1 e^{-i\Delta\beta z} \leftarrow \kappa = C_{12}^{(m)}$$

$$\rightarrow \frac{d}{dz} \left\{ |A_1|^2 + |A_2|^2 \right\} = 0 \quad \text{“Conservation of energy”}$$

General solution:

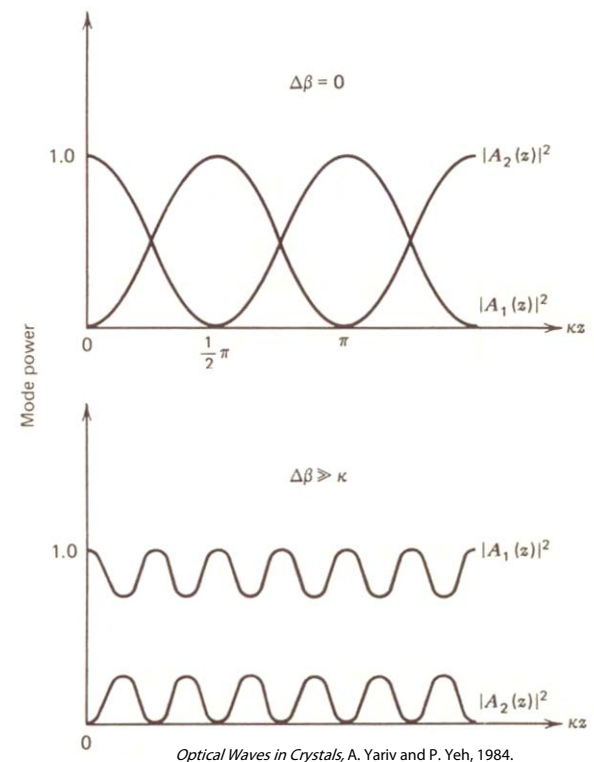
$$A_1(z) = e^{i(\Delta\beta/2)z} \left\{ \left[\cos sz - i \frac{\Delta\beta}{2s} \sin sz \right] A_1(0) - i \frac{\kappa}{s} \sin sz A_2(0) \right\}$$

$$A_2(z) = e^{-i(\Delta\beta/2)z} \left\{ -i \frac{\kappa^*}{s} \sin sz A_1(0) + \left[\cos sz + i \frac{\Delta\beta}{2s} \sin sz \right] A_2(0) \right\}$$

$$\leftarrow s^2 = \kappa\kappa^* + \left(\frac{\Delta\beta}{2} \right)^2$$

Fraction of power exchange:

$$\rightarrow \frac{|\kappa|^2}{|\kappa|^2 + (\Delta\beta/2)^2} \sin^2 \sqrt{|\kappa|^2 + \left(\frac{\Delta\beta}{2} \right)^2} z \quad \leftarrow \Delta\beta = 0$$



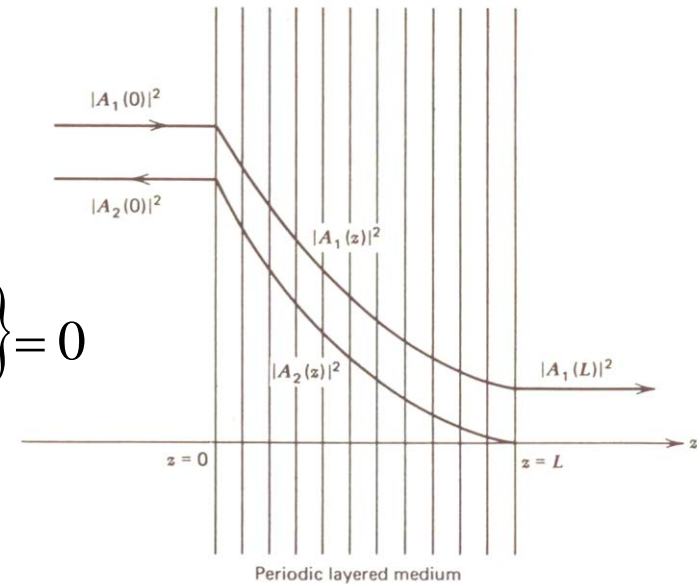
Contradirectional Coupling

Coupled-mode equations:

$$\frac{d}{dz} A_1 = -i\kappa A_2 e^{i\Delta\beta z} \rightarrow \frac{d}{dz} \left\{ |A_1|^2 - |A_2|^2 \right\} = 0$$

$$\frac{d}{dz} A_2 = i\kappa^* A_1 e^{-i\Delta\beta z}$$

“Conservation of energy”



Optical Waves in Crystals, A. Yariv and P. Yeh, 1984.

General solution:

$$A_1(z) = e^{i(\Delta\beta/2)z} \left\{ \frac{s \cosh s(L-z) + i(\Delta\beta/2) \sinh s(L-z)}{s \cosh sL + i(\Delta\beta/2) \sinh sL} A_1(0) + \frac{-i\kappa e^{i(\Delta\beta/2)L} \sinh sz}{s \cosh sL + i(\Delta\beta/2) \sinh sL} A_2(L) \right\}$$

$$A_2(z) = e^{-i(\Delta\beta/2)z} \left\{ \frac{-i\kappa^* \sinh s(L-z)}{s \cosh sL + i(\Delta\beta/2) \sinh sL} A_1(0) + e^{i(\Delta\beta/2)L} \frac{s \cosh sz + i(\Delta\beta/2) \sinh sz}{s \cosh sL + i(\Delta\beta/2) \sinh sL} A_2(L) \right\}$$

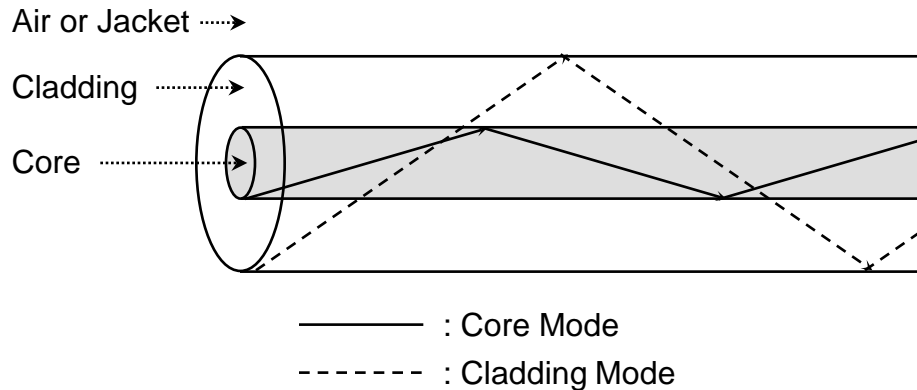
$$\leftarrow s^2 = \kappa\kappa^* - \left(\frac{\Delta\beta}{2} \right)^2$$

Fraction of power exchange:

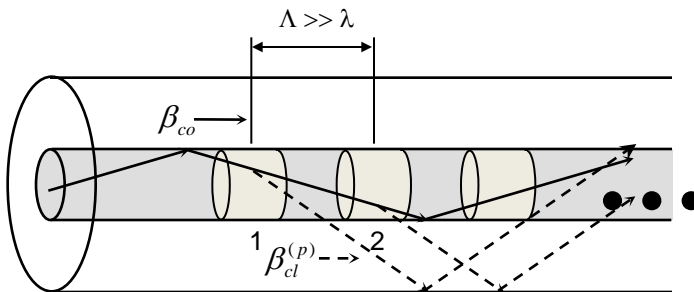
$$\rightarrow \frac{|\kappa|^2 \sinh^2 sL}{s^2 \cosh^2 sL + (\Delta\beta/2)^2 \sinh^2 sL} \quad \leftarrow \Delta\beta = 0$$

Applications to Fiber Grating Analysis (1)

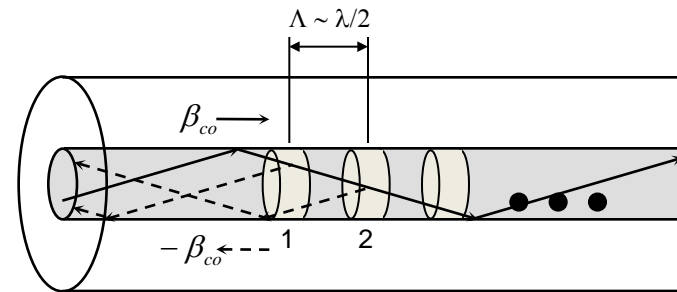
Fiber modes:



Codirectional coupling:

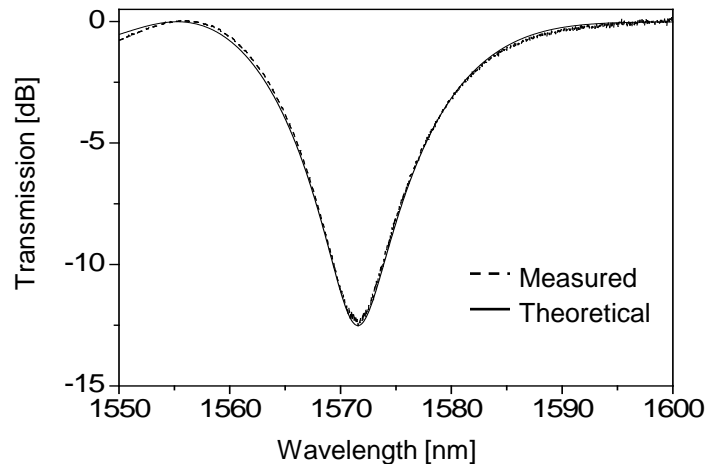
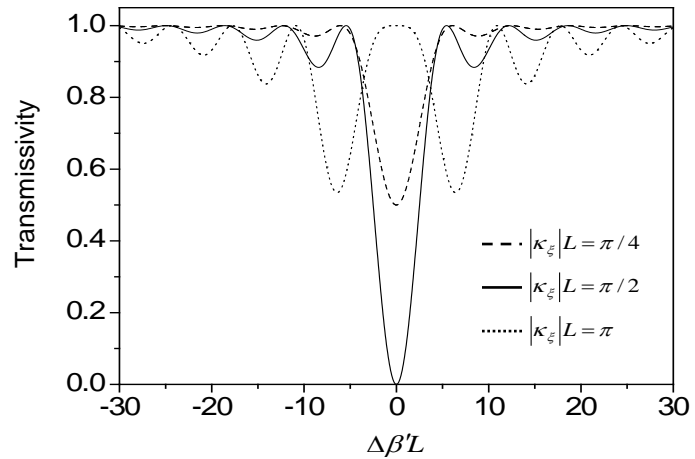


Contradirectional coupling:



Applications to Fiber Grating Analysis (2)

Codirectional coupling:



Contradirectional coupling:

