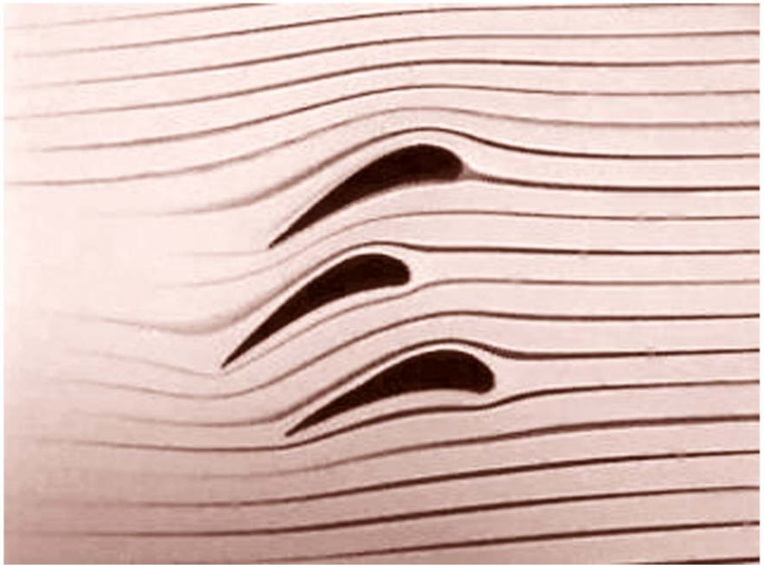
Chapter 2

Kinematics



Chapter 2 Kinematics

Contents

Objectives

- Define methods of flow description
- Classify fluid motions
- Study kinematics of fluid

$$\sum \vec{F} = m\vec{a}$$

2.1 The Velocity Field

velocity, acceleration ~ vector quantities

$\vec{q}$ $\vec{a}$

Cartesian coordinates

x y z

u v w

a_x a_y a_z

2.1 The Velocity Field

2.1.1 Lagrangian approach

~ coordinates of moving particles are represented as function of time

~ follow a particular particle through the flow field → *path line*

At $t = t_0$ coordinates (position) of a particle (a, b, c)

At $t = t$ position of a particle (x, y, z)

$$x = f_1\underline{(a, b, c, t)}$$

$$y = f_2(a, b, c, t)$$

$$z = f_3(a, b, c, t)$$

Independent variables

2.1 The Velocity Field

$$u = \frac{\partial x}{\partial t} \qquad a_x = \frac{\partial u}{\partial t} = \frac{\partial^2 x}{\partial t^2}$$

$$v = \frac{\partial y}{\partial t} \qquad a_y = \frac{\partial v}{\partial t} = \frac{\partial^2 y}{\partial t^2}$$

$$w = \frac{\partial z}{\partial t} \qquad a_z = \frac{\partial w}{\partial t} = \frac{\partial^2 z}{\partial t^2}$$

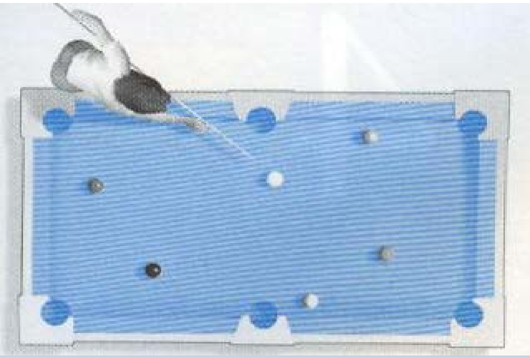
2.1 The Velocity Field

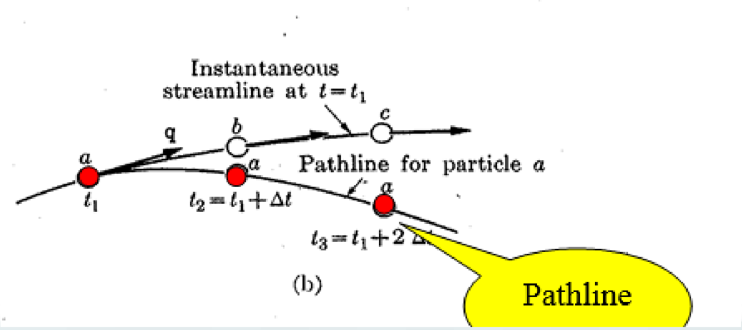
~ commonly used in the solid dynamics

~ convenient to identify a discrete particle, e.g. center of mass of spring - mass system

~ cumbersome when dealing with a fluid as a continuum of particles

→ Due to deformation of fluid, we are not usually concerned with the detailed history of an individual particle, but rather with interrelation of flow properties at individual points in the flow field.





2.1 The Velocity Field

2.1.2 Eulerian method

~ observer fixes attention at discrete points

~ notes flow characteristics in the vicinity of a fixed point as particles pass by

~ focused on the fluid which passes through a control volume that is fixed in space

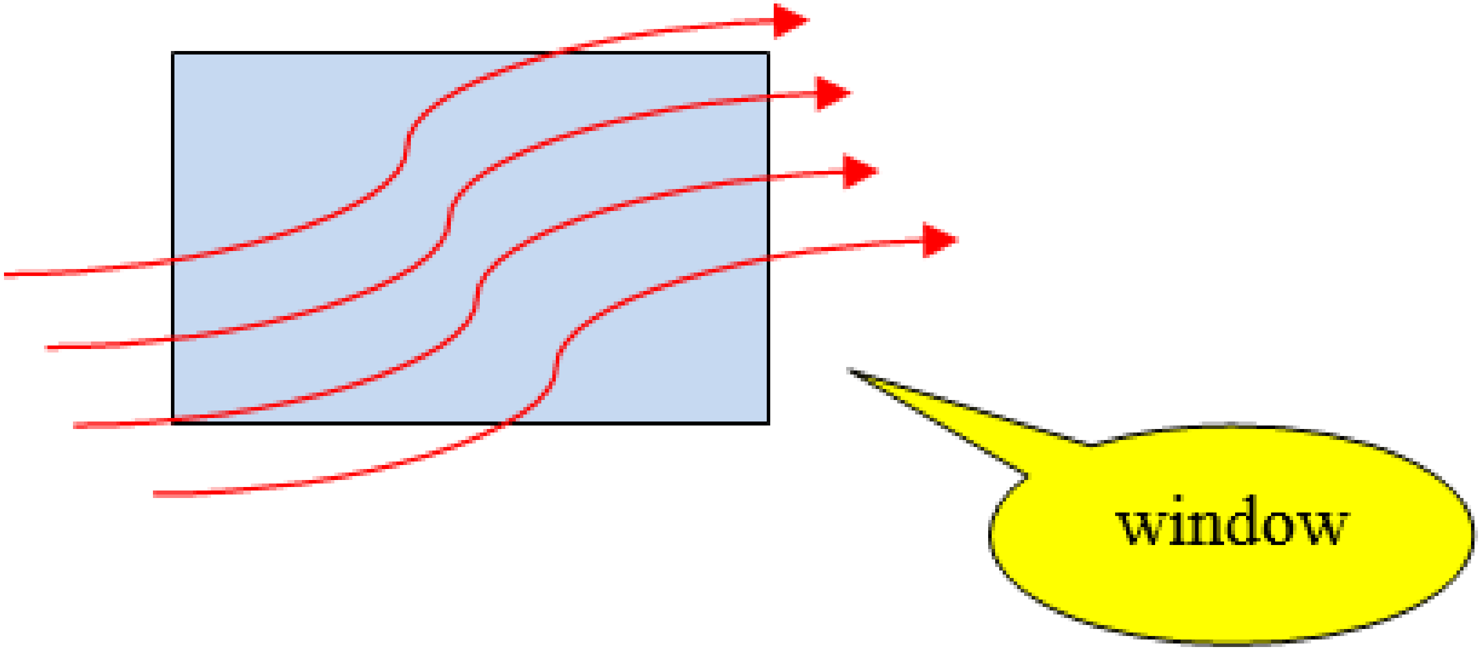
~ familiar framework in which most fluid problems are solved

~ instantaneous picture of the velocities and accelerations of every particle

→ streamline

~ velocities at various points are given as function of time

2.1 The Velocity Field



2.1 The Velocity Field

$$\vec{q} = \vec{i}u + \vec{j}v + \vec{k}w$$

where $u = f_1\underline{(x, y, z, t)}$

$$v = f_2(x, y, z, t)$$

Independent variables

$$w = f_3(x, y, z, t)$$

$x,\ y,\ z,\ t\ =$ independent variables

$\vec{i},\ \vec{j},\ \vec{k}\ =$ unit vectors

2.1 The Velocity Field

2.1.3 Total Derivative

(1) Total change in velocity

= sum of partial derivatives of the four independent variables, x, y, z, t

$$x-\mathrm{dir} \quad : \quad du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

total derivative: $$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= \underbrace{\frac{\partial u}{\partial t}}_{\text{local change due to unsteadiness}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}_{\text{convective change due to translation}}$$

2.1 The Velocity Field

$$y-\text{dir} \quad : \quad \frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}$$

$$z-\text{dir} \quad : \quad \frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}$$

(2) Total rate of density change of compressible fluid

$$\rho = \rho(x,y,z,t)$$

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = \frac{\partial \rho}{\partial t} + u_j\frac{\partial \rho}{\partial x_j}$$

For incompressible fluid, $\frac{d\rho}{dt} = 0$

For steady flow, $\frac{\partial \rho}{\partial t} = 0$

2.1 The Velocity Field

Acceleration

$$a_x = \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = \frac{\partial u}{\partial t} + u_j\frac{\partial u}{\partial x_j}$$

$$a_y = \frac{dv}{dt} = \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = \frac{\partial v}{\partial t} + u_j\frac{\partial v}{\partial x_j}$$

$$a_z = \frac{dw}{dt} = \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = \frac{\partial w}{\partial t} + u_j\frac{\partial w}{\partial x_j}$$

Vector notation

$$\vec{a} = \vec{i}a_x + \vec{j}a_y + \vec{k}a_z$$

$$\vec{a} = \frac{d\vec{q}}{dt} = \underbrace{\frac{\partial \vec{q}}{\partial t}}_{\text{local acceleration}} + \underbrace{(\vec{q}\cdot\nabla)\,\vec{q}}_{\text{convective acceleration}}$$

2.2 Steady versus Uniform motion

i) steady motion: no changes with time at fixed point ⟷ unsteady motion

$\frac{\partial \vec{q}}{\partial t} = 0 \rightarrow$ local acceleration = 0

ii) uniform motion: no changes with space ⟷ non-uniform motion

$(\vec{q} \cdot \nabla)\vec{q} = 0 \rightarrow$ convective acceleration = 0

2.2 Steady versus Uniform motion

- Vector differential operators: ∇ → "del" or "nabla"

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

Gradient: $\nabla f = grad\ f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$

Divergence: $\nabla \cdot \vec{q} = div\ \vec{q} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$

[Re] Vector product

i) dot product → scalar

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\phi$$

2.2 Steady versus Uniform motion

ϕ = angle between the vectors

$$\vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = \vec{k}\cdot\vec{k} = 1 \qquad (\cos 0^\circ = 1)$$

$$\vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{k} = \vec{j}\cdot\vec{i} = \vec{k}\cdot\vec{j} = 0 \qquad (\because \cos 90^\circ = 0)$$

ii) cross product → vector

$$\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\phi$$

Direction = perpendicular to the plane of $\vec{a}\ and\ \vec{b}$ → right-hand rule

$$\vec{q}\cdot\nabla = (\vec{i}u + \vec{j}v + \vec{k}w)\cdot(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z})$$

$$= u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

2.2 Steady versus Uniform motion

$$(\vec{q}\cdot\nabla)\,\vec{q} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)(\vec{i}u + \vec{j}v + \vec{k}w)$$

$$= \left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)\vec{i}$$

$$+ \left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)\vec{j}$$

$$+ \left(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)\vec{k}$$

$$\nabla^2 = \nabla\cdot\nabla = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

2.2 Steady versus Uniform motion

$$\nabla^2 \phi = 0 \quad \rightarrow \quad \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$ → Laplace Eq.

$$grad\ (u + v) = \nabla (u + v) = \nabla u + \nabla v$$

$$div\ (\vec{u} + \vec{v}) = \nabla \cdot (\vec{u} + \vec{v}) = \nabla \cdot \vec{u} + \nabla \cdot \vec{v}$$

$$grad\ (uv) = \nabla (uv) = v \nabla u + u \nabla v$$

$$div\ (u\vec{v}) = \nabla \cdot (u\vec{v}) = \nabla u \cdot \vec{v} + u \nabla \cdot \vec{v}$$

$$div\ grad\ u = \nabla \cdot \nabla u = \nabla^2 u$$

2.3 Flow Lines

2.3.1 Flow lines

3 flow lines: streamline, path line, streak line

(1) **streamline**

= imaginary line connecting a series of points in space at a given instant in such a manner that all particles falling on the line at that instant have velocities whose vectors are tangent to the line

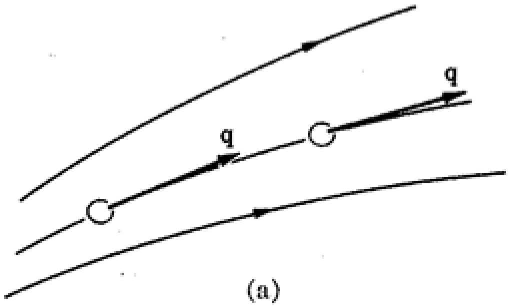
= instantaneous curves which are everywhere tangent to the velocity vector

= a line that is (at a given instant) tangent to the velocity at every point on it

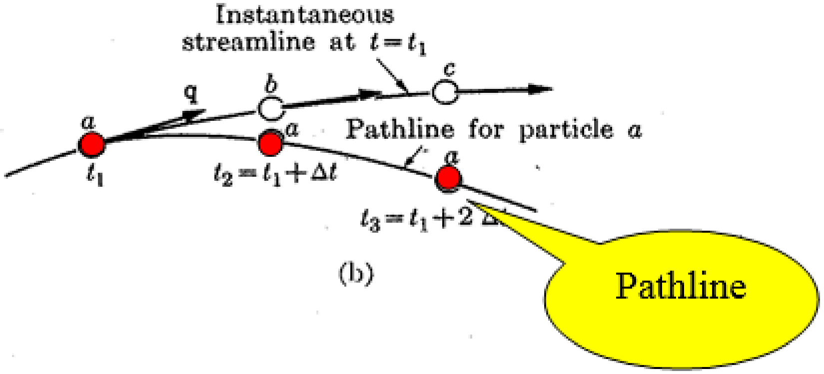
* stream tube = small imaginary tube bounded by streamlines

* stream filament = if cross section of stream tube is infinitesimally small

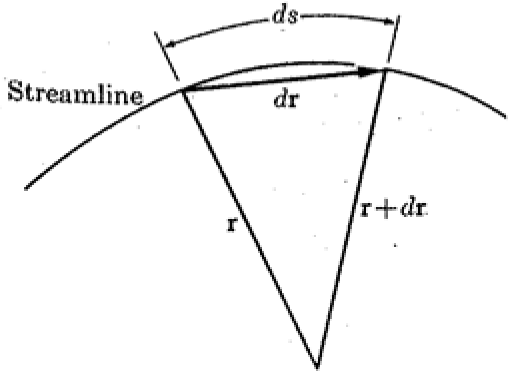
2.3 Flow Lines



(a)



(b)



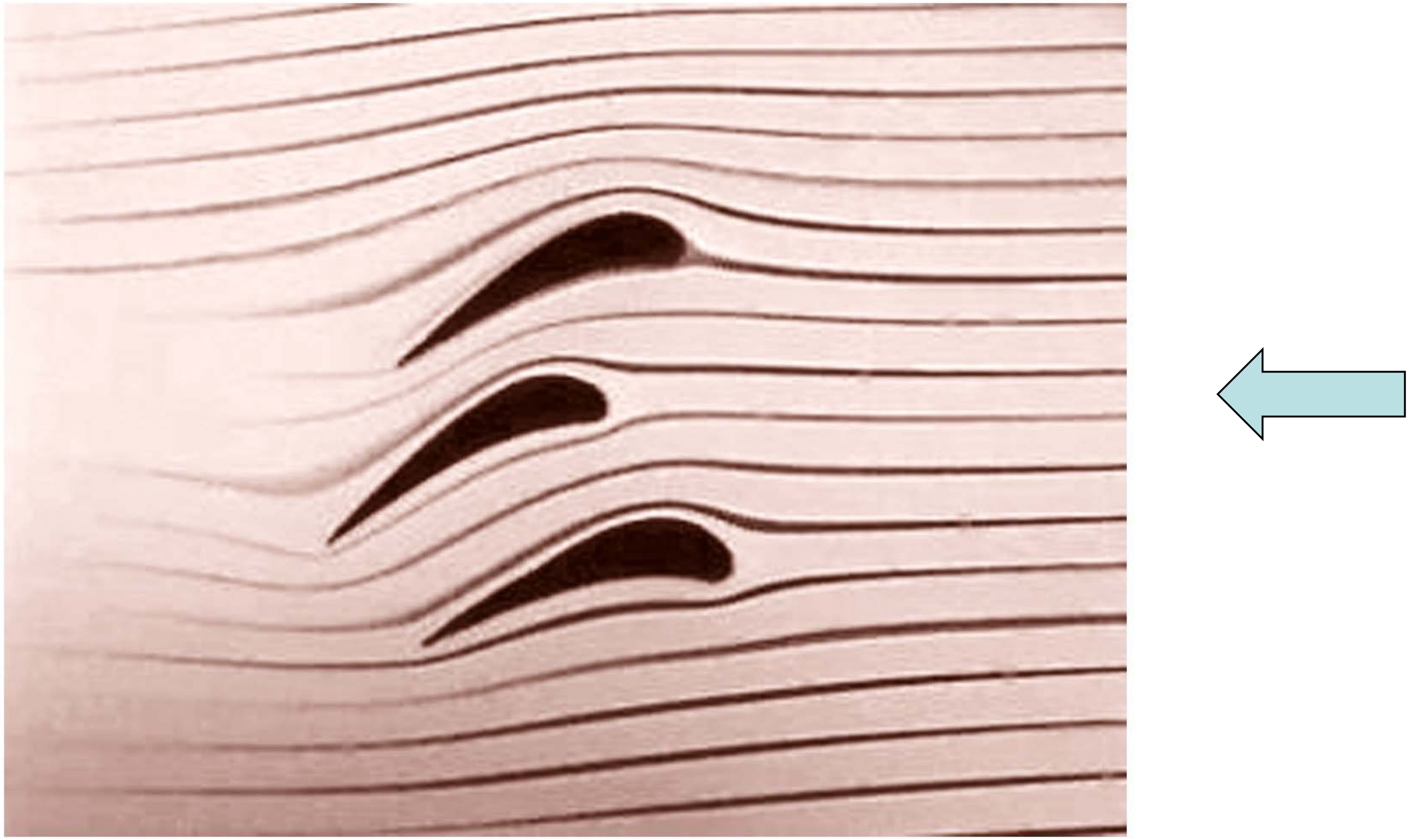
(c)

$d\mathbf{r} = \mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz$

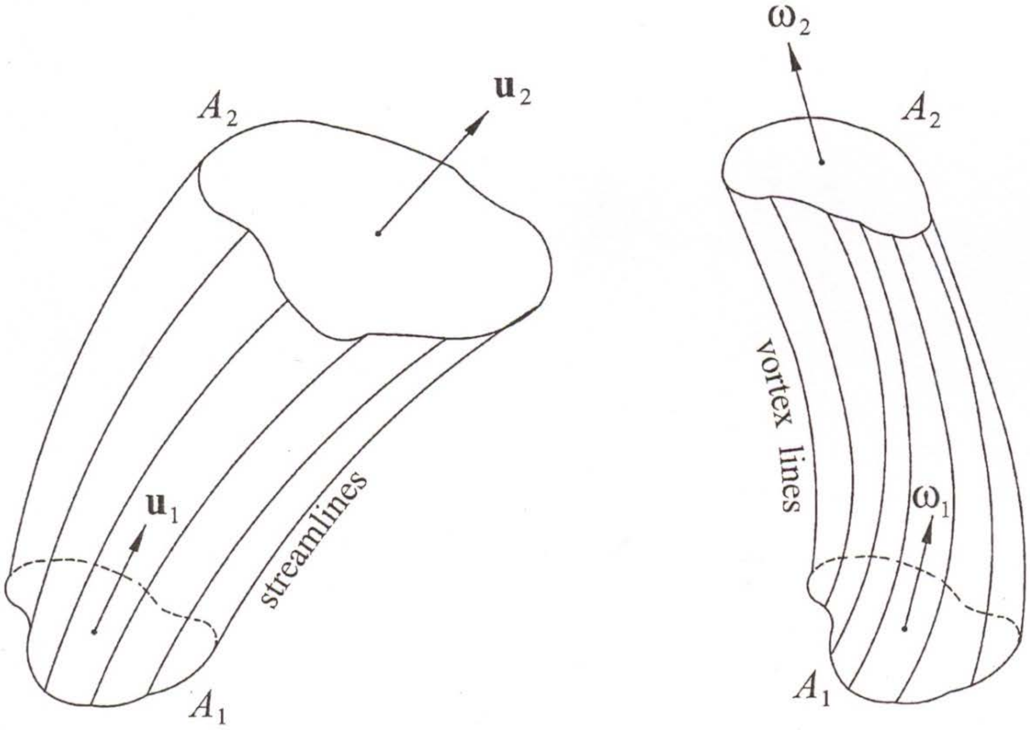
In limit $d\mathbf{r}/ds = \mathbf{n}$ = unit tangent vector

$d\mathbf{r} = \mathbf{n}\,ds$ = element of length along streamline

2.3 Flow Lines



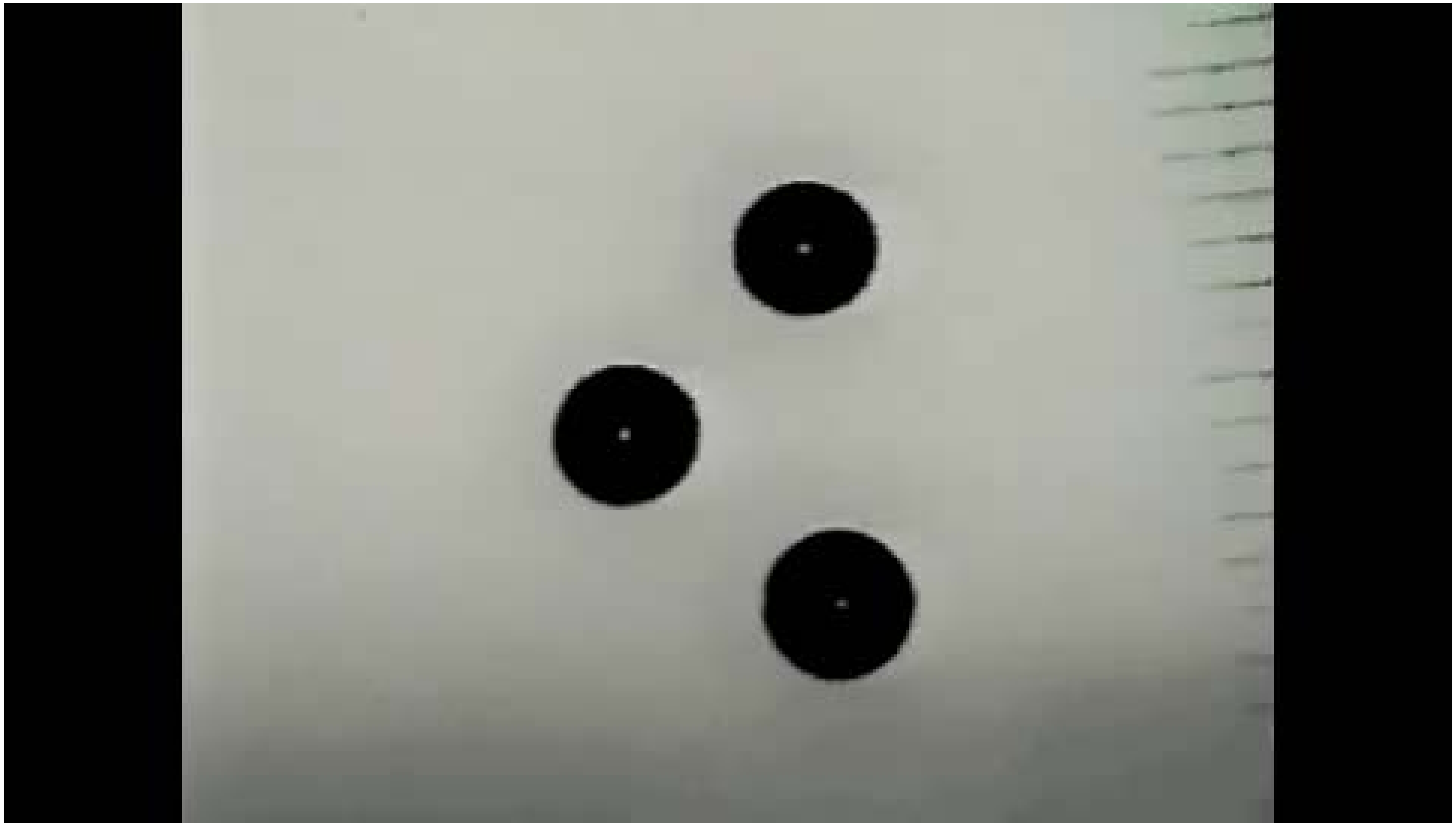
2.3 Flow Lines



(a)

(b)

2.3 Flow Lines



2.3 Flow Lines

(2) **path line**

= trajectory of a particle of fixed identity as time passes

(3) **streak line**

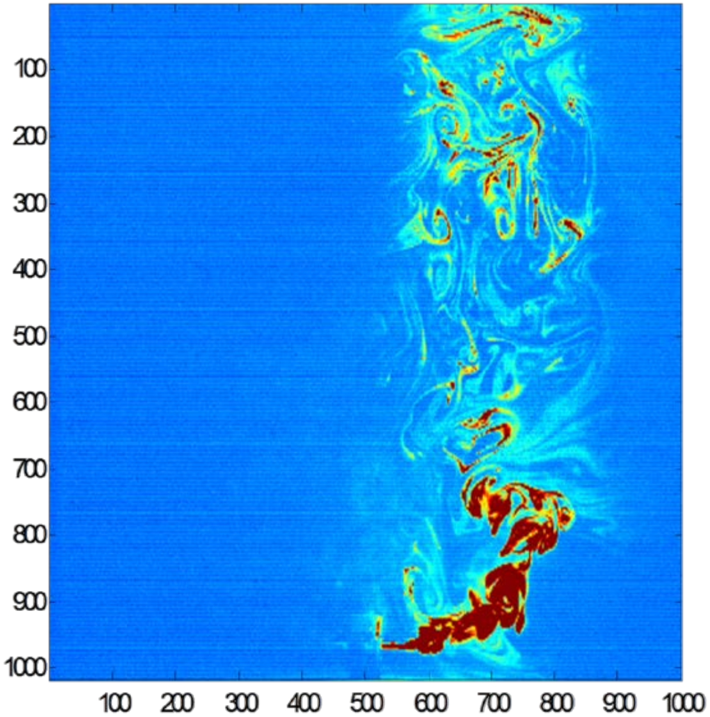
= a line connecting all the particles that have passed successfully through a particular given point (injection point)

= current location of all particles which have passed through a fixed point in space

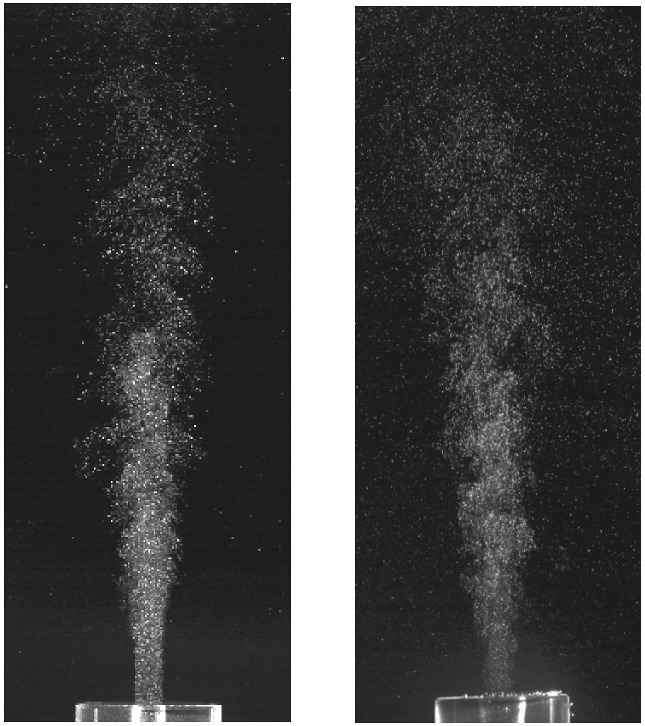
[Ex] dye stream in water, smoke filament in air

* For steady flow, streamline = path line = streak line

2.3 Flow Lines

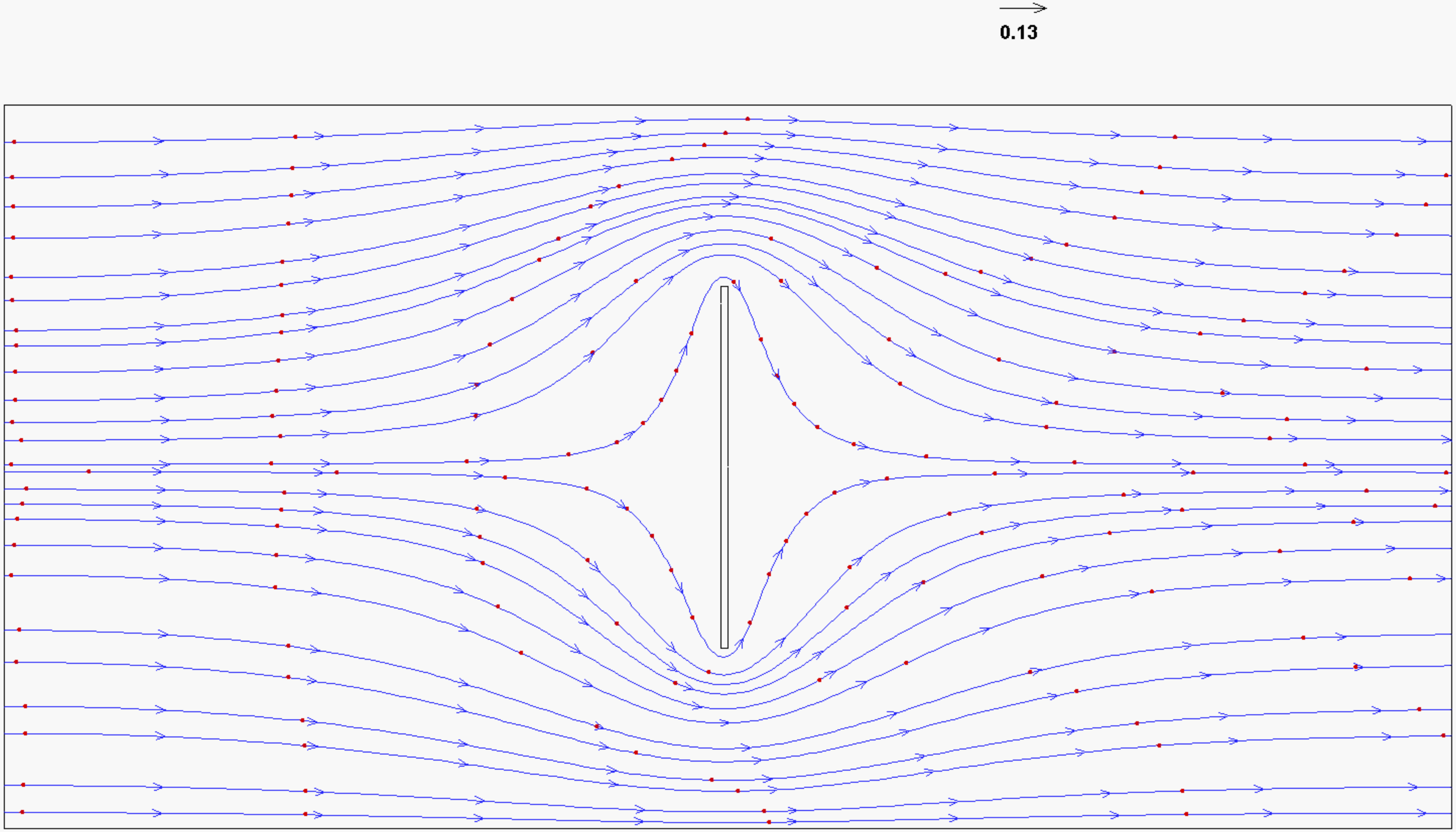


Streak line by LIF



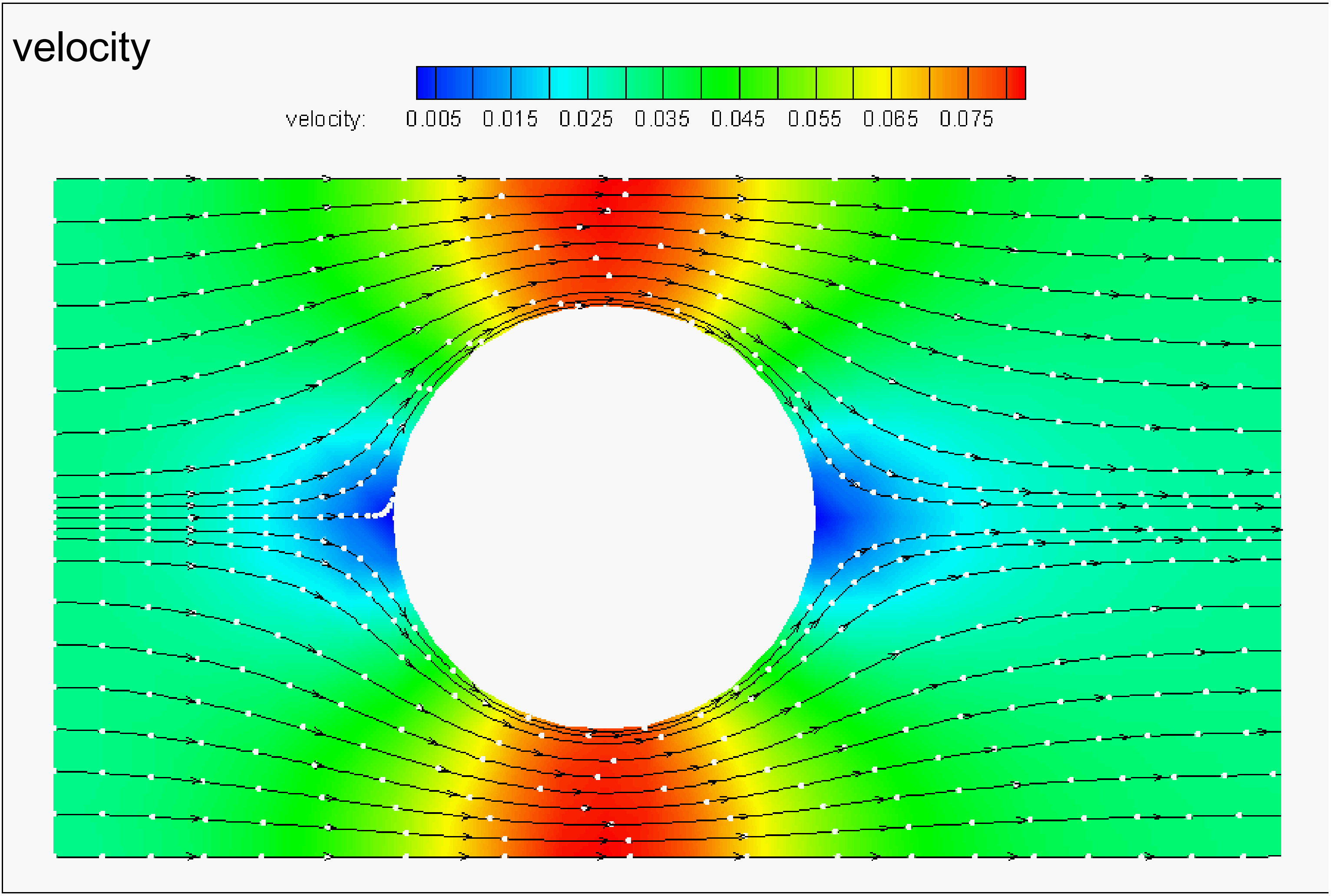
Path line by PIV

2.3 Flow Lines

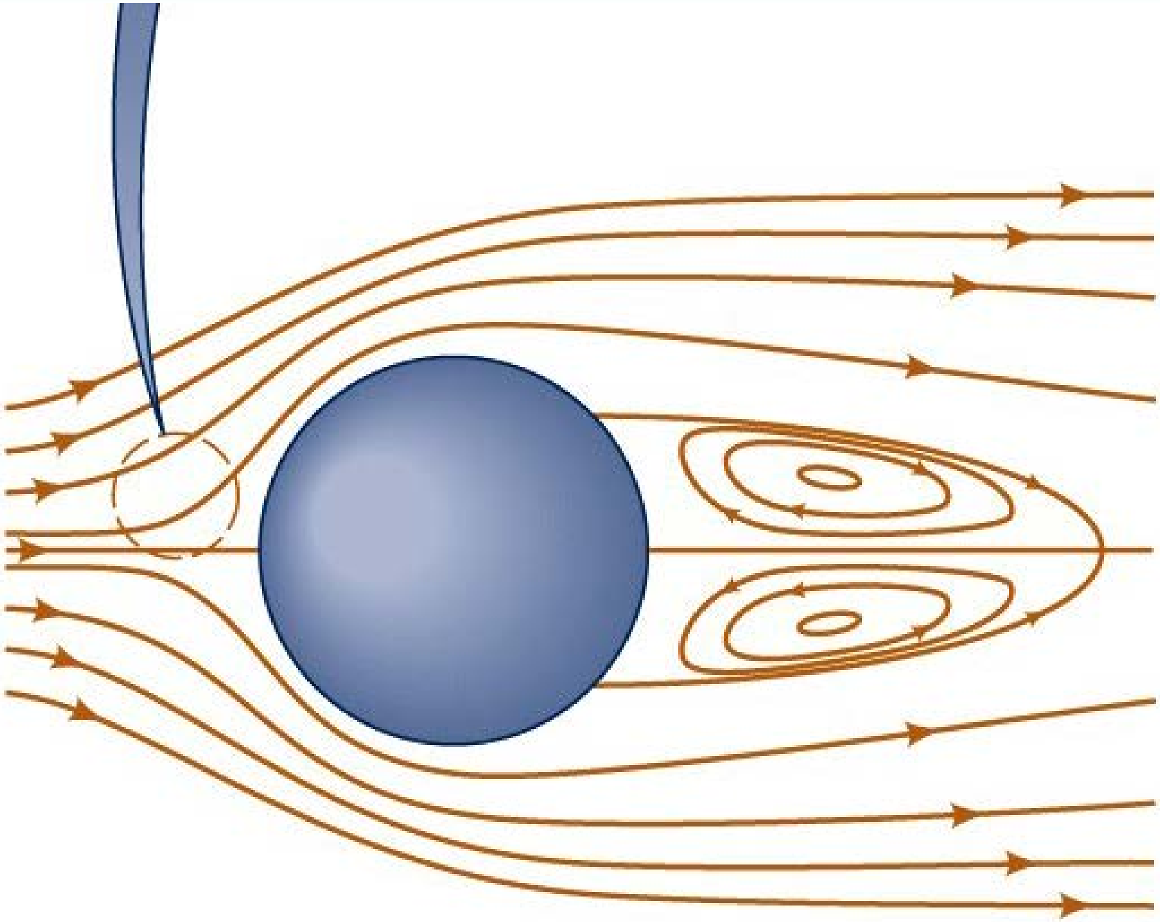


2.3 Flow Lines

At low velocity



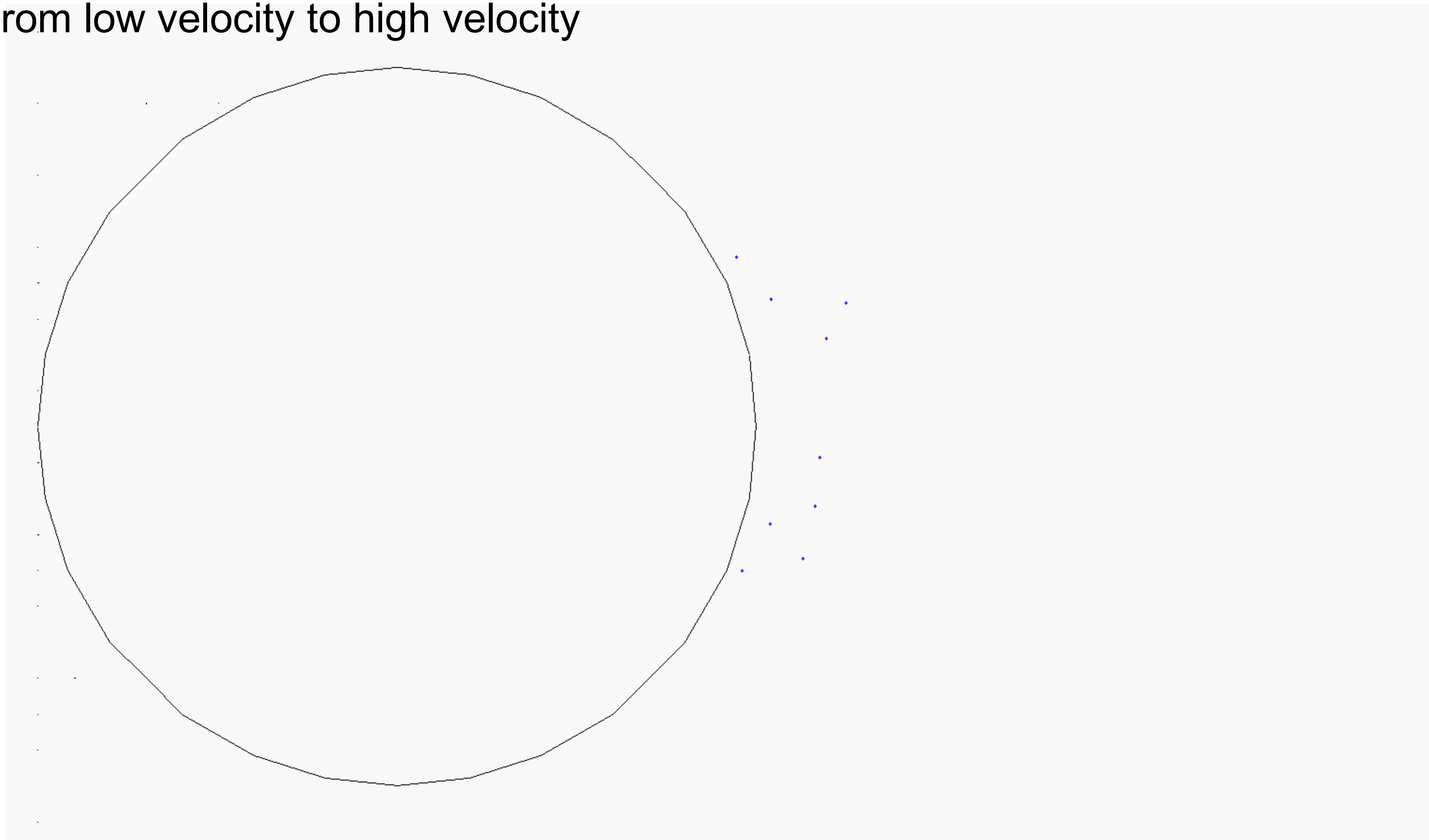
2.3 Flow Lines



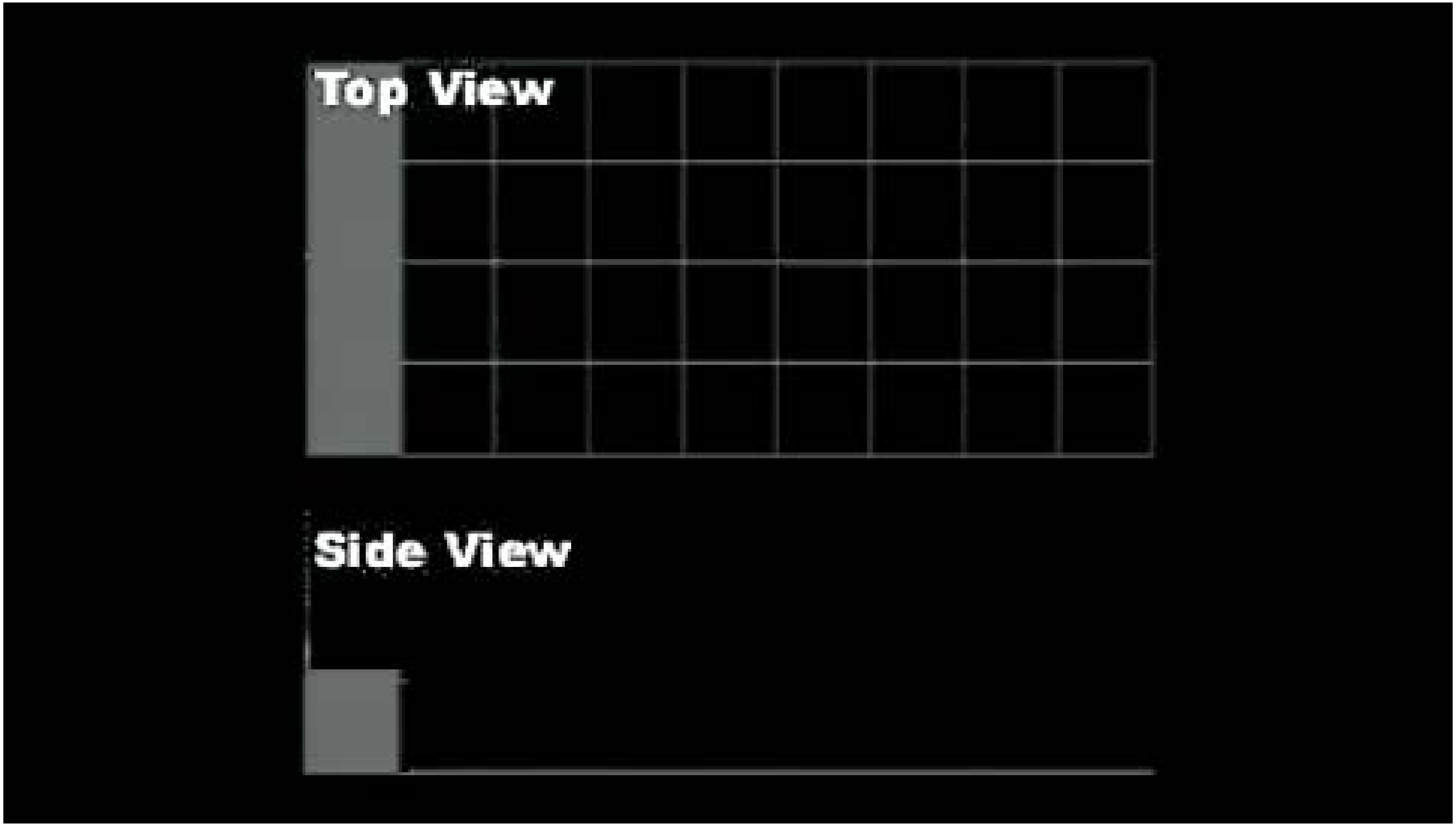
At high velocity

2.3 Flow Lines

From low velocity to high velocity

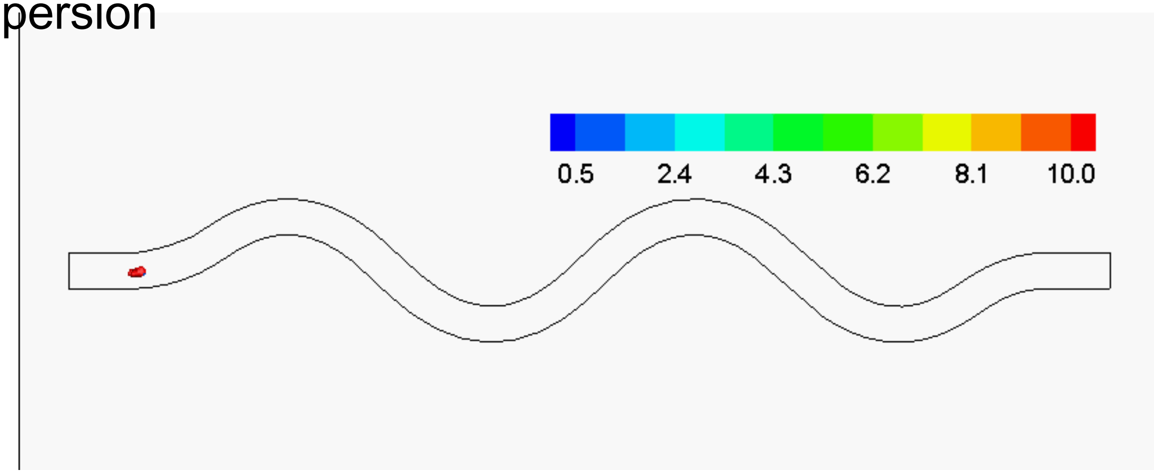


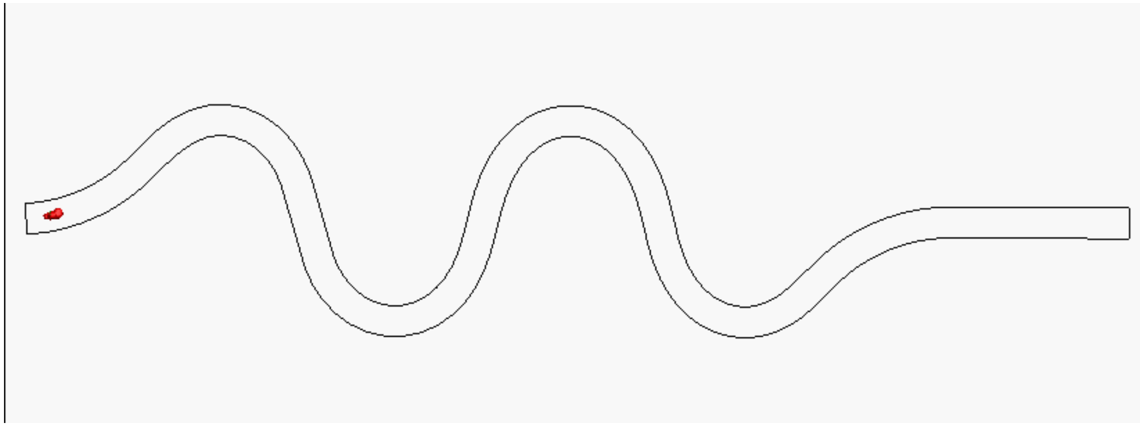
2.3 Flow Lines



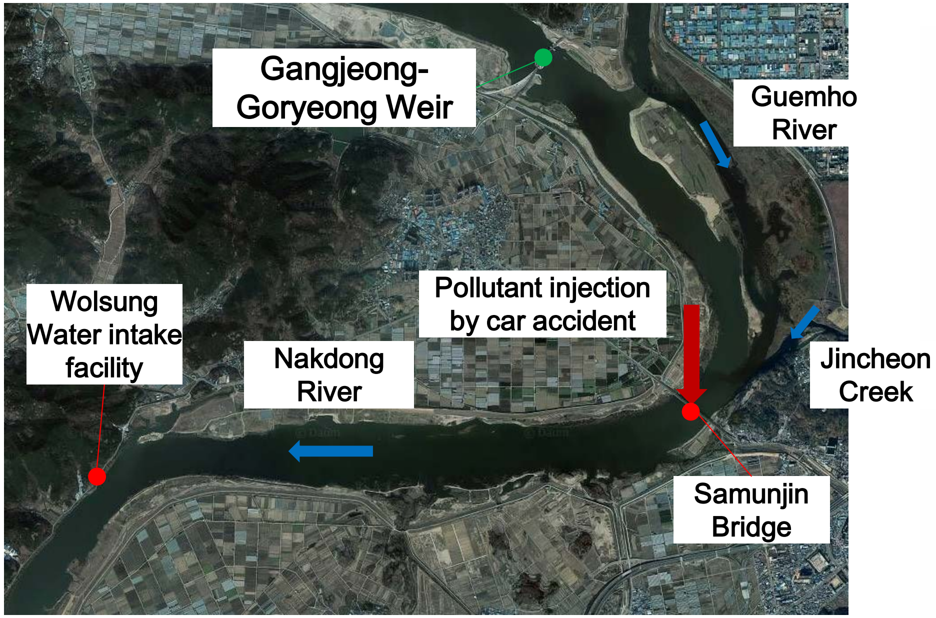
2.3 Flow Lines

Particle dispersion

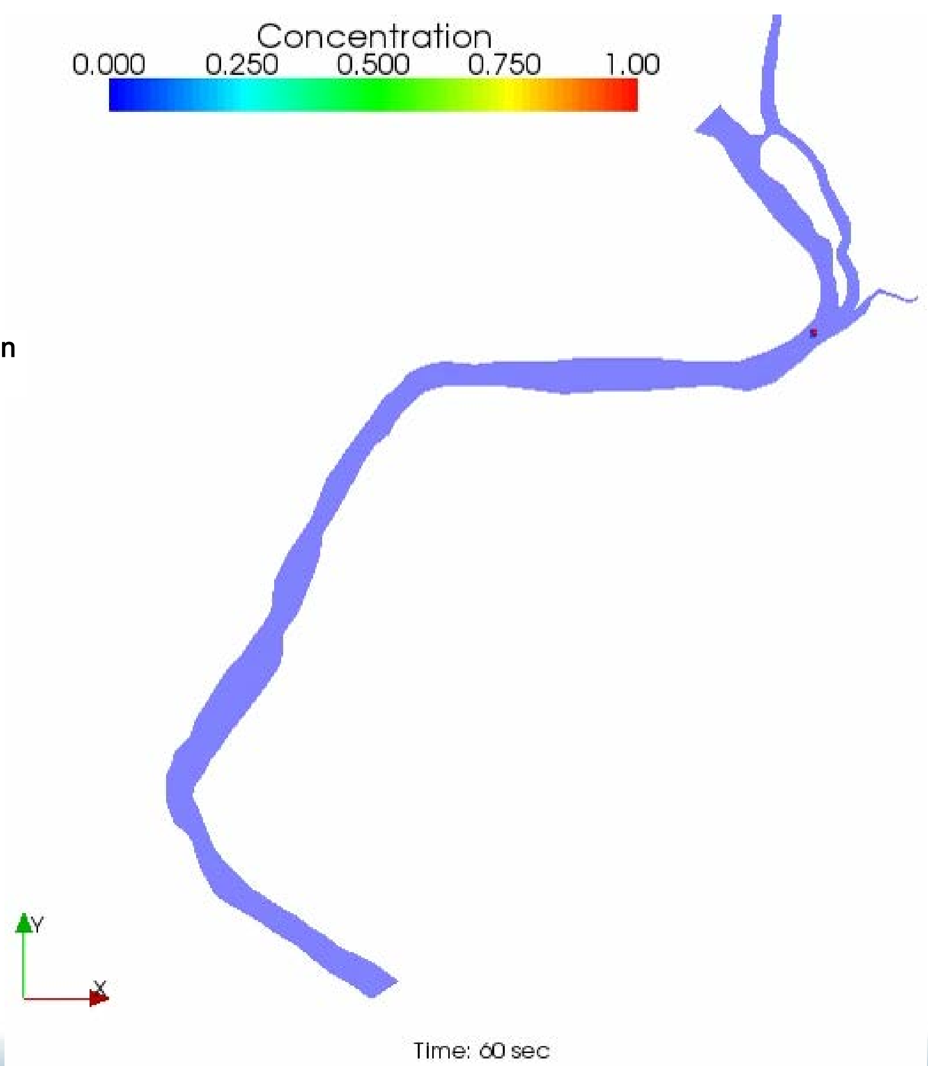




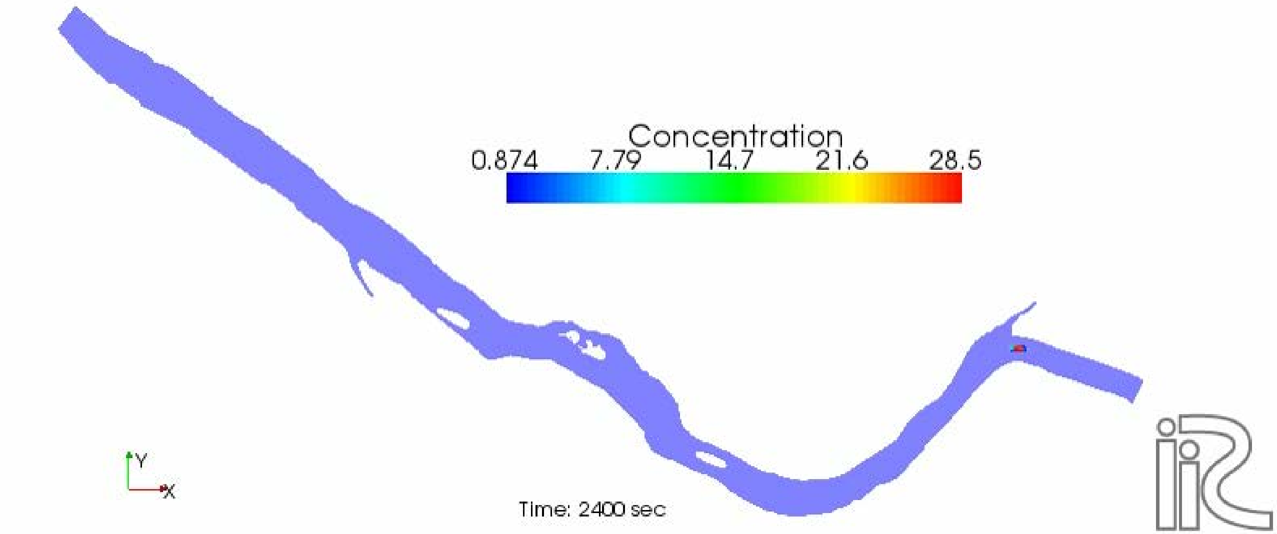
PDM-2D Results: Nakdong River

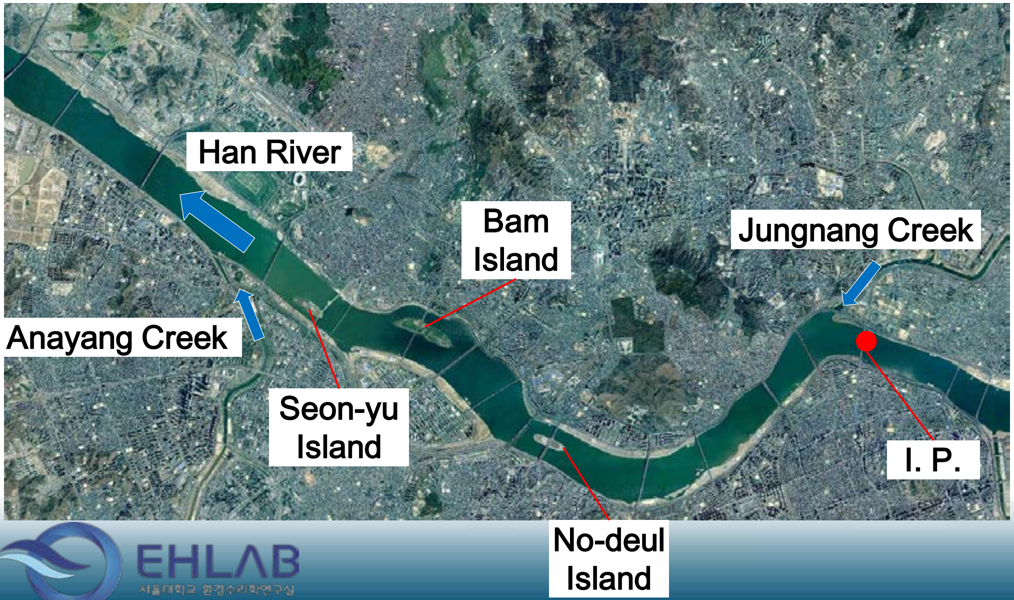


Study area	Gangjeong-Goryeong Weir ~ Dalsung Weir
Discharge	Nakdong River : 410.0 m^3/s Guemho River : 64.5 m^3/s Jincheon Creek : 6.9 m^3/s
Pollutant type	Conservative pollutant
Mass	1 ton
No. of particles	5,000



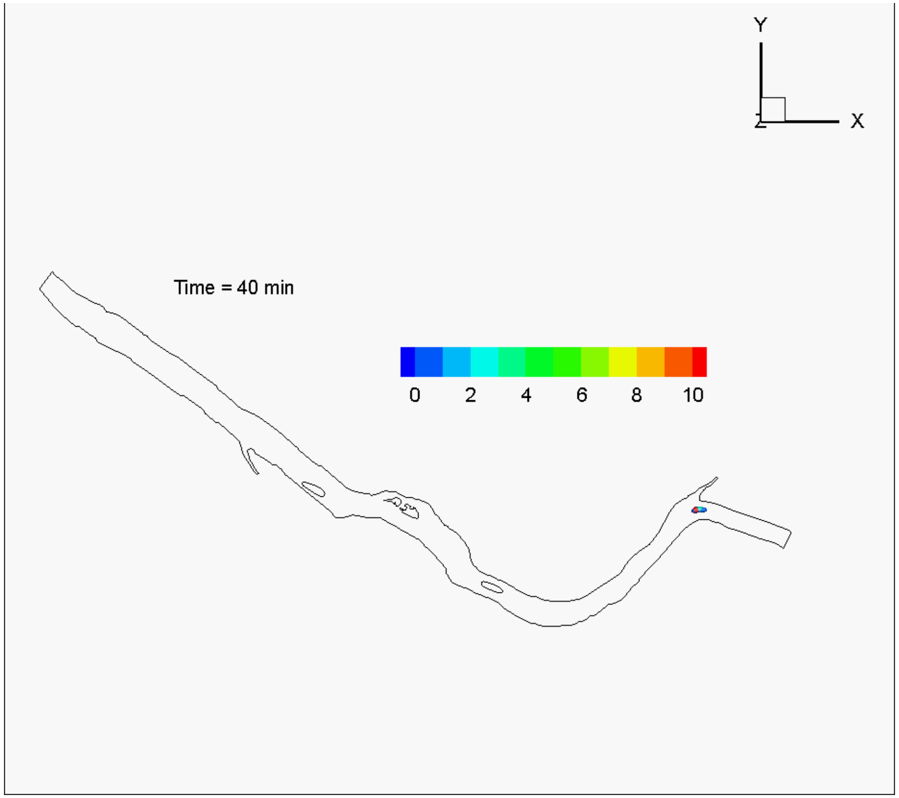
PDM-2D Results: Han River



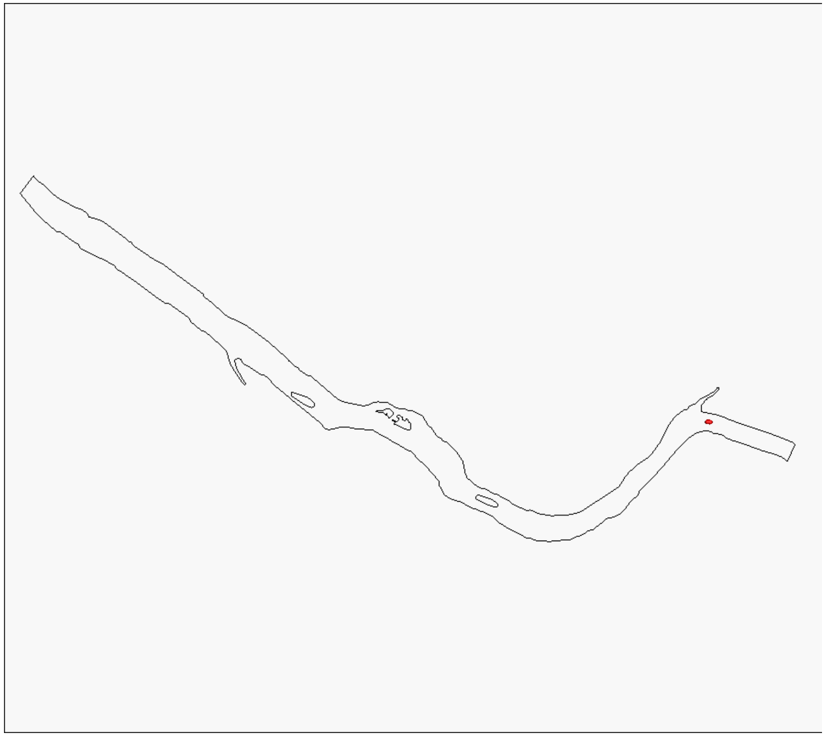


Study area	**Youngdong Bridge ~ Hangju Bridge**
Discharge	Han River : 183.9 m³/s Jungnang Creek : 1.4 m³/s Anyang Creek : 2.2 m³/s
Pollutant type	Conservative pollutant
Mass	1 ton
No. of particles	5,000

2.3 Flow Lines



No Tide



With Tide

2.3 Flow Lines

◈ How can we photo 3 lines?

(1) streamline: spread bunch of reflectors on the flow field, then take a instant shot

(2) path line: put only one particle on the flow field, then take long-time exposure

(3) streak line: take a instant shot of dye injecting from one slot of the dye tanks

2.3 Flow Lines

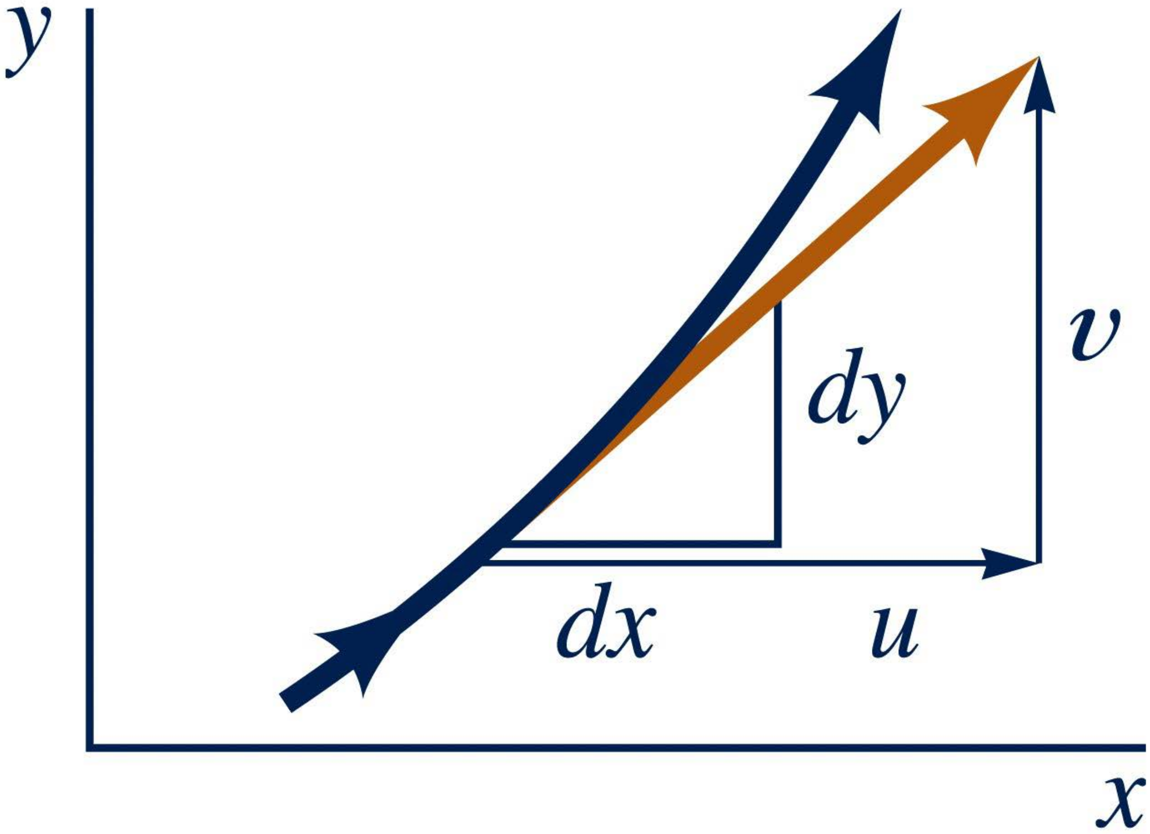
2.3.2 Differential equations for flow lines

(1) **Streamline**

By virtue of definition of a streamline (velocity vector $\vec{q}$ is tangent to the streamline), it's slope in the xy - plane, $\frac{dy}{dx}$, must be equal to that of the velocity, $\frac{v}{u}$.

$$\frac{dy}{dx} = \frac{v}{u}$$

2.3 Flow Lines



2.3 Flow Lines

By similarly treating the projections on the xz plane and on the yz plane

$$\frac{dz}{dx} = \frac{w}{u}; \qquad \frac{dz}{dy} = \frac{w}{v}$$

$$\rightarrow \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

→ Integration of the differential equation for streamline yields equation of streamline. For 2-D Cartesian coordinates

$$\frac{dx}{u} = \frac{dy}{v} \rightarrow \frac{dy}{dx} = \frac{v}{u} \rightarrow v\,dx - u\,dy = 0$$

2.3 Flow Lines

- Vector form of equation of streamline

$$\boxed{\vec{q} \times d\vec{r} = 0}$$

$$\vec{q} = \vec{i}u + \vec{j}v + \vec{k}w$$

$$d\vec{r} = \vec{i}\,dx + \vec{j}\,dy + \vec{k}\,dz$$

$= \vec{n}\,ds =$ element of length along streamline

$$\vec{q} \times d\vec{r} = \vec{i}vdz + \vec{j}wdx + \vec{k}udy - \vec{i}wdy - \vec{j}udz - \vec{k}vdx$$

$$= \vec{i}\,(vdz - wdy) + \vec{j}\,(wdx - udz) + \vec{k}\,(udy - vdx) = 0$$

2.3 Flow Lines

(2) **Path line**

Since the particle is moving with the fluid at its local velocity

$$\frac{dx}{dt} = u; \quad \frac{dy}{dt} = v; \quad \frac{dz}{dt} = w$$

[App] Vector Products

(1) dot product → scalar

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

2.3 Flow Lines

(2) cross product → vector

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \left(a_2 b_3 - a_3 b_2\right)\vec{i} + \left(a_3 b_1 - a_1 b_3\right)\vec{j} + \left(a_1 b_2 - a_2 b_1\right)\vec{k}$$

$$curl\ \vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

2.3 Flow Lines

[Ex] Vorticity: $\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ 3–D flow, $\xi = \nabla \times \vec{V}$

For irrotational flow, $\xi = 0 \;\; ; \;\; \nabla \times \vec{V} = 0$

- $curl\,(u\vec{v}) = \nabla \times (u\vec{v}) = \nabla u \times \vec{v} + u \nabla \times \vec{v}$
- $curl\,grad\,u = \nabla \times \nabla u = 0$
- $divcurl\,\vec{u} = \nabla \cdot \left(\nabla \times \vec{u} \right) = 0$

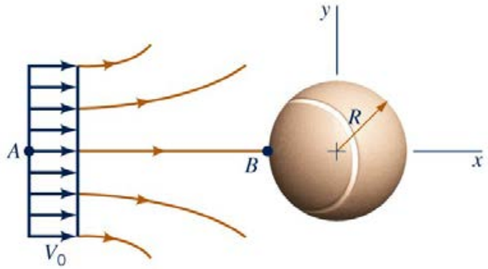
2.3 Flow Lines

Homework Assignment No. 2

Due: 1 week from today

1. The velocity of an inviscid, incompressible fluid as it steadily approaches the stagnation point at the leading edge of a sphere of radius R is

$$\vec{q} = u\vec{i} = u_s\left(1+\frac{R^3}{x^3}\right)\vec{i}$$



What is the fluid acceleration at (a) $x=-3R$, (b) $x=-2R$, and (c) $x=-R$?

(d) When and $u_s = 2\ m/s$ and R = 3 cm, what is the magnitude of the acceleration at $x=-2R$?

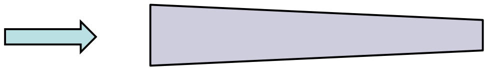
2.3 Flow Lines

2. The velocity field in a flow system is given by

$$\vec{q} = 5\vec{i} + (x + y^2)\vec{j} + 3xy\vec{k}$$

What is the fluid acceleration (a) at (1, 2, 3) and (b) at (-1, -2, -3)?

3. A nozzle is shaped such that the axial-flow velocity increases linearly from 2 to 18 m/s in a distance of 1.20 m. What is the convective acceleration (a) at the inlet and (b) at the exit of the nozzle?



4. Determine the streamline for the two-dimensional steady flow given by

$$\vec{q} = \frac{V_0}{l}(-x\vec{i} + y\vec{j})$$

where V_0 and l are constants. Plot it.