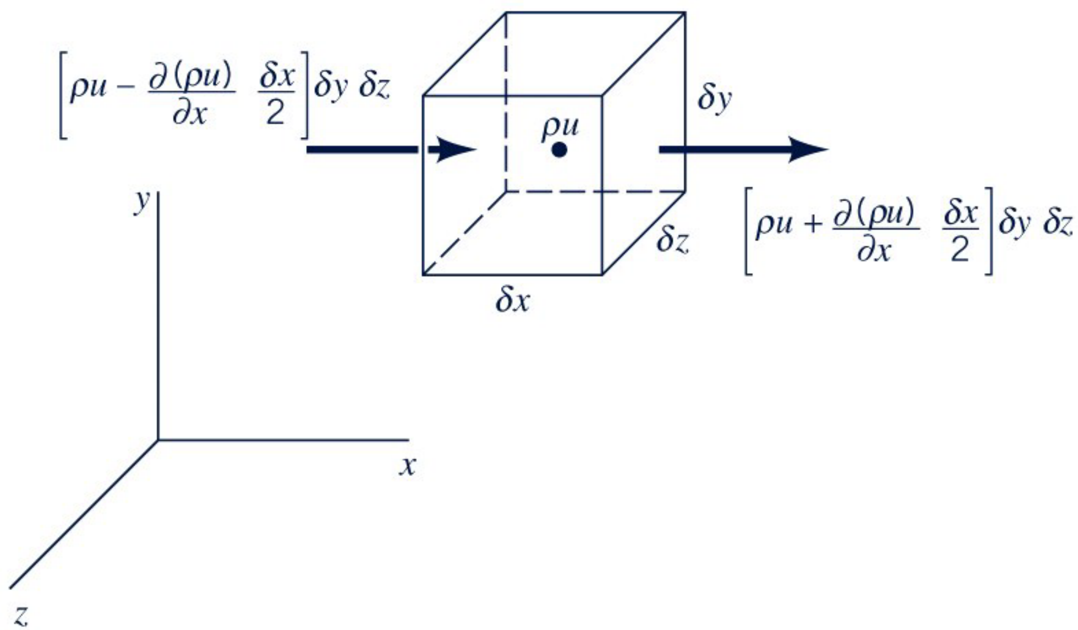
Chapter 6 Equations of Continuity and Motion

Session 6-1 Continuity equation



Chapter 6 Equations of Continuity and Motion

Contents

Chapter 6 Equations of Continuity and Motion

Objectives

- Derive 3D equations of continuity and motion
- Derive Navier-Stokes equation for Newtonian fluid
- Study solutions for simplified cases of laminar flow
- Derive Bernoulli equation for irrotational motion and frictionless flow
- Study solutions for vortex motions

6.1 Continuity Equation

- Derivation of 3-D Eq.

$$\begin{cases} \text{conservation of mass} \rightarrow \text{continuity eq.} \\ \text{conservation of momentum} \rightarrow \text{eq. of motion} \rightarrow \text{Navier-Strokes eq.} \end{cases}$$

Consider infinitesimal control volume ($\Delta x \Delta y \Delta z$) → point form

[Cf] Finite control volume – arbitrary CV → integral form

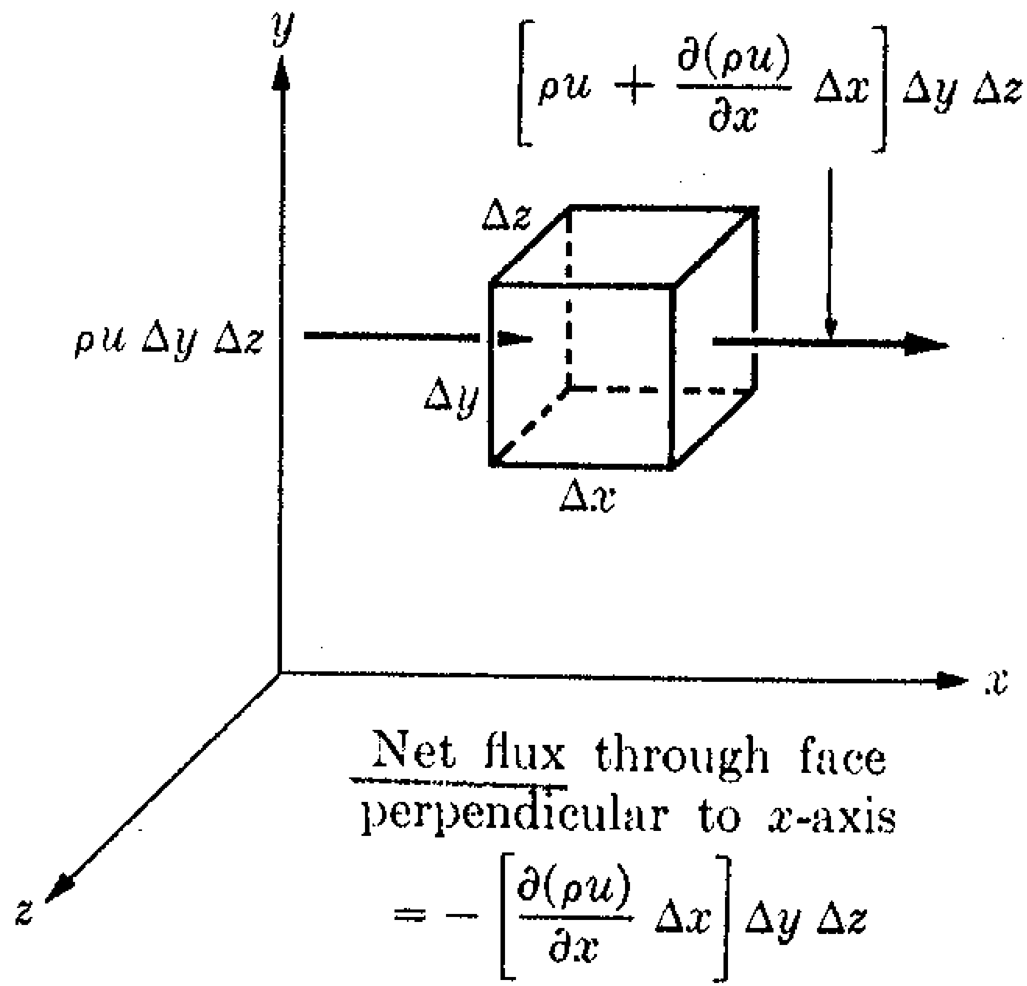
Apply principle of conservation of matter to the CV

→ sum of net flux = time rate change of mass inside C.V.

1) mass flux per unit time (mass flow)

$$= \frac{mass}{time} = \rho \frac{vol}{time} = \rho Q = \rho u \Delta A$$

6.1 Continuity Equation



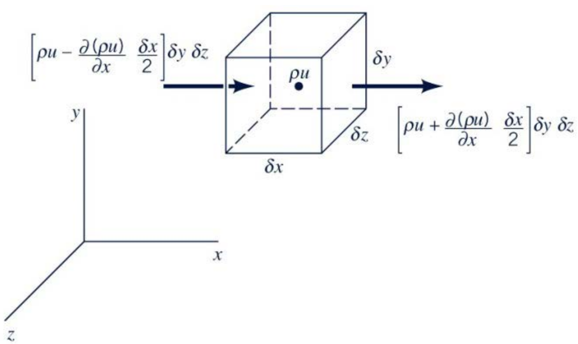
6.1 Continuity Equation

- net flux through face perpendicular to x -axis

= flux in –flux out

$$= \rho u \Delta y \Delta z - \left(\rho u + \frac{\partial (\rho u)}{\partial x} \Delta x \right) \Delta y \Delta z = -\frac{\partial (\rho u)}{\partial x} \Delta x \Delta y \Delta z$$

- net flux through face perpendicular to y -axis



$$= -\frac{\partial (\rho v)}{\partial y} \Delta x \Delta y \Delta z$$

- net flux through face perpendicular to z -axis

$$= -\frac{\partial (\rho w)}{\partial z} \Delta x \Delta y \Delta z \qquad \text{(A)}$$

6.1 Continuity Equation

2) time rate change of mass inside C.V.

$$= \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z) \qquad \text{(B)}$$

Thus, equating (A) and (B) gives

$$\frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z) = -\frac{\partial(\rho u)}{\partial x}\Delta x \Delta y \Delta z - \frac{\partial(\rho v)}{\partial y}\Delta x \Delta y \Delta z - \frac{\partial(\rho w)}{\partial z}\Delta x \Delta y \Delta z$$

$$LHS = \frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z) = \rho \frac{\partial}{\partial t}(\Delta x \Delta y \Delta z) + \Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t}$$

6.1 Continuity Equation

Since C.V. is fixed → $\dfrac{\partial(\Delta x \Delta y \Delta z)}{\partial t} = 0$

$$\therefore \quad LHS = \Delta x \Delta y \Delta z \frac{\partial \rho}{\partial t}$$

Cancelling terms makes

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{q} = 0} \qquad (6.1)$$

→ Continuity Eq. for compressible fluid in unsteady flow (point form)

6.1 Continuity Equation

The 2nd term of Eq. (6.1) can be expressed as

$$\nabla \cdot (\rho \vec{q}) = \underbrace{\vec{q} \nabla \rho}_{\mathrm{I}} + \underbrace{\rho \nabla \cdot \vec{q}}_{\mathrm{II}}$$

(I): $\vec{q} \nabla \rho = (u\vec{i} + v\vec{j} + w\vec{k}) \left(\dfrac{\partial \rho}{\partial x}\vec{i} + \dfrac{\partial \rho}{\partial y}\vec{j} + \dfrac{\partial \rho}{\partial z}\vec{k} \right)$

gradient

$$= u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}$$

divergence

(II): $\rho \nabla \cdot \vec{q} = \rho \left(\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} \right)$

$$\therefore \nabla \cdot (\rho \vec{q}) = u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \qquad \text{(i)}$$

6.1 Continuity Equation

Substituting (i) into Eq (6.1) yields

$$\underbrace{\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}}_{\boxed{\frac{d\rho}{dt}}} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \rho\vec{q} = 0} \qquad (6.1)$$

$$\boxed{\frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0} \qquad (6.2a)$$

$$\boxed{\frac{d\rho}{dt} + \rho(\nabla \cdot \vec{q}) = 0} \qquad (6.2b)$$

6.1 Continuity Equation

[Re] Total derivative (total rate of density change)

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x}\frac{dx}{dt} + \frac{\partial \rho}{\partial y}\frac{dy}{dt} + \frac{\partial \rho}{\partial z}\frac{dz}{dt}$$

$$= \frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}$$

1) For steady-state conditions

$$\rightarrow \frac{\partial \rho}{\partial t} = 0$$

6.1 Continuity Equation

Then (6.1) becomes

$$\boxed{\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=\nabla\cdot(\rho\vec{q})=0} \tag{6.3}$$

2) For incompressible fluid (whether or not flow is steady)

$$\rightarrow \frac{d\rho}{dt}=0 \tag{6.4}$$

Then (6.2) becomes

$$\boxed{\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\nabla\cdot\vec{q}=0} \tag{6.5}$$

6.1 Continuity Equation

[Re] Continuity equation derived using a finite CV method

Eq. (4.5a):

$$\int_{CV} \frac{\partial \rho}{\partial t} dV + \oint_{CS} \rho \vec{q} \cdot d\vec{A} = 0 \qquad (4.5)$$

→ volume-averaged (integrated) form

- Gauss' theorem:

volume integral ↔ surface integral

– reduce dimensions by 1 (3D → 2D)

$$\int_{v} \left(\nabla \cdot \vec{X} \right) dV = \int_{A} \vec{X} \cdot d\vec{A}$$

6.1 Continuity Equation

Thus,

$$2nd\ term = \oint_{CS} \rho \vec{q} \cdot d\vec{A} = \int_{CV} \nabla \cdot (\rho \vec{q}) dV$$

Eq. (4.5) becomes

$$\int_{CV} \frac{\partial \rho}{\partial t} dV + \int_{CV} \nabla \cdot (\rho \vec{q}) dV = \int_{CV} \left(\underline{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{q})} \right) dV = 0 \qquad \text{(A)}$$

Since integrands must be equal to zero.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{q}) = 0$$

→ same as Eq. (6.1) → point form

6.1 Continuity Equation

- 1D Continuity equation

$$(A):\ \int \frac{\partial \rho}{\partial t} dA + \int \frac{\partial \rho u}{\partial x} dA = 0$$

$$\frac{\partial}{\partial t}\int \rho dA + \frac{\partial}{\partial x}\int \rho u dA = 0$$

For incompressible fluid flow

$$\rho \frac{\partial}{\partial t}\underbrace{\int dA}_{\boxed{A}} + \rho \frac{\partial}{\partial x}\underbrace{\int u dA}_{\boxed{VA}} = 0$$

where V = cross-sectional average velocity

6.1 Continuity Equation

$$\therefore \frac{\partial A}{\partial t} + \frac{\partial VA}{\partial x} = 0$$

Consider lateral inflow/outflow

$$\frac{\partial A}{\partial t} + \frac{\partial VA}{\partial x} = \int_{\sigma} q d\sigma$$

where q = flow through σ

For steady flow; $\frac{\partial A}{\partial t} = 0$

$$\therefore \quad \frac{\partial VA}{\partial x} = 0$$

$$VA = const. = Q$$

6.1 Continuity Equation

[Re] Continuity equation in polar (cylindrical) coordinates

u, r - radial

v, θ - azimuthal

w, z - axial

For compressible fluid of unsteady flow

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (\rho u r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho v)}{\partial \theta} + \frac{\partial (\rho w)}{\partial z} = 0}$$

For incompressible fluid

$$\frac{1}{r}\frac{\partial (u r)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0$$

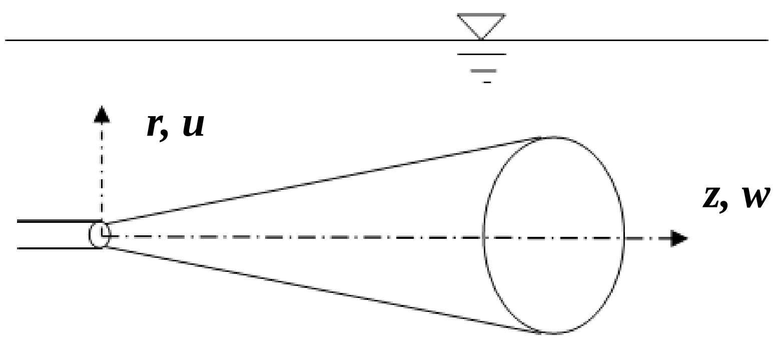
6.1 Continuity Equation

For incompressible fluid and flow of axial symmetry

$$\frac{\partial \rho}{\partial t}=0, \quad \frac{\partial \rho}{\partial r}=\frac{\partial \rho}{\partial \theta}=\frac{\partial \rho}{\partial z}=0, \qquad \frac{\partial(\rho v)}{\partial \theta}=0$$

$$\therefore \frac{1}{r}\frac{\partial(ur)}{\partial r}+\frac{\partial w}{\partial z}=0$$ → 2-D boundary layer flow

Example: submerged jet



6.1 Continuity Equation

[Re] Green's Theorem

1) Transformation of double integrals into line integrals

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dxdy = \oint_C \left(F_1 dx + F_2 dy \right)$$

$$\iint_R \left(curl\ \vec{F} \right) \cdot \vec{k} dxdy = \oint_C \vec{F} \cdot d\vec{r}$$

$$\vec{F} = F_1 \vec{i} + F_2 \vec{j}$$

2) 1st form of Green's theorem

$$\iiint_T \left(f \nabla^2 g + grad\ f \cdot grad\ g \right) dV = \iint_S f \frac{\partial g}{\partial n} dA$$

6.1 Continuity Equation

3) 2nd form of Green's theorem

$$\iiint_T \left(f\nabla^2 g + g\nabla^2 f \right) dV = \iint_S \left(f\frac{\partial g}{\partial n} - g\frac{\partial f}{\partial x} \right) dA$$

[Re] Divergence theorem of Gauss

→ transformation between <u>volume integrals</u> and <u>surface integrals</u>

$$\iiint_T div\vec{F}\ dV = \iint_S \vec{F} \cdot \vec{n}\ dA$$

where n = outer unit normal vector of S

$$\overrightarrow{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

6.1 Continuity Equation

$$\vec{n} = \cos\alpha\vec{i} + \cos\beta\vec{j} + \cos\gamma\vec{k}$$

$$\iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dxdydz$$

$$= \iint_S \left(F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma \right) dA$$

By the way

$$\iint_S \vec{F} \cdot \vec{n} dA = \iint_S \left(F_1 dydz + F_2 dzdx + F_3 dxdy \right)$$

$$\therefore \quad \iiint_T \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dxdydz$$

$$= \iint_S \left(F_1 dydz + F_2 dzdx + F_3 dxdy \right) \qquad (6.6)$$

6.2 Stream Function in 2-D, Incompressible Flows

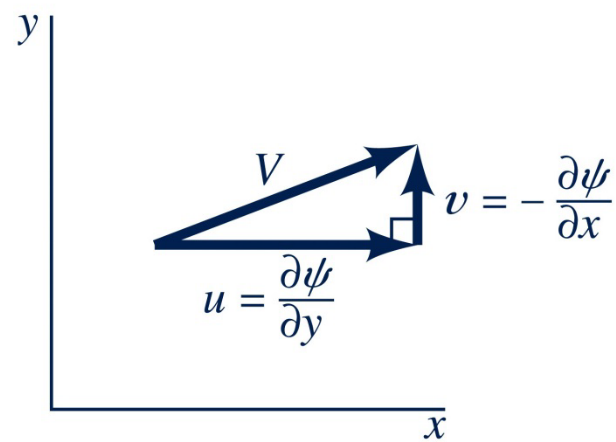
- 2-D incompressible continuity eq. is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (6.7)$$

In 2D flowfield, define stream function $\psi(x, y)$ as

$$u=-\frac{\partial \psi}{\partial y} \qquad (6.8)$$

$$v=\frac{\partial \psi}{\partial x} \qquad (6.9)$$



- We can a simplified equation by having to determine only one unknown function.

6.2 Stream Function in 2-D, Incompressible Flows

Then LHS of Eq. (6.7) becomes

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right)=-\cancel{\frac{\partial^2 \psi}{\partial x \partial y}}+\cancel{\frac{\partial^2 \psi}{\partial x \partial y}}=0 \qquad (6.10)$$

→ <u>Thus, continuity equation is satisfied.</u>

1) Apply stream function to the equation for <u>a stream line</u> in 2-D flow

Eq. (2.10): $vdx-udy=0$ (6.11)

Substitute (6.8) into (6.11)

$$\frac{\partial \psi}{\partial x}dx+\frac{\partial \psi}{\partial y}dy=d\psi=0 \qquad (6.12)$$

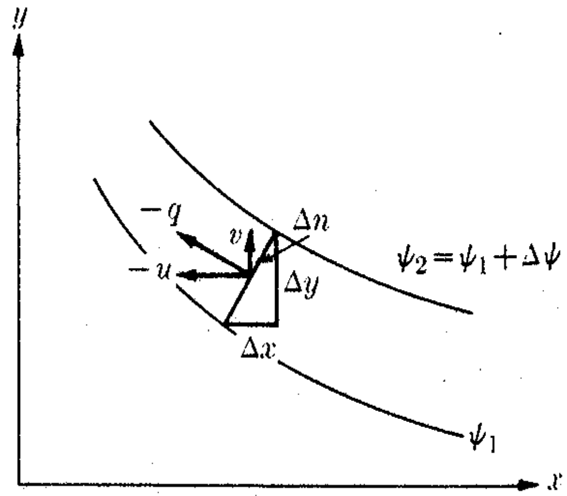
6.2 Stream Function in 2-D, Incompressible Flows

ψ = constant (6.13)

→ The stream function is constant along a streamline.

2) Apply stream function to the law of conservation of mass

$$-qdn = -udy + vdx \qquad (6.14)$$

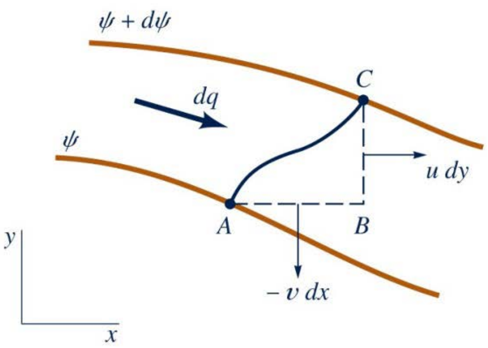


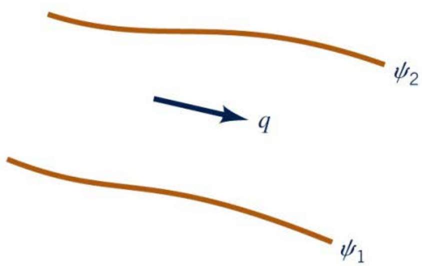
Substitute (6.8) into (6.14)

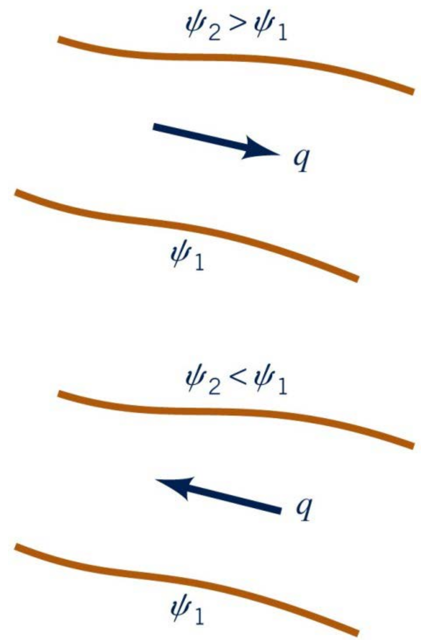
$$-qdn = \frac{\partial \psi}{\partial y}dy + \frac{\partial \psi}{\partial x}dx = d\psi \qquad (6.15)$$

6.2 Stream Function in 2-D, Incompressible Flows

→ Change in ψ ($d\psi$) between adjacent streamlines is equal to the volume rate of flow per unit width.







- Stream function in cylindrical coordinates

$$v_r = -\frac{\partial \psi}{r \partial \theta} \quad \text{radial} \qquad v_\theta = \frac{\partial \psi}{\partial r} \quad \text{azimuthal}$$

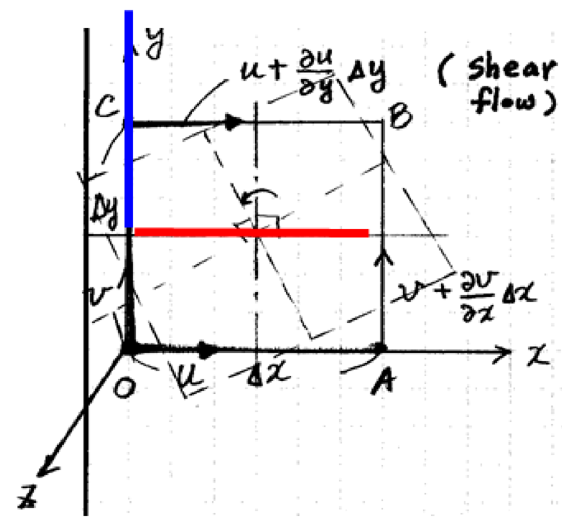
6.3 Rotational and Irrotational Motion

6.3.1 Rotation and vorticity

Assume the rate of rotation of fluid element Δx and Δy about Z-axis is positive when it rotates counterclockwise.

- time rate of rotation of Δx-face about z-axis

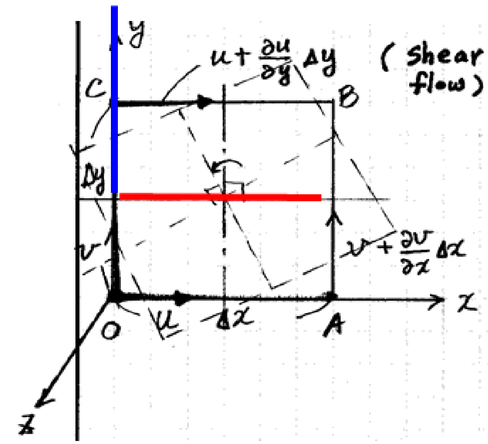
$$= \frac{1}{\Delta t} \frac{\left[\left\{ v + \left(\frac{\partial v}{\partial x} \right) \Delta x \right\} - v \right] \Delta t}{\Delta x} = \frac{\partial v}{\partial x}$$



6.3 Rotational and Irrotational Motion

- time rate of rotation of Δy-face about z- axis

$$= -\frac{1}{\Delta t}\frac{\left[u+\left(\frac{\partial u}{\partial y}\Delta y\right)-u\right]\Delta t}{\Delta y} = -\frac{\partial u}{\partial y}$$



net rate of rotation = average of rotations of Δx-and Δy-face

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

6.3 Rotational and Irrotational Motion

Doing the same way for x-axis, and y-axis

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

$$\omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \quad (6.16a)$$

1) Rotation

$$\vec{\omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\vec{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\vec{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k}$$

$$= \frac{1}{2}\left(\nabla \times \vec{q}\right) = \frac{1}{2} curl\ \vec{q} \quad (6.16b)$$

6.3 Rotational and Irrotational Motion

Magnitude:

$$|\vec{\omega}| = \sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$$

a) Ideal fluid → irrotational flow

$$\nabla \times \vec{q} = 0$$

$$\omega_x = \omega_y = \omega_z = 0$$

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \qquad (6.17)$$

b) Viscous fluid → rotational flow

$$\nabla \times \vec{q} \neq 0$$

6.3 Rotational and Irrotational Motion

2) Vorticity

$$\vec{\zeta} = curl\ \vec{q} = \nabla \times \vec{q} = 2\vec{\omega}$$

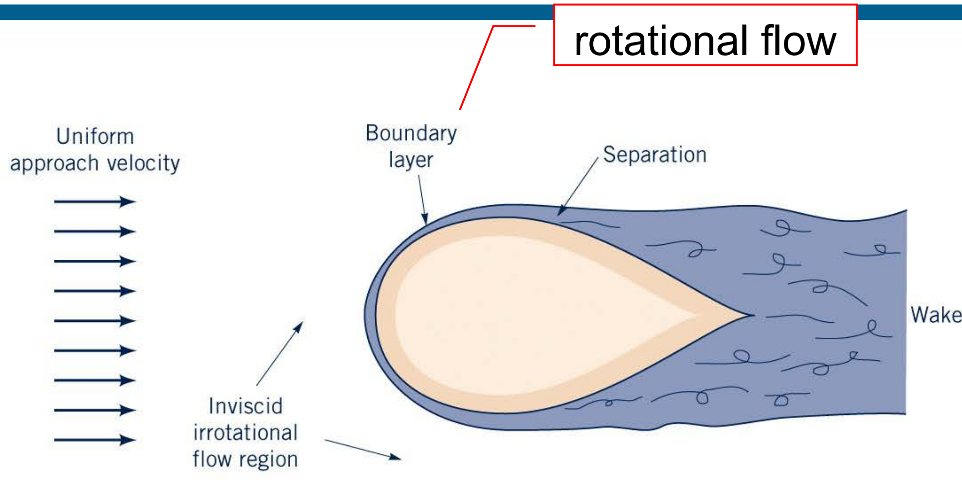
- Rotation in cylindrical coordinates

$$\omega_r = \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}\right)$$

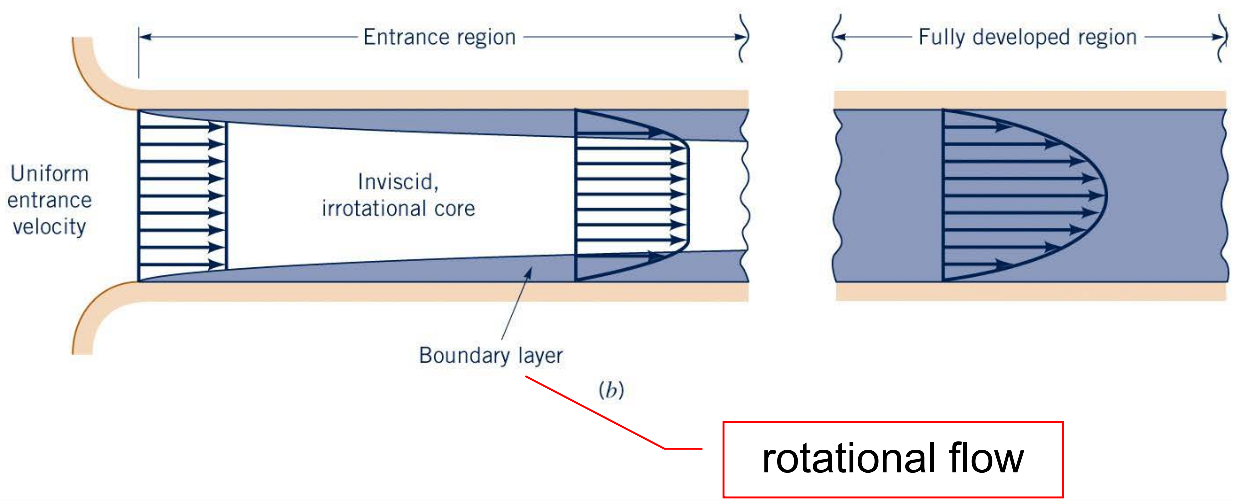
$$\omega_\theta = \frac{1}{2}\left(\frac{\partial v_r}{\partial z} - \frac{\partial vz}{\partial r}\right)$$

$$\omega_z = \frac{1}{2}\left(-\frac{1}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r}\right)$$

6.3 Rotational and Irrotational Motion

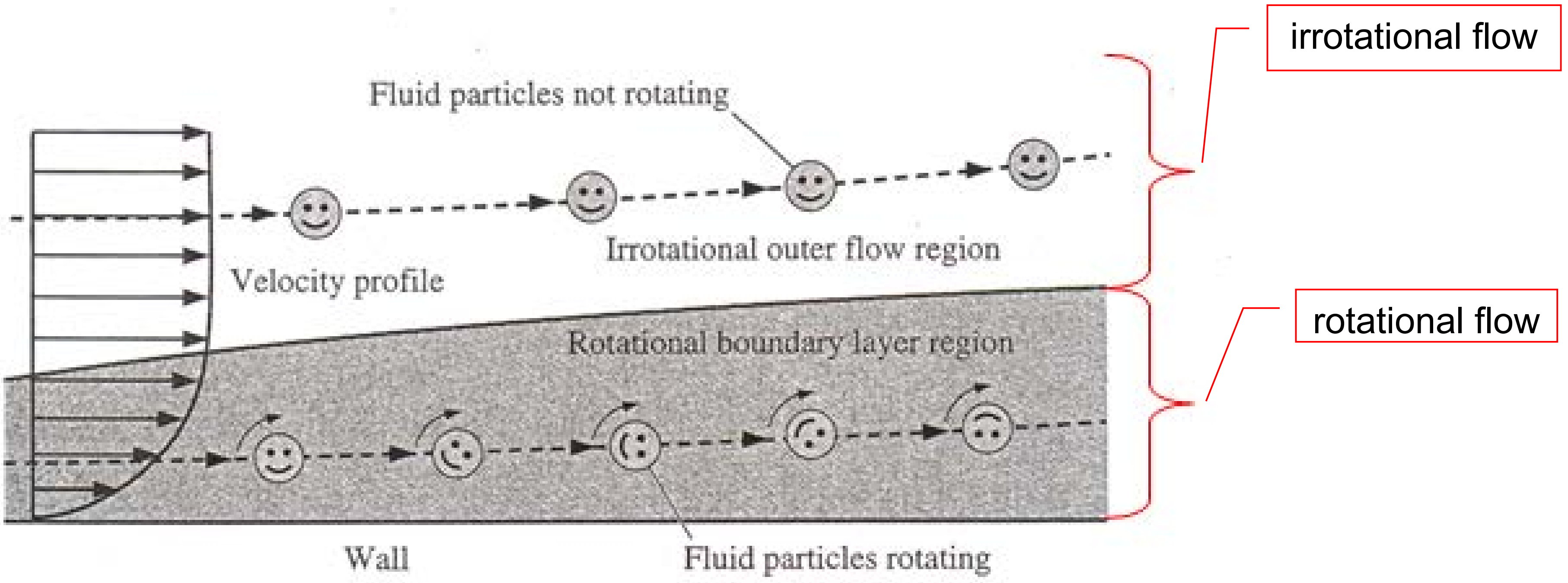


(a)



(b)

6.3 Rotational and Irrotational Motion

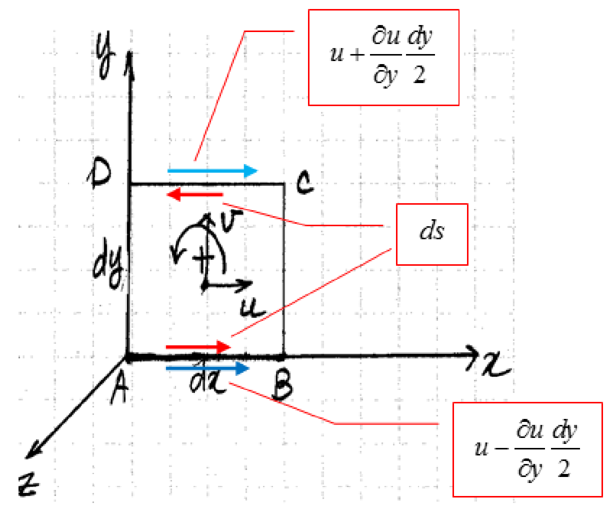


6.3 Rotational and Irrotational Motion

6.3.2 Circulation

Γ = line integral of the tangential velocity component about any closed contour *S*

$$\Gamma = \oint \vec{q} \cdot d\vec{s} \qquad (6.19)$$



6.3 Rotational and Irrotational Motion

– take line integral from A to B, C, D, A ~ infinitesimal CV

$$d\Gamma \cong \left[u - \frac{\partial u}{\partial y}\frac{dy}{2} \right] dx + \left[v + \frac{\partial v}{\partial x}\frac{dx}{2} \right] dy - \left[u + \frac{\partial u}{\partial y}\frac{dy}{2} \right] dx - \left[v - \frac{\partial v}{\partial x}\frac{dx}{2} \right] dy$$

$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy$$

$$d\Gamma \cong \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dxdy$$

$$\Gamma = \iint_A \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dA = \iint_A 2\omega_z dA = \iint_A \left(\nabla \times \vec{q} \right)_z dA \qquad (6.20)$$

6.3 Rotational and Irrotational Motion

For irrotational flow,

circulation $\Gamma = 0$ (if there is no singularity vorticity source).

[Re] Fluid motion and deformation of fluid element

Motion $\begin{cases} \text{translation} \\ \text{rotation} \end{cases}$

Deformation $\begin{cases} \text{linear deformation} \\ \text{angular deformation} \end{cases}$

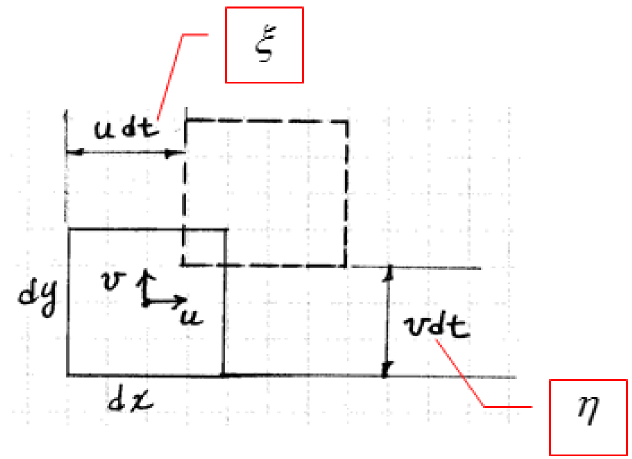
6.3 Rotational and Irrotational Motion

(1) Motion

1) Translation: ξ, η

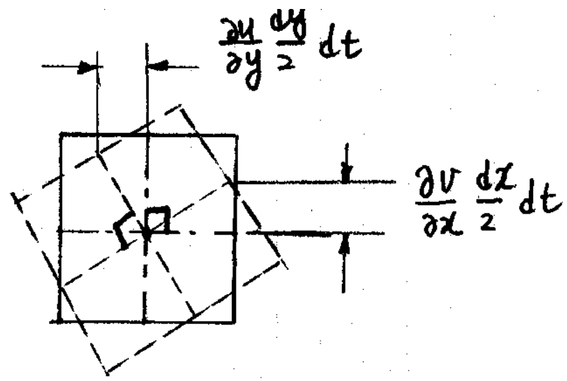
$$\xi = u\ dt,\ \ u = \frac{d\xi}{dt}$$

$$\eta = v\ dt,\ \ v = \frac{d\eta}{dt}$$



2) Rotation ← Shear flow

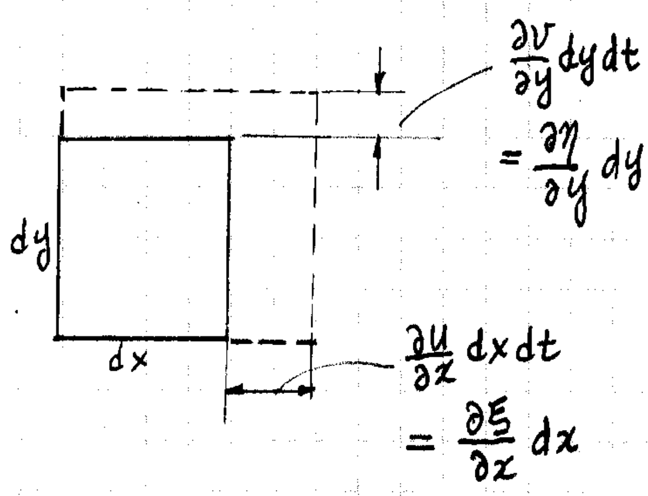
$$\omega_x = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$



6.3 Rotational and Irrotational Motion

(2) Deformation

1) Linear deformation – normal strain



$$\varepsilon_x = \frac{\partial \xi}{\partial x}$$

$$\varepsilon_y = \frac{\partial \eta}{\partial y}$$

i) For compressible fluid, changes in temperature or pressure cause change in volume.

ii) For incompressible fluid, if length in 2-D increases, then length in another 1-D decreases in order to make total volume unchanged.

6.3 Rotational and Irrotational Motion

2) Angular deformation– shear strain

$$\gamma_{xy} = \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial x}$$