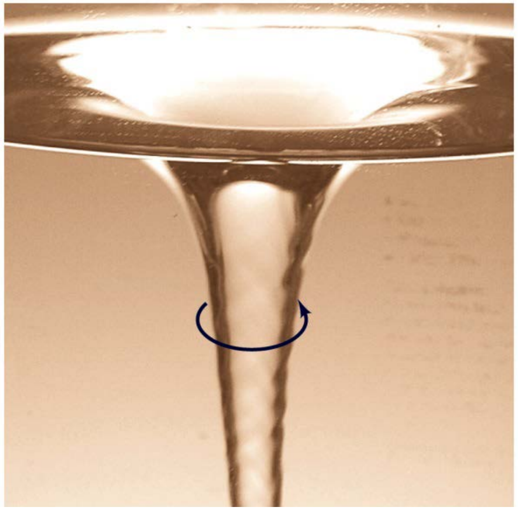
Chapter 6 Equations of Continuity and Motion

Session 6-3 Motions of viscous and inviscid fluids



Chapter 6 Equations of Continuity and Motion

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6.6 Irrotational Motion

○ In Ch. 4, 1st law of thermodynamics → 1D Energy eq.

→ Bernoulli eq. for steady flow of an incompressible fluid with zero friction (ideal fluid)

○ In Ch. 6, Newton's 2nd law → Momentum eq. → Eq. of motion (6.4) → Bernoulli eq.

Integration assuming irrotational flow (6.3)

○ Irrotational flow = Potential flow

6.6 Irrotational Motion

6.6.1 Velocity potential and stream function

If $\phi(x, y, z, t)$ is any <u>scalar quantity</u> having continuous first and second derivatives, then by a fundamental vector identity

$$\rightarrow curl(grad\ \phi) \equiv \underline{\nabla \times (\nabla \phi) \equiv 0} \qquad (6.46)$$

[Detail] vector identity

$$\nabla \phi = grad\ \phi = \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}$$

6.6 Irrotational Motion

$$curl(grad\ \phi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial \phi}{\partial x} & \dfrac{\partial \phi}{\partial y} & \dfrac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial \phi^2}{\partial y \partial z} - \frac{\partial \phi^2}{\partial y \partial z}\right) + \vec{j}\left(\frac{\partial \phi^2}{\partial z \partial x} - \frac{\partial \phi^2}{\partial z \partial x}\right) + \vec{k}\left(\frac{\partial \phi^2}{\partial x \partial y} - \frac{\partial \phi^2}{\partial x \partial y}\right) \Rightarrow 0$$

6.6 Irrotational Motion

By the way, for irrotational flow

Eq.(6.17) : $\nabla \times \vec{q} = 0$ (A)

Thus, from (6.46) and (A), we can say that for irrotational flow there must exist a scalar function ϕ whose gradient is equal to the velocity vector $\vec{q}$.

$$grad\ \phi = \vec{q} \quad \text{(B)}$$

Now, let's define the positive direction of flow is the direction in which is decreasing, ϕ then

$$\boxed{\vec{q} = -grad\ \phi(x, y, z, t) = -\nabla \phi} \quad (6.47)$$

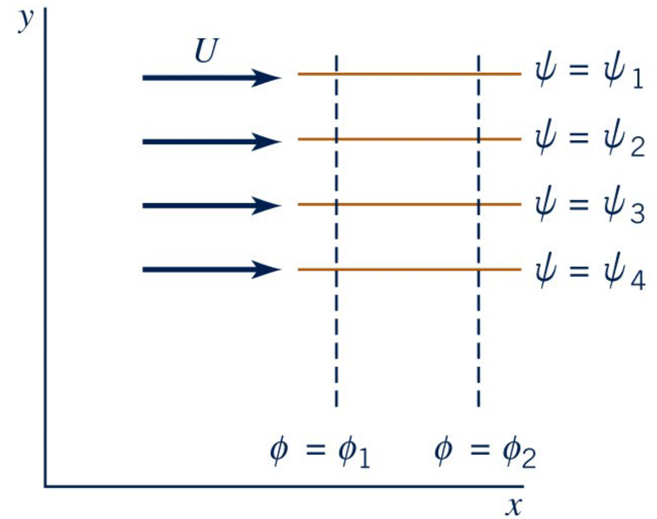
6.6 Irrotational Motion

where ϕ = velocity potential

$$\boxed{u = -\frac{\partial \phi}{\partial x}, \ v = -\frac{\partial \phi}{\partial y}, \ w = -\frac{\partial \phi}{\partial z}} \qquad (6.47a)$$

→ Velocity potential exists only for irrotational flows; however stream function is not subject to this restriction.

→ irrotational flow = potential flow for both compressible and incompressible fluids



6.6 Irrotational Motion

(1) Continuity equation for incompressible fluid

Eq. (6.5): $\nabla \cdot \vec{q} = 0$ (C)

Substitute (6.47) into (C)

$$\therefore \nabla \cdot \left(-\nabla \phi \right) = -\nabla^2 \phi = 0 \quad \rightarrow \text{Laplace Eq.} \tag{6.48}$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \leftarrow \text{Cartesian coordinates} \tag{6.49}$$

$$\nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \leftarrow \text{Cylindrical coordinates} \tag{6.50}$$

6.6 Irrotational Motion

[Detail] velocity potential in cylindrical coordinates

$$v_r = -\frac{\partial \phi}{\partial r}, \quad v_\theta = -\frac{\partial \phi}{r\partial \theta}, \quad v_z = -\frac{\partial \phi}{\partial z}$$

(2) For 2-D incompressible irrotational motion

- Velocity potential

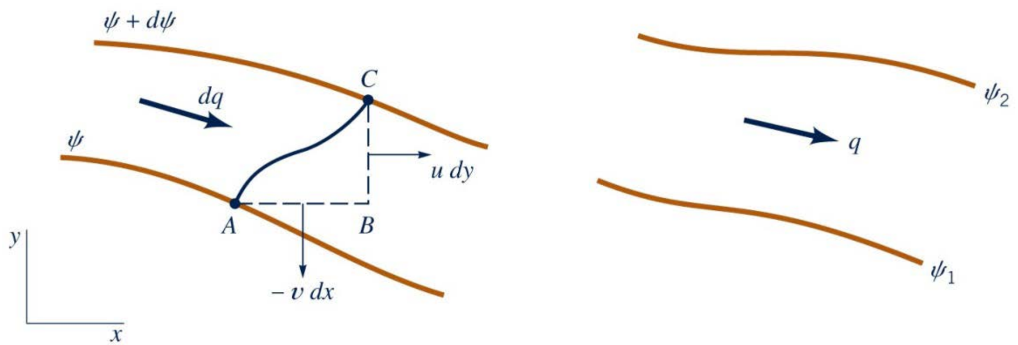
$$u = -\frac{\partial \phi}{\partial x}$$
$$v = -\frac{\partial \phi}{\partial y} \qquad (6.51)$$

6.6 Irrotational Motion

- Stream function: Eq. (6.8)

$$u = -\frac{\partial \psi}{\partial y}$$

$$v = \frac{\partial \psi}{\partial x} \qquad (6.52)$$



$$\therefore \left.\begin{aligned} \frac{\partial \psi}{\partial y} &= \frac{\partial \phi}{\partial x} \\ \frac{\partial \psi}{\partial x} &= -\frac{\partial \phi}{\partial y} \end{aligned}\right\} \rightarrow \text{Cauchy-Riemann equation} \qquad (6.53)$$

6.6 Irrotational Motion

Now, substitute stream function, (6.8) into irrotational flow, (6.17)

Eq. (6.17) : $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \leftarrow \left[rotation = 0 \quad \nabla \times \vec{q} = 0 \right]$

$$\therefore \; -\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial^2 \psi}{\partial x^2} \rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \rightarrow \text{Laplace eq.} \qquad \text{(D)}$$

Also, for 2-D flow, velocity potential satisfies the Laplace eq.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \text{(E)}$$

6.6 Irrotational Motion

→ Both ϕ and ψ satisfy the Laplace eq. for 2-D incompressible irrotational motion.

→ ϕ and ψ may be interchanged.

→ Lines of constant ϕ and ψ must form an orthogonal mesh system

→ Flow net

- Flow net analysis

Along a streamline, $\psi =$ constant.

Eq. for a streamline, Eq. (2.10)

$$\left.\frac{dy}{dx}\right|_{\psi = const.} = \frac{v}{u} \qquad (6.54)$$

6.6 Irrotational Motion

BTW along lines of constant velocity potential

$$\rightarrow d\phi = 0$$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = 0 \qquad \text{(F)}$$

Substitute Eq. (6.47a)

$$\left.\frac{dy}{dx}\right|_{\phi=const.} = -\frac{\partial \phi / \partial x}{\partial \phi / \partial y} = -\frac{u}{v} \qquad (6.55)$$

6.6 Irrotational Motion

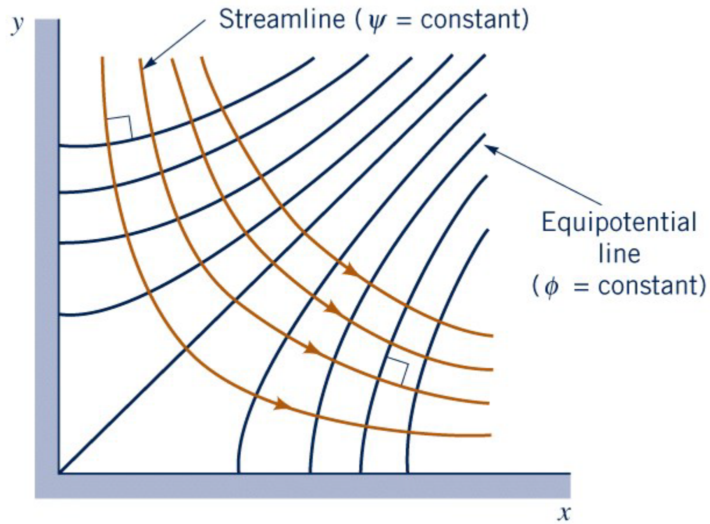
From Eqs. (6.54) and (6.55)

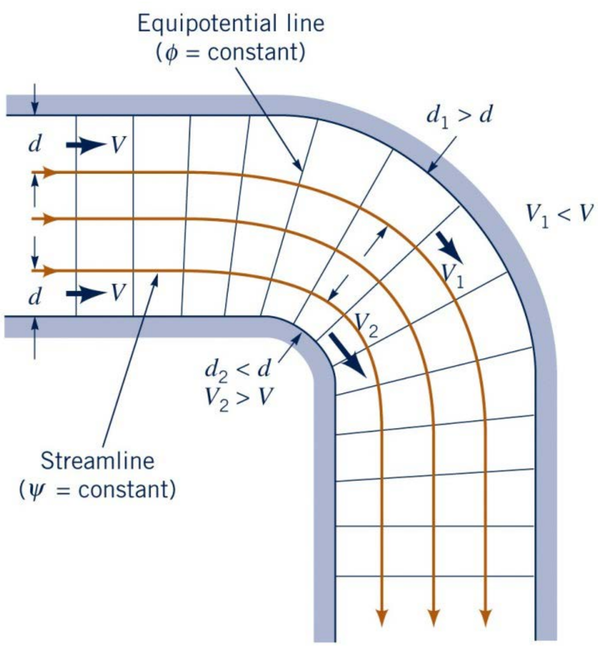
$$\left.\frac{dy}{dx}\right|_{\psi=const.} = -\left.\frac{dx}{dy}\right|_{\phi=const.} \qquad (6.56)$$

→ Slopes are the negative reciprocal of each other.

→ Flow net analysis (graphical method) can be used when a solution of the Laplace equation is difficult for complex boundaries.

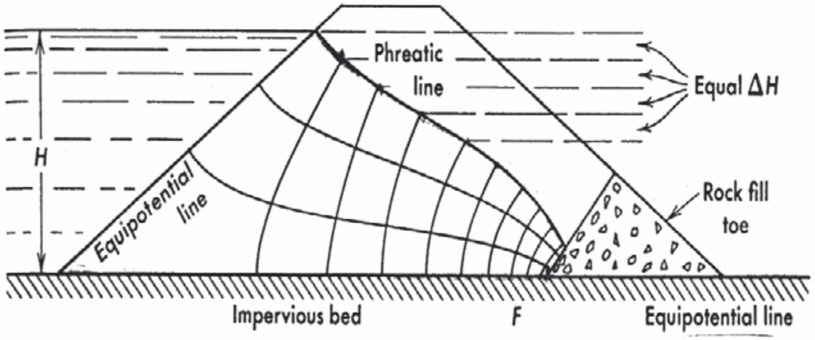
6.6 Irrotational Motion

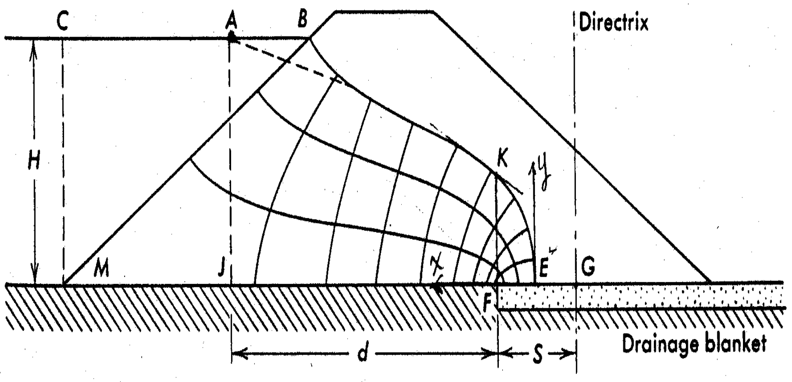


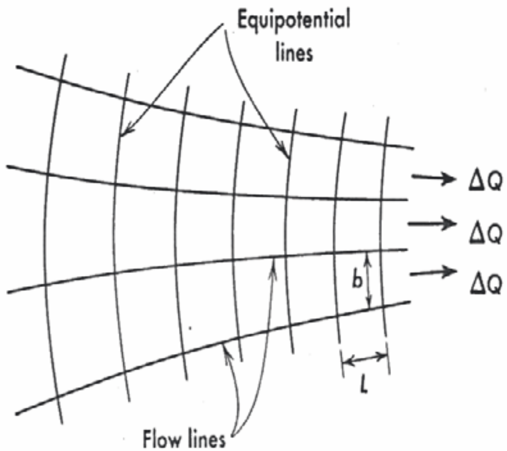


6.6 Irrotational Motion

- Seepage of earth dam







$$Q = \sum \Delta Q = n_f K \Delta H = \frac{n_f}{n_d} KH$$

n_f = number of flowlines;

n_d = number of equipotential lines;

K = permeability coefficient (m/s)

6.6 Irrotational Motion

Potential flows

1. Uniform flow

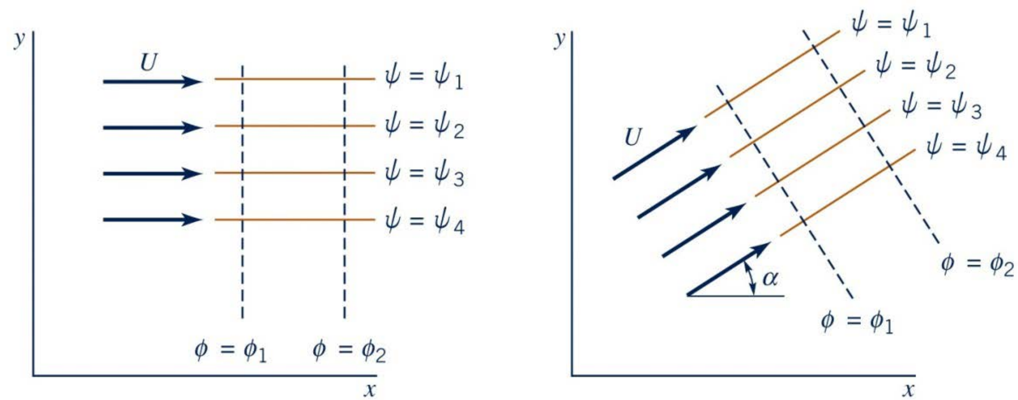
→ streamlines are all straight and parallel, and the magnitude of the velocity is constant

$$\frac{\partial \phi}{\partial x} = U, \quad \frac{\partial \phi}{\partial y} = 0$$

$$\boxed{\phi = Ux + C}$$

$$\frac{\partial \psi}{\partial y} = U, \quad \frac{\partial \psi}{\partial x} = 0$$

$$\boxed{\psi = Uy + C'}$$



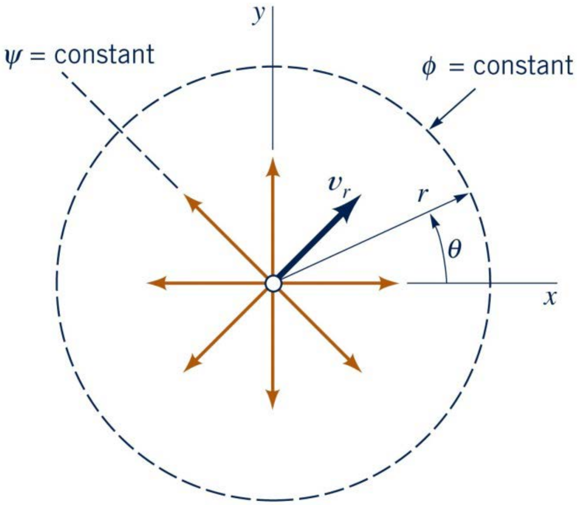
6.6 Irrotational Motion

2. Source and Sink

- Fluid flowing radially outward from a line through the origin perpendicular to the *x*-*y* plane
- Let *m* be the volume rate of flow emanating from the line (per unit length)

$$(2\pi r)v_r = m$$

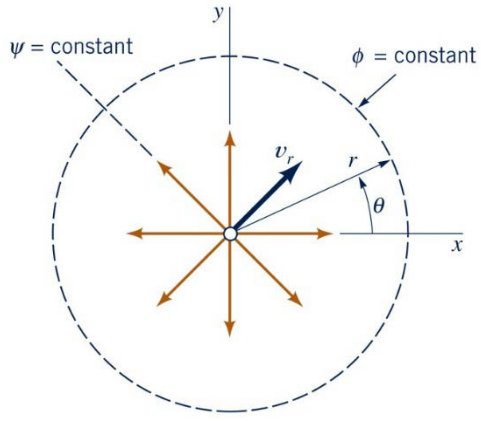
$$v_r = \frac{m}{2\pi r}$$



6.6 Irrotational Motion

If m is positive, the flow is radially outward → source

If m is negative, the flow is radially inward → sink

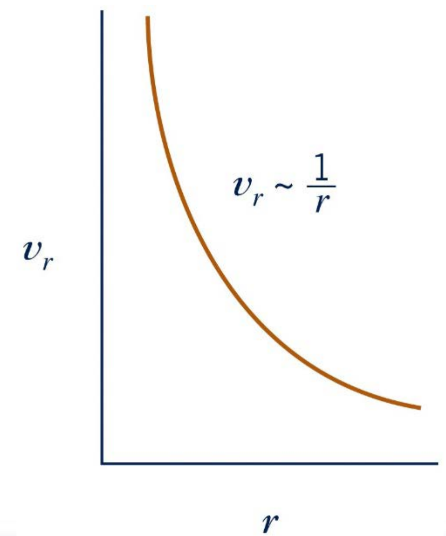


$$\frac{\partial\phi}{\partial r}=\frac{m}{2\pi r},\ \frac{1}{r}\frac{\partial\phi}{\partial\theta}=0$$

$$\boxed{\phi=\frac{m}{2\pi}\ln r}$$

$$v_r=\frac{1}{r}\frac{\partial\psi}{\partial\theta}=\frac{m}{2\pi r}$$

$$\boxed{\psi=\frac{m}{2\pi}\theta}$$



The <u>streamlines are radial lines</u>,

and <u>equipotential lines are concentric circles</u>.

6.6 Irrotational Motion

3. Vortex (Sec. 6.8)

Flow field in which the streamlines are concentric circles

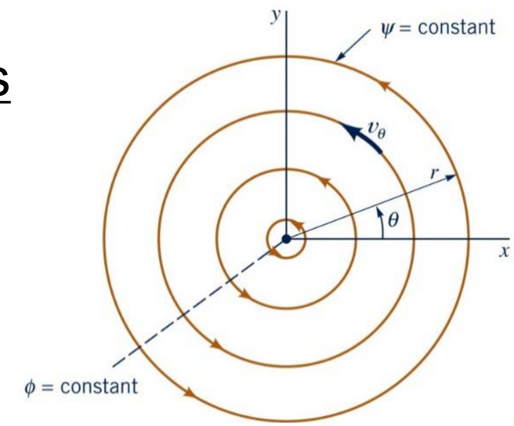
In cylindrical coordinate

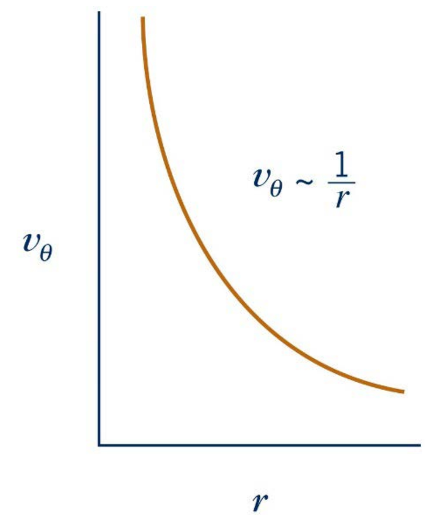
$$\phi = K\theta$$

$$\psi = -K \ln r$$

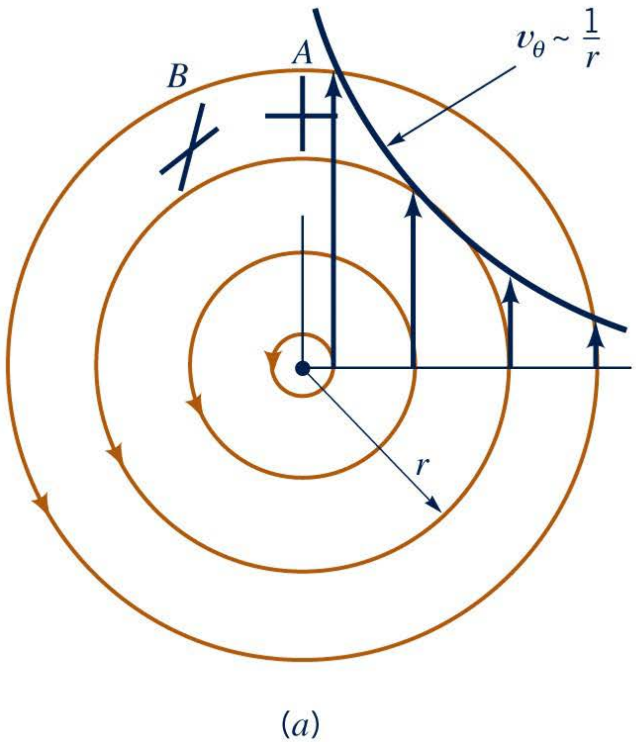
$$\boxed{v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = \frac{K}{r}}$$

The tangential velocity varies inversely with distance from the origin.

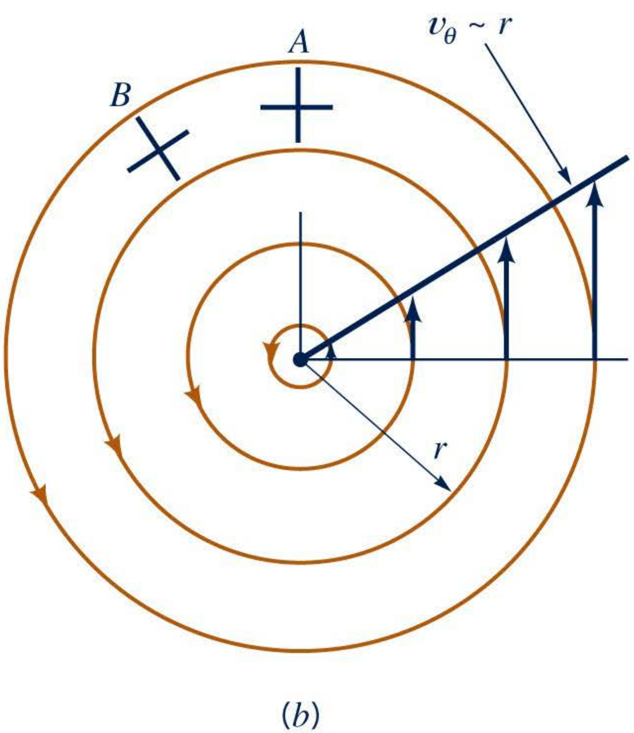




6.6 Irrotational Motion



Free vortex



Forced vortex

6.6 Irrotational Motion

[Appendix II] Potential flow problem

Find velocity potential ϕ

- Find $\psi \rightarrow$ Find flow pattern
- Find velocity
- Find kinetic energy

Find velocity, Find kinetic energy $\rightarrow$ Find pressure, force

Bernoulli eq.

6.6 Irrotational Motion

6.6.2 The Bernoulli equation for irrotational incompressible fluids

(1) For irrotational incompressible fluids

Substitute Eq. (6.17) into Eq. (6.28)

Eq. (6.17) : $\nabla \times \vec{q} = 0$

$$\left.\begin{aligned} \frac{\partial w}{\partial y} &= \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} &= \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} &= \frac{\partial u}{\partial y} \end{aligned}\right\}$$ irrotational flow

6.6 Irrotational Motion

Eq. (6.28): Navier-Stokes eq. (x -comp.) for incompressible fluid

$$\boxed{\frac{1}{2}\frac{\partial v^2}{\partial x}} \qquad \boxed{\frac{1}{2}\frac{\partial w^2}{\partial x}}$$

$$\frac{\partial u}{\partial t}+\underbrace{u\frac{\partial u}{\partial x}}_{\downarrow\ \frac{1}{2}\frac{\partial u^2}{\partial x}}+\underbrace{v\frac{\partial u}{\partial y}}_{\downarrow\ v\frac{\partial v}{\partial x}}+\underbrace{w\frac{\partial u}{\partial z}}_{\downarrow\ w\frac{\partial w}{\partial x}}=-g\frac{\partial h}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left[\frac{\partial^2 u}{\partial x^2}+\underbrace{\frac{\partial^2 u}{\partial y^2}}_{\downarrow\ \frac{\partial^2 v}{\partial y\partial x}}+\underbrace{\frac{\partial^2 u}{\partial z^2}}_{\downarrow\ \frac{\partial^2 w}{\partial z\partial x}}\right]$$

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u^2}{2}+\frac{v^2}{2}+\frac{w^2}{2}\right)=-g\frac{\partial h}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) \qquad (6.57)$$

6.6 Irrotational Motion

Substitute $q^2 = u^2 + v^2 + w^2$ and continuity eq. for incompressible fluid into Eq. (6.57)

Continuity eq., Eq. (6.5): $\nabla \cdot \vec{q} = \dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$

Then, viscous force term can be dropped.

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q^2}{2}\right) = -g\frac{\partial h}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \rightarrow x\text{- Eq.}$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left[\frac{q^2}{2} + gh + \frac{p}{\rho}\right] = 0$$

6.6 Irrotational Motion

$$y-Eq. \qquad \frac{\partial v}{\partial t}+\frac{\partial}{\partial y}\left[\frac{q^2}{2}+gh+\frac{p}{\rho}\right]=0 \qquad (6.58)$$

$$z-Eq. \qquad \frac{\partial w}{\partial t}+\frac{\partial}{\partial z}\left[\frac{q^2}{2}+gh+\frac{p}{\rho}\right]=0 \qquad (6.59)$$

Introduce velocity potential ϕ

$$\boxed{u=-\frac{\partial \phi}{\partial x}, v=-\frac{\partial \phi}{\partial y}, w=-\frac{\partial \phi}{\partial z}}$$

$$\frac{\partial u}{\partial t}=-\frac{\partial^2 \phi}{\partial t \partial x}, \quad \frac{\partial v}{\partial t}=-\frac{\partial^2 \phi}{\partial t \partial y}, \quad \frac{\partial w}{\partial t}=-\frac{\partial^2 \phi}{\partial t \partial z} \qquad \text{(A)}$$

6.6 Irrotational Motion

Substituting (A) into (6.59) yields

$$\frac{\partial}{\partial x}\left[-\frac{\partial \phi}{\partial t}+\frac{q^2}{2}+gh+\frac{p}{\rho}\right]=0 \qquad x-Eq.$$

$$\frac{\partial}{\partial y}\left[-\frac{\partial \phi}{\partial t}+\frac{q^2}{2}+gh+\frac{p}{\rho}\right]=0 \qquad y-Eq.$$

$$\frac{\partial}{\partial z}\left[-\frac{\partial \phi}{\partial t}+\frac{q^2}{2}+gh+\frac{p}{\rho}\right]=0 \qquad z-Eq. \qquad \text{(B)}$$

6.6 Irrotational Motion

Integrating (B) leads to Bernoulli eq.

$$-\frac{\partial \phi}{\partial t}+\frac{q^2}{2}+gh+\frac{p}{\rho}=F(t) \qquad (6.60)$$

~ valid throughout the entire field of irrotational motion

For a steady flow; $\frac{\partial \phi}{\partial t}=0$

$$\boxed{\frac{q^2}{2}+gh+\frac{p}{\rho}=const.} \qquad (6.61)$$

6.6 Irrotational Motion

→ Bernoulli eq. for a steady, irrotational flow of an incompressible fluid

Dividing (6.61) by g (acceleration of gravity) gives the head terms

$$\frac{q^2}{2g} + h + \frac{p}{\gamma} = const.$$

$$\frac{q_1^2}{2g} + h_1 + \frac{p_1}{\gamma} = \frac{q_2^2}{2g} + h_2 + \frac{p_2}{\gamma} = H \tag{6.62}$$

H = total head at a point; constant for entire flow field of irrotational motion

(for both along and normal to any streamline)

→ point form of 1- D Bernoulli Eq.

p, H, q = values at particular point → point values in flow field

6.6 Irrotational Motion

[Cf] Eq. (4.26)

$$\frac{p_1}{\gamma} + h_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + h_2 + \frac{V_2^2}{2g} = H$$

H = constant along a stream tube

→ 1-D form of 1-D Bernoulli eq.

p, h, V = cross-sectional average values at each section → average values

• Assumptions made in deriving Eq. (6.62)

→ incompressibility + steadiness + irrotational motion+ constant viscosity (Newtonian fluid)

6.6 Irrotational Motion

In Eq. (6.57), viscosity term dropped out because $\nabla \cdot \vec{q} = 0$ (continuity Eq.).

→ Thus, Eq. (6.62) can be applied to either a viscous or inviscid fluid.

• Viscous flow

Velocity gradients result in viscous shear.

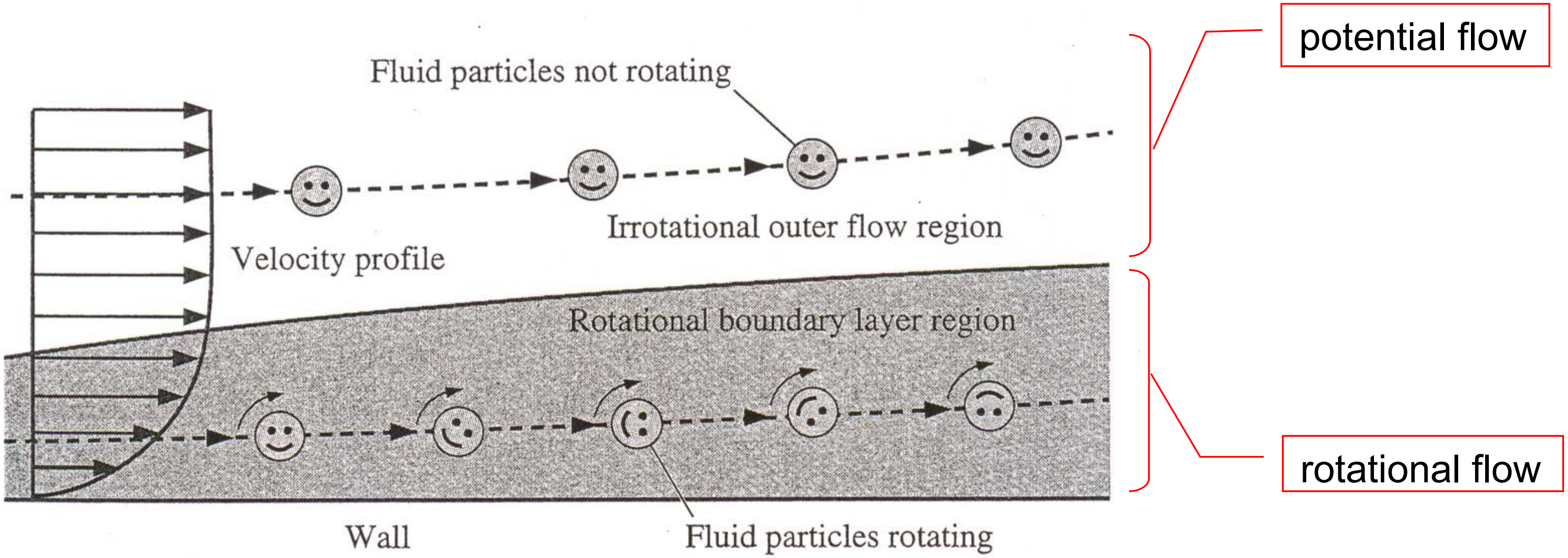
→ Viscosity causes a spread of vorticity (forced vortex).

→ Flow becomes rotational.

→ H in Eq. (6.62) varies throughout the fluid field.

→ Irrotational motion takes place only in a few special cases (irrotational vortex).

6.6 Irrotational Motion



6.6 Irrotational Motion

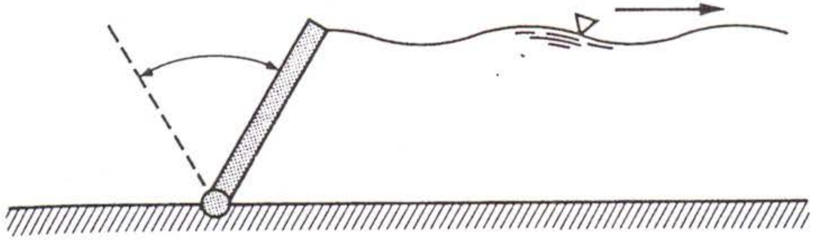
• Irrotational motion can never become rotational as long as only gravitational and pressure force acts on the fluid particles (without shear forces).

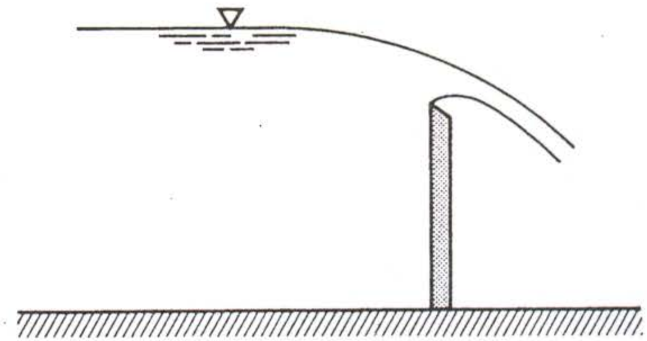
→ In real fluids, nearly irrotational flows may be generated if the motion is primarily a result of pressure and gravity forces.

[Ex] free surface wave motion generated by pressure forces (Fig. 6.8)

flow over a weir under gravity forces (Fig. 6.9)

6.6 Irrotational Motion





6.6 Irrotational Motion

• Vortex motion

i) Forced vortex - rotational flow

~ generated by the transmission of tangential shear stresses

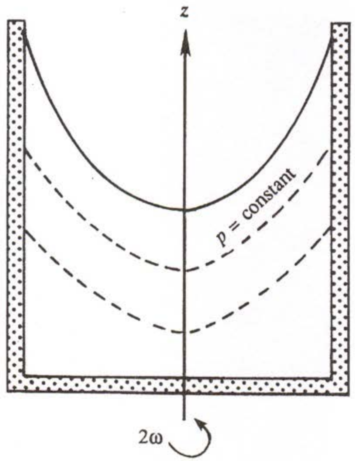
→ rotating cylinder

ii) Free vortex - irrotational flow

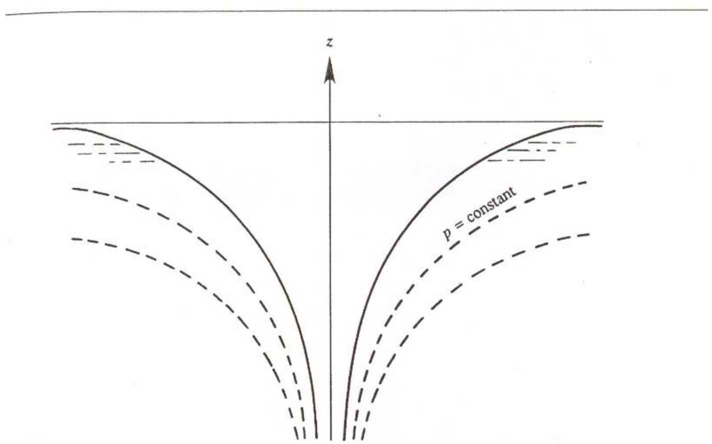
~ generated by the gravity and pressure

→ drain in the tank bottom, tornado, hurricane

6.6 Irrotational Motion



Forced vortex



Free vortex

6.6 Irrotational Motion

- Boundary layer flow (Ch. 8)

i) Flow within thin boundary layer - viscous flow- rotational flow

→ use boundary layer theory

ii) Flow outside the boundary layer - irrotational (potential) flow

→ use potential flow theory

6.7 Frictionless Flow

6.7.1 The Bernoulli equation for flow along a streamline

For inviscid flow

→ Assume no frictional (viscous) effects but compressible fluid flows

→ Bernoulli eq. can be obtained by integrating Navier-Stokes equation along a streamline.

Eq. (6.24a): N-S eq. for ideal compressible fluid ($\mu = 0$)

$$\rho \vec{g} - \nabla p + \cancel{\mu \nabla^2 \vec{q}} + \cancel{\frac{\mu}{3} \nabla (\nabla \cdot \vec{q})} = \rho \frac{\partial \vec{q}}{\partial t} + \rho \left(\vec{q} \cdot \nabla \right) \vec{q}$$

$$\vec{g} - \frac{\nabla p}{\rho} = \frac{\partial \vec{q}}{\partial t} + \left(\vec{q} \cdot \nabla \right) \vec{q} \qquad (6.63)$$

6.7 Frictionless Flow

→ Euler's equation of motion for inviscid (ideal) fluid flow

$$\vec{g} = -g\nabla h$$

Substituting (6.26a) into (6.63) leads to

$$-g\nabla h - \frac{\nabla p}{\rho} = \frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q} \qquad (6.64)$$

$$\vec{i}dx + \vec{j}dy + \vec{k}dz$$

Multiply $d\vec{r}$ (element of streamline length) and integrate along the streamline

$$-g\int \nabla h\cdot d\vec{r} - \int \frac{1}{\rho}\nabla p\cdot d\vec{r} = \int\left(\frac{\partial \vec{q}}{\partial t}\right)\cdot d\vec{r} + \int\left[(\vec{q}\cdot\nabla)\vec{q}\right]\cdot d\vec{r} + C(t) \qquad (6.65)$$

6.7 Frictionless Flow

$$-gh-\int \frac{dp}{\rho}=\int \left(\frac{\partial \vec{q}}{\partial t}\right)\cdot d\vec{r}+\int \underbrace{\left[\left(\vec{q}\cdot\nabla\right)\vec{q}\right]\cdot d\vec{r}}_{\mathrm{I}}+C\left(t\right) \qquad (6.66)$$

$$I=\left[\left(\vec{q}\cdot\nabla\right)\vec{q}\right]\cdot d\vec{r}=d\vec{r}\cdot\left[\left(\vec{q}\cdot\nabla\right)\vec{q}\right]=\vec{q}\cdot\left[\underbrace{\left(d\vec{r}\cdot\nabla\right)}_{\mathrm{II}}\vec{q}\right]$$

By the way,

$$II=d\vec{r}\cdot\nabla=\frac{\partial(\ )}{\partial x}dx+\frac{\partial(\ )}{\partial y}dy+\frac{\partial(\ )}{\partial z}dz$$

$$\therefore\ \left(d\vec{r}\cdot\nabla\right)\vec{q}=\frac{\partial\vec{q}}{\partial x}dx+\frac{\partial\vec{q}}{\partial y}dy+\frac{\partial\vec{q}}{\partial z}dz=d\vec{q}$$

6.7 Frictionless Flow

$$I = \vec{q} \cdot d\vec{q} = d\left(\frac{q^2}{2}\right)$$

$$\therefore \int \left[\left(\vec{q} \cdot \nabla \right) \vec{q} \right] \cdot d\vec{r} = \int d\left(\frac{q^2}{2}\right) = \frac{q^2}{2}$$

Thus, Eq. (6.66) becomes

$$\int \frac{dp}{\rho} + gh + \frac{q^2}{2} + \int \left(\frac{\partial q}{\partial t}\right) \cdot d\vec{r} = -C\left(t\right) \qquad (6.67)$$

For steady motion, $\dfrac{\partial \vec{q}}{\partial t} = 0; C\left(t\right) \rightarrow C$

6.7 Frictionless Flow

$$\int \frac{dp}{\rho} + gh + \frac{q^2}{2} = const. \quad \text{along a streamline} \qquad (6.68)$$

For incompressible fluids, $\rho =$ const.

$$\frac{p}{\rho} + gh + \frac{q^2}{2} = const.$$

Divide by g

$$\boxed{\frac{p}{\gamma} + h + \frac{q^2}{2g} = C \quad \text{along a streamline}} \qquad (6.69)$$

6.7 Frictionless Flow

→ Bernoulli equation for steady, frictionless, incompressible fluid flow

→ Eq. (6.69) is identical to Eq. (6.22). Constant *C* is varying from one streamline to another in a rotational flow, Eq. (6.69); it is invariant throughout the fluid for irrotational flow, Eq. (6.22).

6.7.2 Summary of Bernoulli equation forms

• Bernoulli equations for steady, incompressible flow

1) For irrotational flow

$$H = \frac{p}{\gamma} + h + \frac{q^2}{2g} = \text{constant throughout the flow field} \qquad (6.70)$$

6.7 Frictionless Flow

2) For frictionless flow (rotational)

$$H = \frac{p}{\gamma} + h + \frac{q^2}{2g} = \text{ constant along a streamline} \tag{6.71}$$

3) For 1-D frictionless flow (rotational)

$$H = \frac{p}{\gamma} + h + Ke\frac{V^2}{2g} = \text{ constant along finite pipe} \tag{6.72}$$

4) For steady flow with friction ~ include head loss h_L

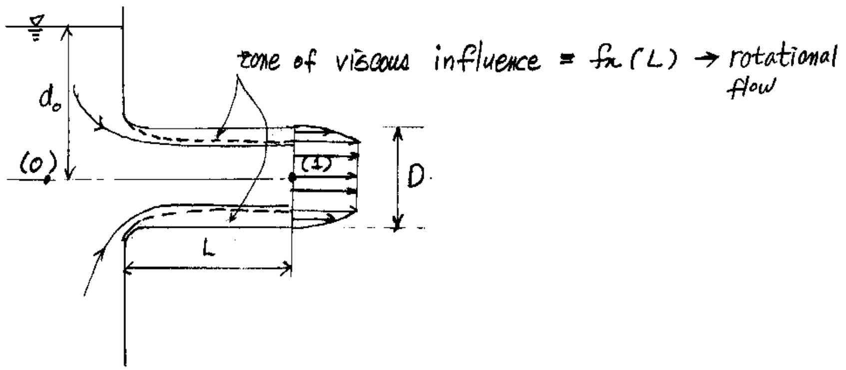
$$\frac{p_1}{\gamma} + h_1 + \frac{q_1^2}{2g} = \frac{p_2}{\gamma} + h_2 + \frac{q_2^2}{2g} + h_L \tag{6.73}$$

6.7 Frictionless Flow

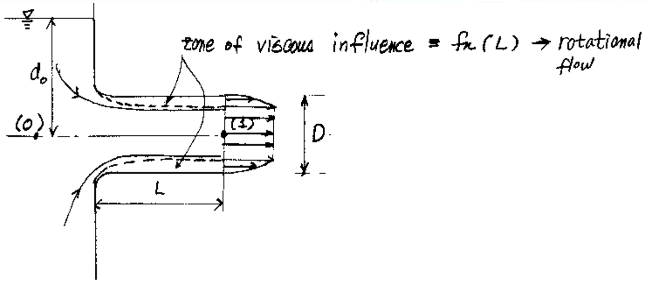
6.7.3 Applications of Bernoulli's equation to flows of real fluids

(1) Efflux from a short tube

- Zone of viscous action (boundary layer): frictional effects cannot be neglected.
- Flow in the reservoir and central core of the tube: primary forces are <u>pressure and gravity</u> forces. → <u>irrotational</u> flow
- Apply Bernoulli eq. along the centerline streamline between (0) and (1)



6.7 Frictionless Flow



$$\frac{p_0}{\gamma} + z_0 + \frac{q_0^2}{2g} = \frac{p_1}{\gamma} + z_1 + \frac{q_1^2}{2g}$$

p_0 = hydrostatic pressure = γd_0 , $p_1 = p_{atm} \rightarrow \; p_{1_{gage}} = 0$

$$z_0 = z_1$$

q_0 = 0 (neglect velocity at the large reservoir)

$$\therefore \; \frac{q_1^2}{2g} = d_0 \qquad q_1 = \sqrt{2gd_0} \qquad \rightarrow \text{Torricelli's result} \qquad (6.74)$$

If we neglect thickness of the zone of viscous influence

$$Q = \frac{\pi D^2}{4} q_1$$

6.7 Frictionless Flow

(2) Stratified flow

During summer months, large reservoirs and lakes become thermally stratified.

→ At thermocline, temperature changes rapidly with depth.

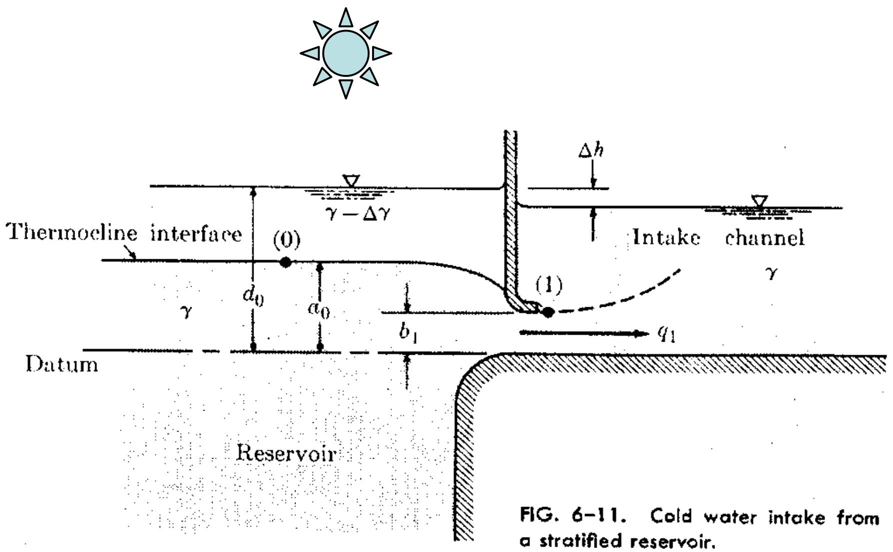
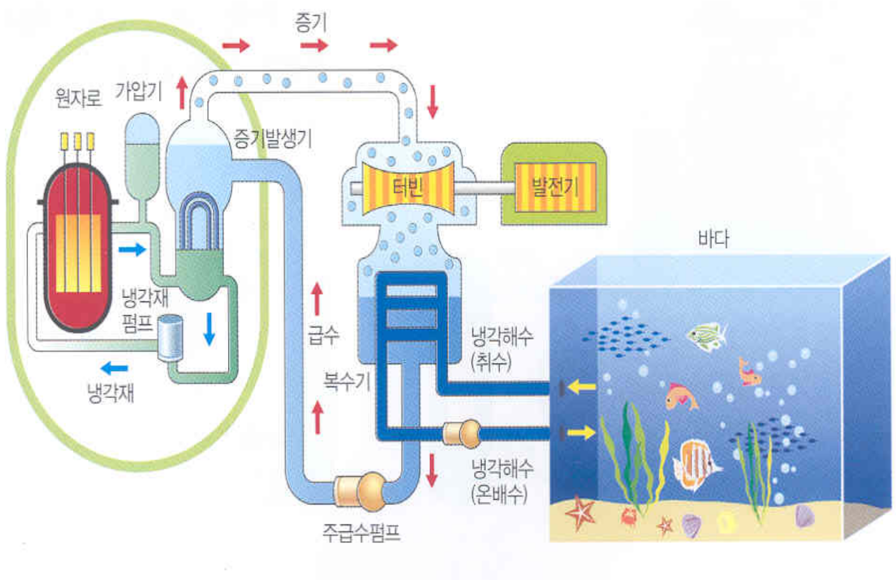
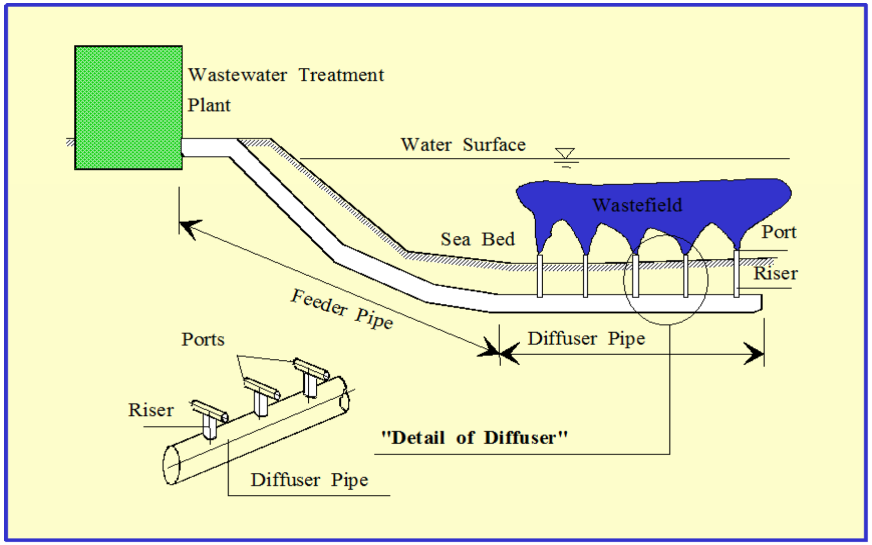


FIG. 6-11. Cold water intake from a stratified reservoir.

- Selective withdrawal: Colder water is withdrawn into the intake channel with a velocity q_1 (uniform over the height b_1) in order to provide cool condenser water for thermal (nuclear) power plant.

6.7 Frictionless Flow





6.7 Frictionless Flow

Apply Bernoulli eq. between points (0) and (1)

$$\frac{p_0}{\gamma} + a_0 + \frac{q_0^{\;2}}{2g} = \frac{p_1}{\gamma} + b_1 + \frac{q_1^2}{2g} \qquad (6.75)$$

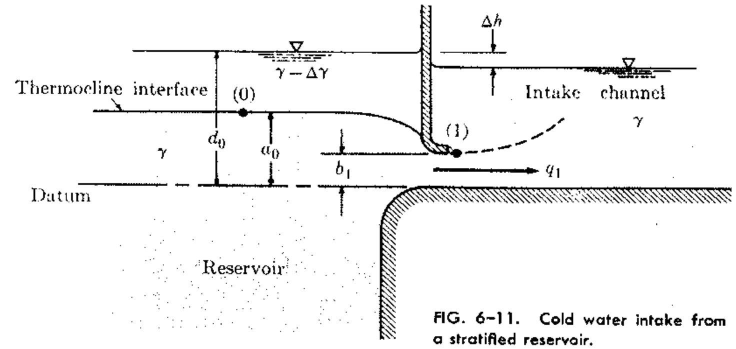


FIG. 6-11. Cold water intake from a stratified reservoir.

$$q_0 \cong 0$$

p_0 = hydrostatic pressure = $(\gamma - \Delta\gamma)(d_0 - a_0)$

$$p_1 = \gamma\left(d_0 - \Delta h - b_1\right)$$

$$\therefore \quad \frac{q_1^{\;2}}{2g} = \Delta h - \frac{\Delta\gamma}{\gamma}\left(d_0 - a_0\right) \qquad (6.76)$$

$$q_1 = \left[2g\left\{\Delta h - \frac{\Delta\gamma}{\gamma}\left(d_0 - a_0\right)\right\}\right]^{\frac{1}{2}} \qquad (6.77)$$

6.7 Frictionless Flow

For isothermal (unstratified) case, $a_0 = d_0$

$$q_1 = \sqrt{2g\Delta h} \quad \rightarrow \text{Torricelli's result} \qquad (6.78)$$

(3) Velocity measurements with the Pitot tube (Henri Pitot, 1732)

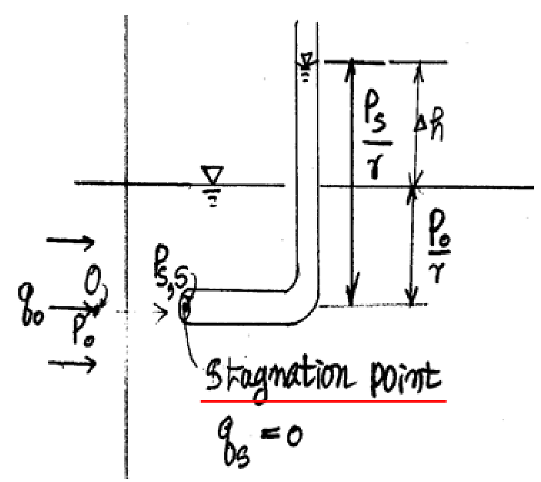
→ Measure velocity from stagnation or impact pressure

$$\frac{p_0}{\gamma} + h_0 + \frac{q_0^{\ 2}}{2g} = \frac{p_s}{\gamma} + h_s + \cancel{\frac{q_s^{\ 2}}{2g}}$$

$$h_0 = h_s, \quad q_s = 0 \qquad (6.79)$$

$$\therefore \quad \frac{q_0^{\ 2}}{2g} = \frac{p_s - p_0}{\gamma} = \Delta h \qquad (6.80)$$

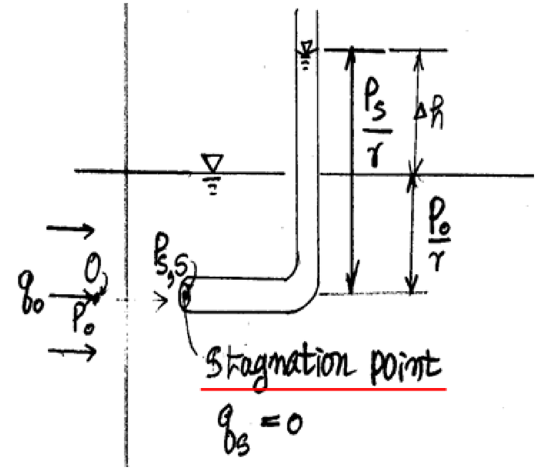
$$q_0 = \sqrt{2g\Delta h} \qquad (6.81)$$



6.7 Frictionless Flow

- Pitot-static tube

$$q_0 = \sqrt{\frac{2(p_s - p_0)}{\rho}} \qquad \text{(A)}$$



By the way,

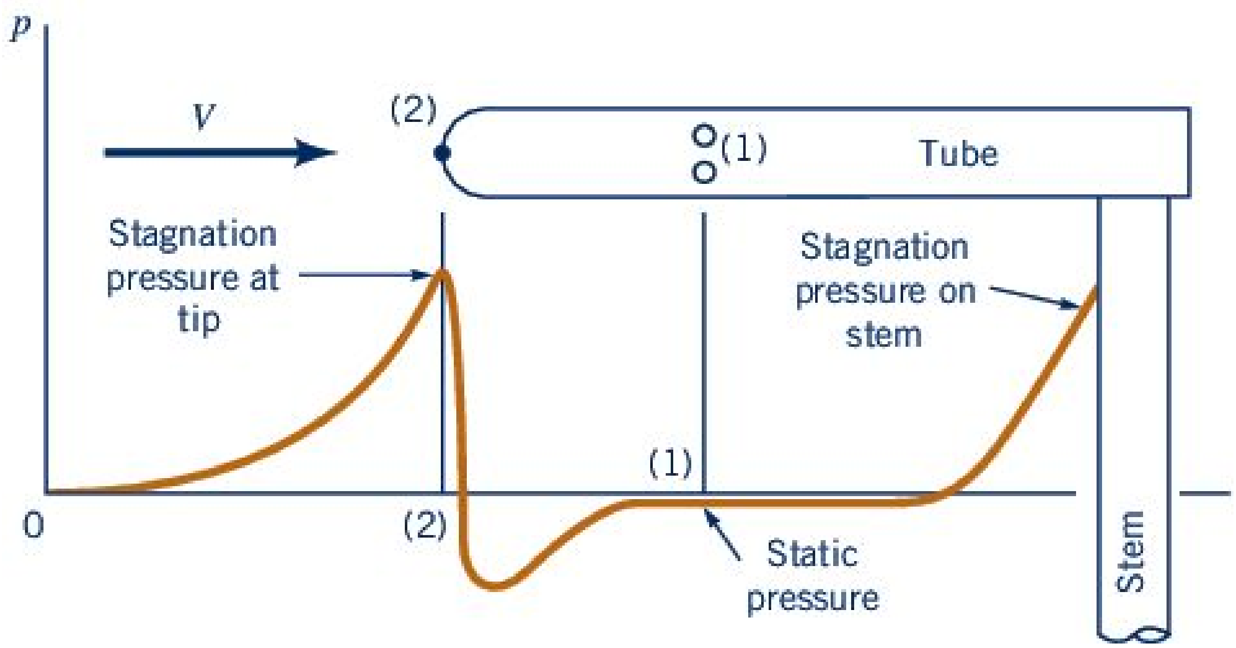
$$p_1 = p_s + \gamma\Delta h = p_2 = p_0 + \gamma_m \Delta h$$

$$p_s - p_0 = \Delta h(\gamma_m - \gamma) \qquad \text{(B)}$$

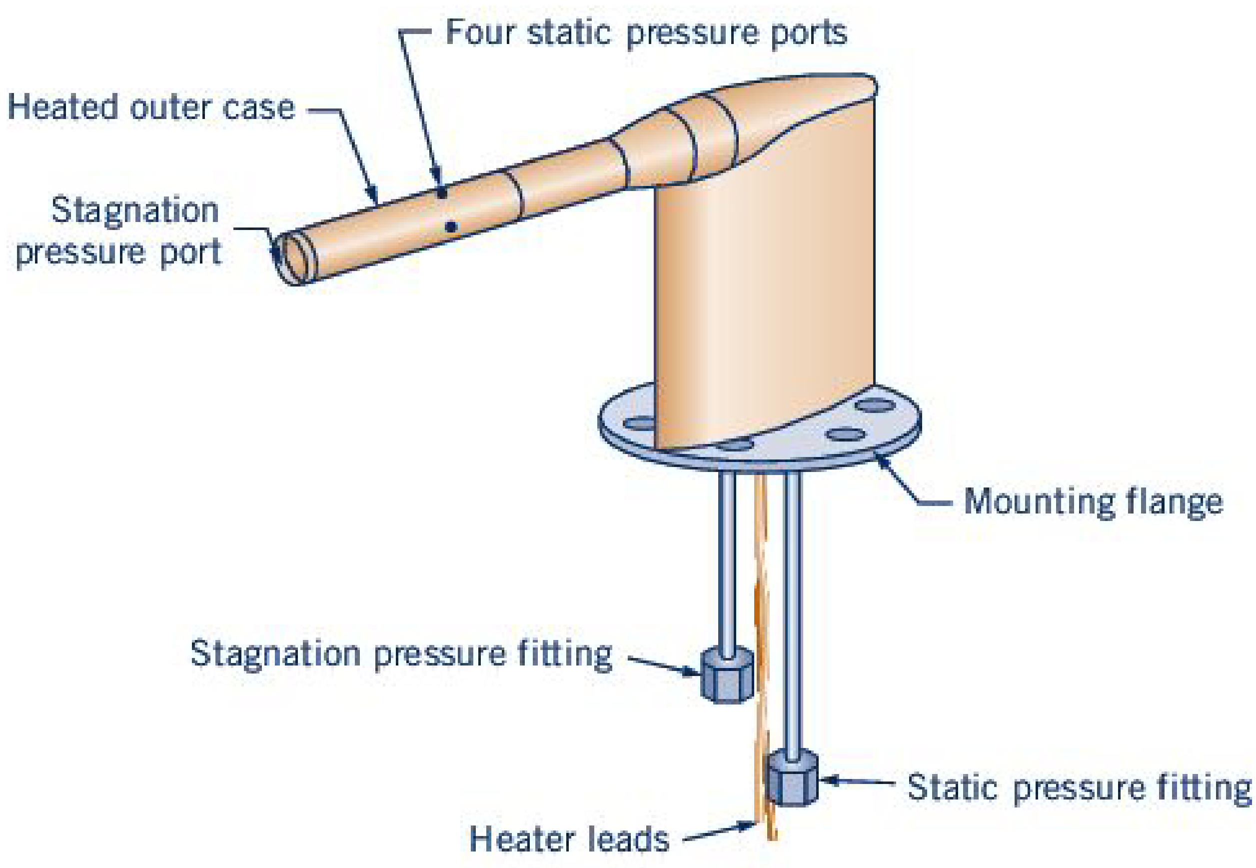
Combine (A) and (B)

$$q_0 = \sqrt{\frac{2\Delta h(\gamma_m - \gamma)}{\rho}}$$

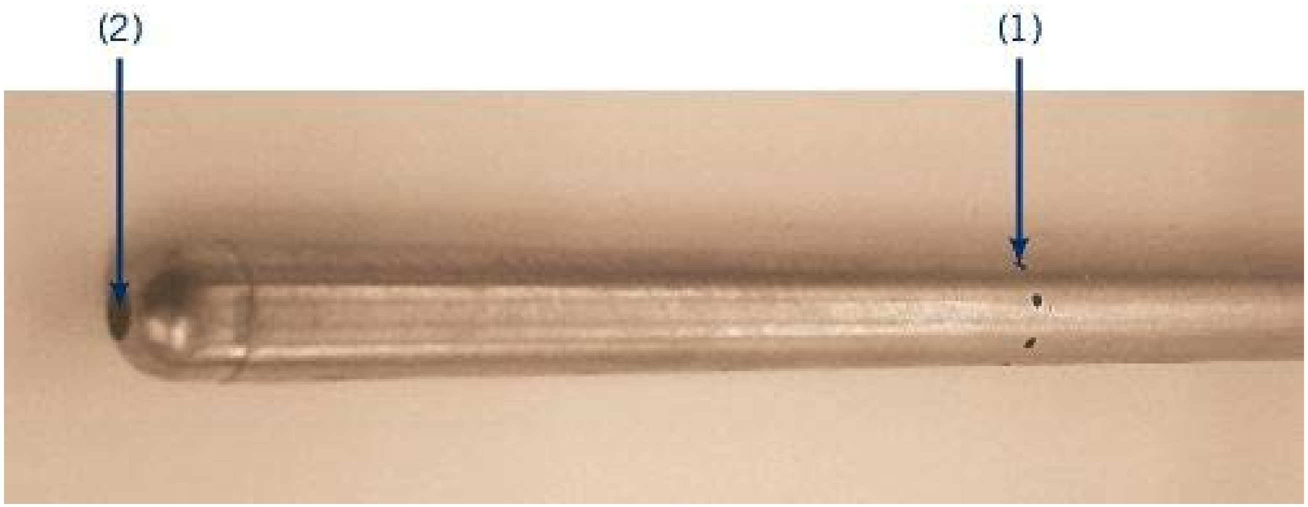
6.7 Frictionless Flow



6.7 Frictionless Flow



6.7 Frictionless Flow



6.8 Vortex Motion

- vortex = fluid motion in which streamlines are concentric circles

For steady flow of an incompressible fluid, apply Navier-Stokes equations in cylindrical coordinates

Assumptions:

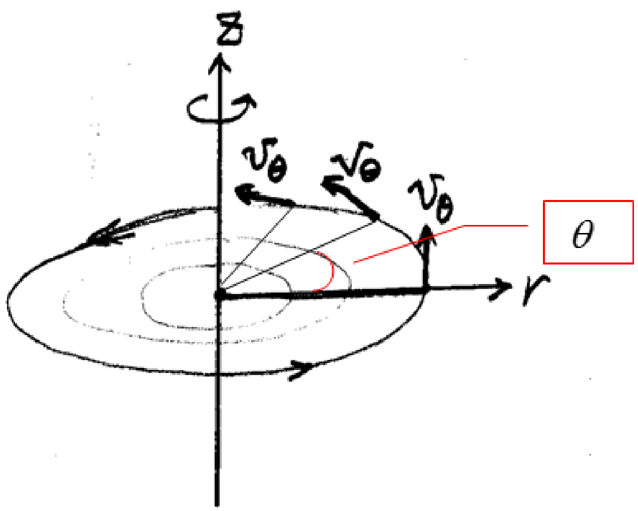
$$\frac{\partial(\ )}{\partial t} = 0$$

$v_\theta \neq 0$

$$v_r = 0; \quad v_z = 0; \quad \frac{\partial v_\theta}{\partial z} = 0$$

$$\frac{\partial p}{\partial \theta} = 0$$

$$\frac{\partial p}{\partial z} = \frac{\partial p}{\partial h}$$ (h =vertical direction)



6.8 Vortex Motion

Continuity Eq.: Eq. (6.30)

$$\frac{1}{r}\frac{\partial}{\partial r}\cancel{\left(rv_r\right)}+\frac{1}{r}\frac{\partial}{\partial \theta}\left(v_\theta\right)+\frac{\partial}{\partial z}\cancel{\left(v_z\right)}=0$$

$$\frac{1}{r}\frac{\partial}{\partial \theta}\left(v_\theta\right)=0 \rightarrow \frac{\partial v_\theta}{\partial \theta}=0 \qquad (6.82)$$

Navier-Stokes Eq.: Eq. (6.29)

1) r-comp.

$$\rho\left(\cancel{\frac{\partial v_r}{\partial t}}+v_r\cancel{\frac{\partial v_r}{\partial r}}+\frac{v_\theta}{r}\cancel{\frac{\partial v_r}{\partial \theta}}-\frac{{v_\theta}^2}{r}+v_z\cancel{\frac{\partial v_r}{\partial z}}\right)$$

6.8 Vortex Motion

$$= -\frac{\partial p}{\partial r} + \mu\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left[\cancel{rv_r}\right]\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_r}{\partial \theta^2}} - \frac{2}{r^2}\cancel{\frac{\partial v_\theta}{\partial \theta}} + \cancel{\frac{\partial^2 v_r}{\partial z^2}}\right\} + \cancel{\rho g_r}$$

$$\frac{v_\theta{}^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r} \qquad (6.83a)$$

2) θ -comp.

$$\rho\left(\cancel{\frac{\partial v_\theta}{\partial t}} + \cancel{v_r}\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\cancel{\frac{\partial v_\theta}{\partial \theta}} - \cancel{\frac{v_r v_\theta}{r}} + \cancel{v_z}\frac{\partial v_\theta}{\partial z}\right)$$

$$= -\frac{1}{r}\cancel{\frac{\partial p}{\partial \theta}} + \mu\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left[rv_\theta\right]\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_\theta}{\partial \theta^2}} - \frac{2}{r^2}\cancel{\frac{\partial v_r}{\partial \theta}} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}}\right\} + \cancel{\rho g_\theta}$$

6.8 Vortex Motion

$$\therefore \; 0 = \frac{\mu}{\rho}\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r v_\theta\right)\right] \tag{6.83b}$$

3) z-comp.

$$\rho\left(\cancel{\frac{\partial v_z}{\partial t}} + \cancel{v_r}\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\cancel{\frac{\partial v_z}{\partial \theta}} - \cancel{\frac{v_r v_\theta}{r}} + v_z\cancel{\frac{\partial v_z}{\partial z}}\right)$$

$$= -\frac{\partial p}{\partial z} + \mu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\cancel{\frac{\partial v_z}{\partial r}}\right) + \frac{1}{r^2}\cancel{\frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}\right\} + \rho g_z$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} + g_z = -\frac{1}{\rho}\frac{\partial p}{\partial h} - g \tag{6.83c}$$

6.8 Vortex Motion

Integrate θ -Eq. (6.83) w.r.t. *r*

$$C_1 = \frac{1}{r}\frac{\partial}{\partial r}\left(rv_\theta\right)$$

$$rC_1 = \frac{\partial}{\partial r}\left(rv_\theta\right)$$

Integrate again

$$\left.\begin{aligned} \frac{r^2}{2}C_1 + C_2 &= rv_\theta \qquad (A) \\ v_\theta &= \frac{C_1}{2}r + \frac{C_2}{r} \qquad (B) \end{aligned}\right\} \text{need 2 BCs} \qquad (6.84)$$

6.8 Vortex Motion

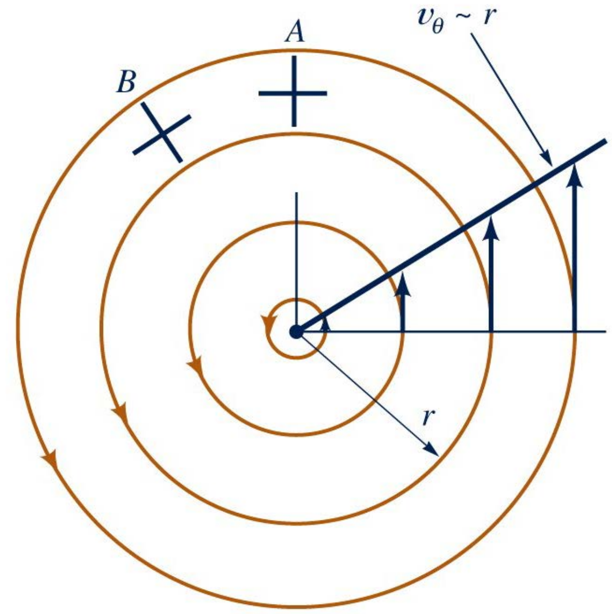
z -Eq.

$$\frac{\partial p}{\partial h} = -\rho g = -\gamma$$

$p = -\gamma h$ → hydrostatic pressure distribution

6.8 Vortex Motion

6.8.1 Forced Vortex - rotational flow



Consider cylindrical container of radius R is rotated at a constant angular velocity Ω about a vertical axis

Substitute BCs into Eq. (6.84)

i) $r = 0, \quad v_\theta = 0$

$$\rightarrow (A):\ 0 + C_2 = 0 \qquad \therefore\ C_2 = 0$$

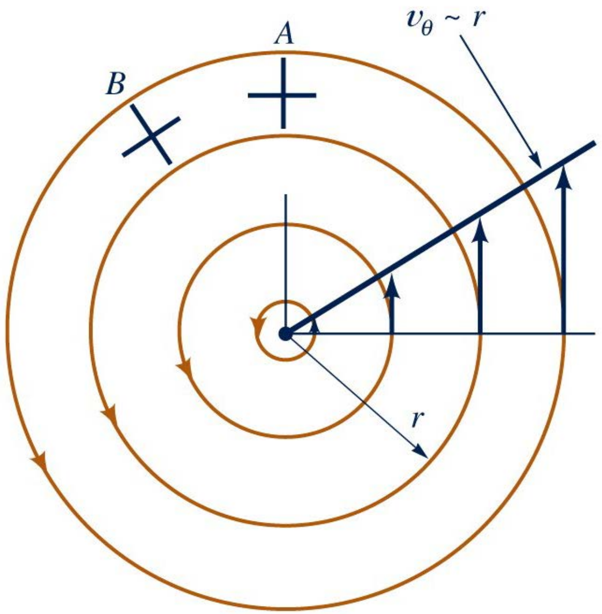
ii) $r = R,\ v_\theta = R\Omega$

$$\rightarrow (B): R\Omega = \frac{C_1}{2} R \qquad \therefore\ C_1 = 2\Omega$$

6.8 Vortex Motion

Eq. (B) becomes

$$\boxed{v_\theta = \frac{2\Omega}{2} r = \Omega r}$$ → solid-body rotation



$$r - Eq.: \ \frac{\Omega^2 r^2}{r} = \frac{1}{\rho}\frac{\partial p}{\partial r} \quad \rightarrow \quad \frac{\partial p}{\partial r} = \rho\Omega^2 r \qquad \text{(C)}$$

$$z - Eq.: \ \frac{\partial p}{\partial h} = -\gamma \qquad \text{(D)}$$

6.8 Vortex Motion

Consider total derivative *dp*

$$dp = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial h}dh = \rho\Omega^2 r dr - \gamma dh$$

Integrate once

$$p = \rho\Omega^2 \frac{r^2}{2} - \gamma h + C_3$$

Incorporate B.C. to decide C_3

$r = 0; \quad h = h_0$ and $p = p_0$

$$p_0 = 0 - \gamma h_0 + C_3 \quad \therefore \quad C_3 = p_0 + \gamma h_0$$

6.8 Vortex Motion

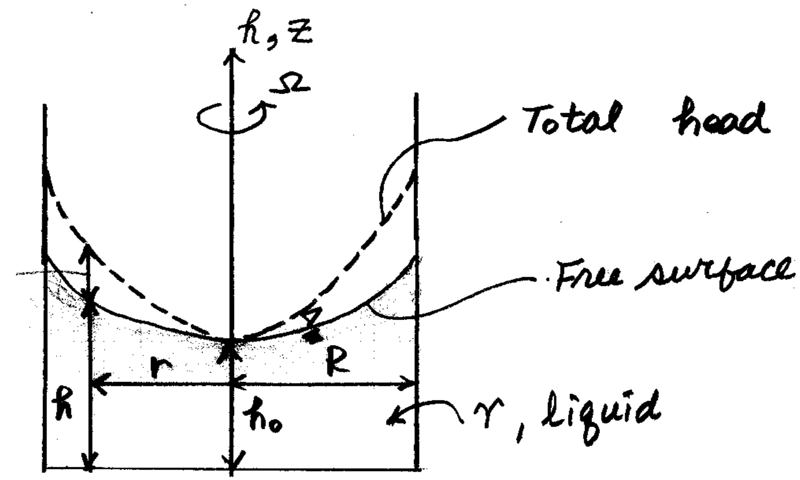
$$p - p_0 = \rho \frac{\Omega^2 r^2}{2} - \gamma (h - h_0)$$

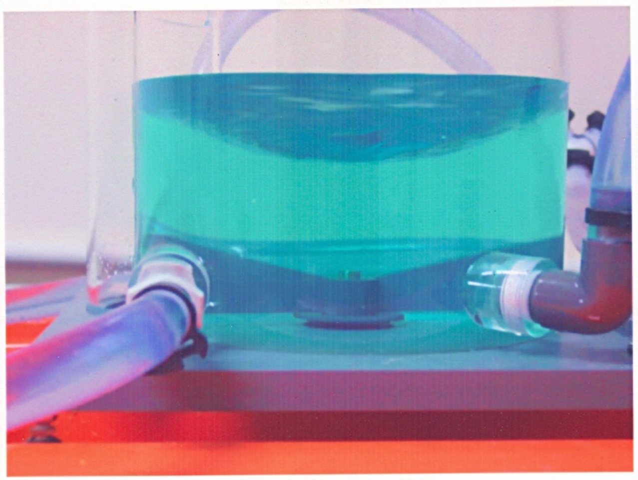
At free surface

$$p = p_0$$

$$h = h_0 + \frac{\Omega^2}{2g} r^2$$

→ paraboloid of revolution





6.8 Vortex Motion

- Rotation components in cylindrical coordinates

Eq. (6.18):

$$\omega_z = \frac{1}{2}\left(\cancel{-\frac{1}{r}\frac{\partial v_r}{\partial \theta}} + \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} \right)$$

$$= \frac{1}{2}\left(\frac{r\Omega}{r} + \frac{\partial}{\partial r}(r\Omega) \right) = \frac{1}{2}(\Omega + \Omega) = \Omega$$

vorticity $= 2\omega_z = 2\Omega \neq 0$

→ rotational flow

→ Forced vortex is generated by the transmission of tangential shear stresses.

6.8 Vortex Motion

- Total head

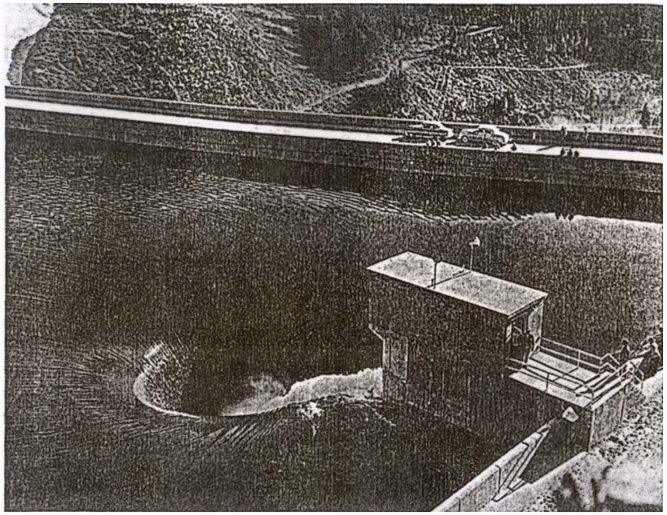
$$H = \frac{p}{\gamma} + h + \frac{{v_\theta}^2}{2g} \neq \text{ const.}$$

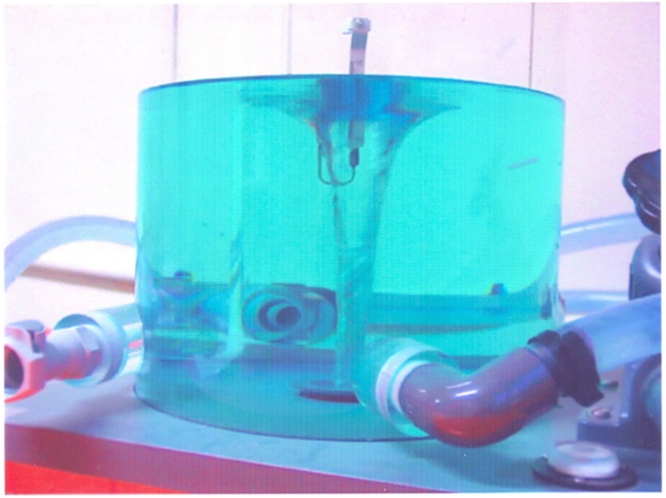
→ increases with radius

6.8 Vortex Motion

6.8.2 Irrotational or free vortex

Free vortex: drain hole vortex, tornado, hurricane, morning glory spillway





6.8 Vortex Motion

For irrotational flow,

$$\frac{p}{\gamma} + h + \frac{{v_\theta}^2}{2g} = \text{const.} \qquad \rightarrow \text{throughout the fluid field}$$

Differentiate w.r.t *r*

$$\frac{1}{\gamma}\frac{\partial p}{\partial r} + \cancel{\frac{\partial h}{\partial r}} + \frac{1}{g} v_\theta \frac{\partial v_\theta}{\partial r} = 0$$

z coincides with h

$$\left(\frac{\partial h}{\partial r} = \frac{\partial h}{\partial \theta} = 0, \frac{\partial h}{\partial z} = 1 \right)$$

$$\therefore \quad \frac{\partial p}{\partial r} = -\rho v_\theta \frac{\partial v_\theta}{\partial r} \qquad \text{(A)}$$

6.8 Vortex Motion

Eq (6.83a): r-Eq. of N-S Eq.

$$\frac{\partial p}{\partial r} = \rho \frac{{v_\theta}^2}{r} \qquad \text{(B)}$$

Equate (A) and (B)

$$-\rho v_\theta \frac{\partial v_\theta}{\partial r} = \rho \frac{{v_\theta}^2}{r} \quad \rightarrow \quad -\frac{\partial v_\theta}{\partial r} r = v_\theta$$

Integrate using separation of variables

$$\int \frac{1}{v_\theta} \partial v_\theta = \int -\frac{1}{r} \partial r$$

6.8 Vortex Motion

$$\ln v_\theta = -\ln r + C$$

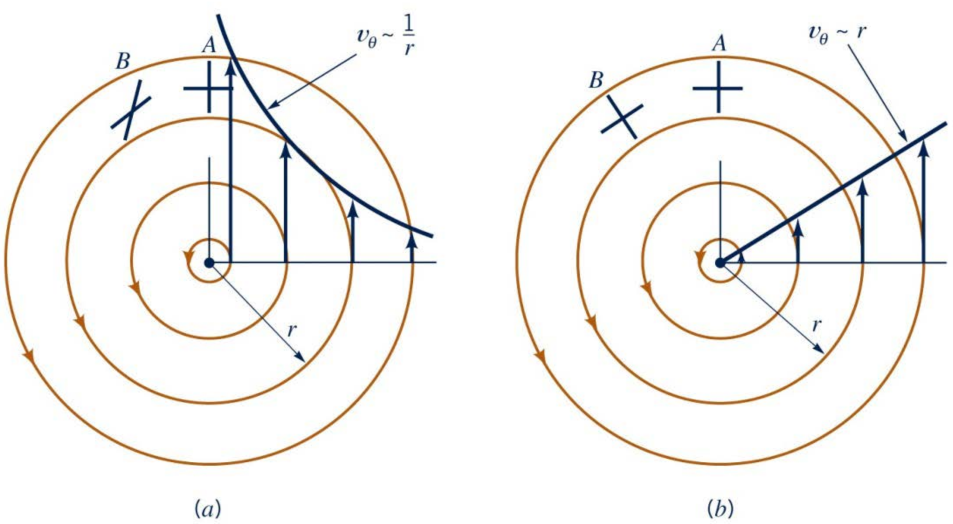
$$\ln v_\theta + \ln r = \ln\left(v_\theta r\right) = C$$

$v_\theta r = C_4$ ~ constant angular momentum

$$\boxed{v_\theta = \frac{C_4}{r}}$$

[Cf] Forced vortex

$$v_\theta = \Omega r$$



6.8 Vortex Motion

- Radial pressure gradient

(B):

$$\frac{\partial p}{\partial r} = \rho \frac{v_\theta{}^2}{r} = \rho \frac{(v_\theta r)^2}{r^3} = \rho \frac{C_4{}^2}{r^3}$$

- Total derivative

$$\boxed{\frac{\partial p}{\partial h} = -\gamma}$$

$$dp = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial h} dh = \rho \frac{C_4{}^2}{r^3} dr - \gamma dh$$

6.8 Vortex Motion

Integrate once

$$p = -\rho \frac{C_4^{\ 2}}{2r^2} - \gamma h + C_5$$

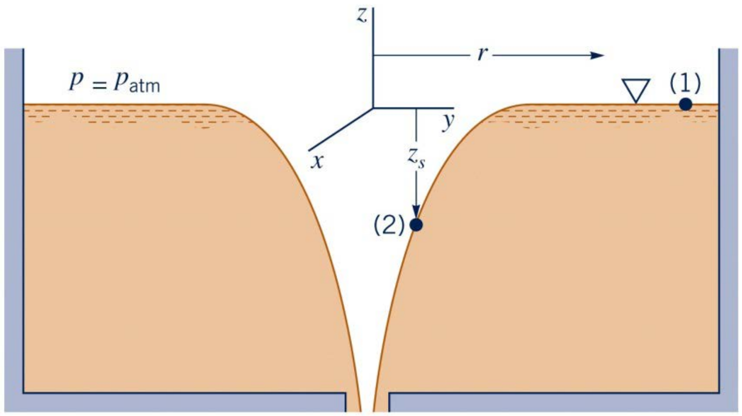
B.C.: $r = \infty:\quad h = h_0$ and $p = p_0$

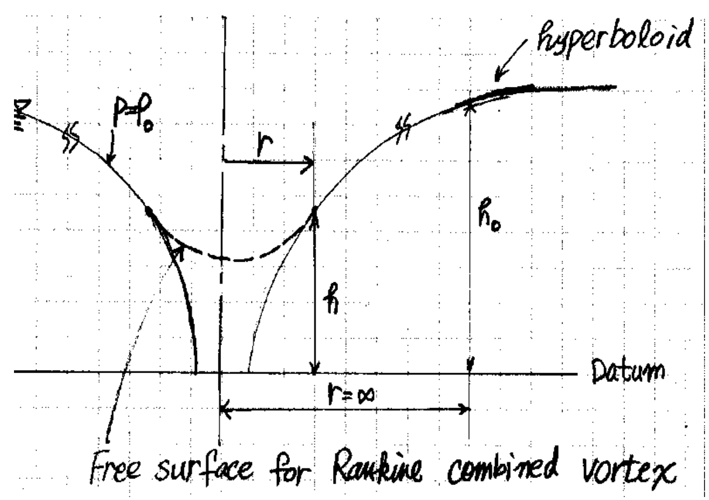
Substitute B.C. into Eq. (6.93)

$$p_0 = -\gamma h_0 + C_5$$

$$C_5 = p_0 + \gamma h_0$$

$$\boxed{p - p_0 = \gamma\left(h_0 - h\right) - \rho \frac{C_4^{\ 2}}{2r^2}}$$





6.8 Vortex Motion

[Cf] Forced vortex: $p - p_0 = \frac{\rho}{2}\Omega^2 r^2 + \gamma\left(h_0 - h\right)$

• Locus of free surface is given when $p = p_0$

$$h = h_0 - \frac{C_4^{\ 2}}{2gr^2} \rightarrow \text{hyperboloid of revolution}$$

[Cf] Forced vortex: $h = h_0 + \frac{\Omega^2}{2g} r^2$

• Circulation

$$\boxed{ds = r\,d\theta}$$

$$\Gamma = \oint \vec{q} \cdot d\vec{s} = \int_0^{2\pi} v_\theta r\,d\theta = \left[C_4 \theta\right]_0^{2\pi} = 2\pi C_4 \neq 0$$

$$\boxed{v_\theta r = C_4}$$

6.8 Vortex Motion

→ Even though flow is irrotational, circulation for a contour enclosing the origin is not zero because of the singularity point.

- Stream function, ψ

$$\boxed{C_4 = \frac{\Gamma}{2\pi}}$$

$$v_\theta = \frac{\partial \psi}{\partial r} = \frac{C_4}{r} = \frac{\Gamma}{2\pi r}$$

$$\psi = \frac{\Gamma}{2\pi}\int \frac{dr}{r} = \frac{\Gamma}{2\pi}\ln r$$

where Γ = vortex strength

6.8 Vortex Motion

- Vorticity component ω_z

$$\omega_z = -\frac{1}{r}\cancel{\frac{\partial v_r}{\partial \theta}} + \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r}$$

Substitute $v_\theta = \dfrac{C_4}{r}$

$$\omega_z = \frac{C_4}{r^2} + \frac{\partial}{\partial r}\left(\frac{C_4}{r}\right) = \frac{C_4}{r^2} - \frac{C_4}{r^2} = 0$$

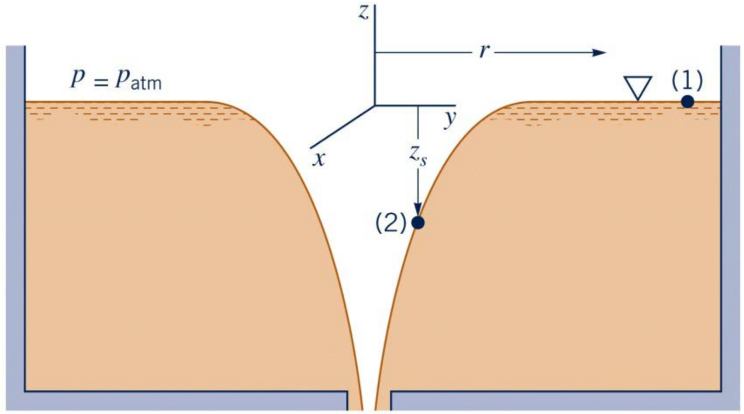
→ Irrotational motion

6.8 Vortex Motion

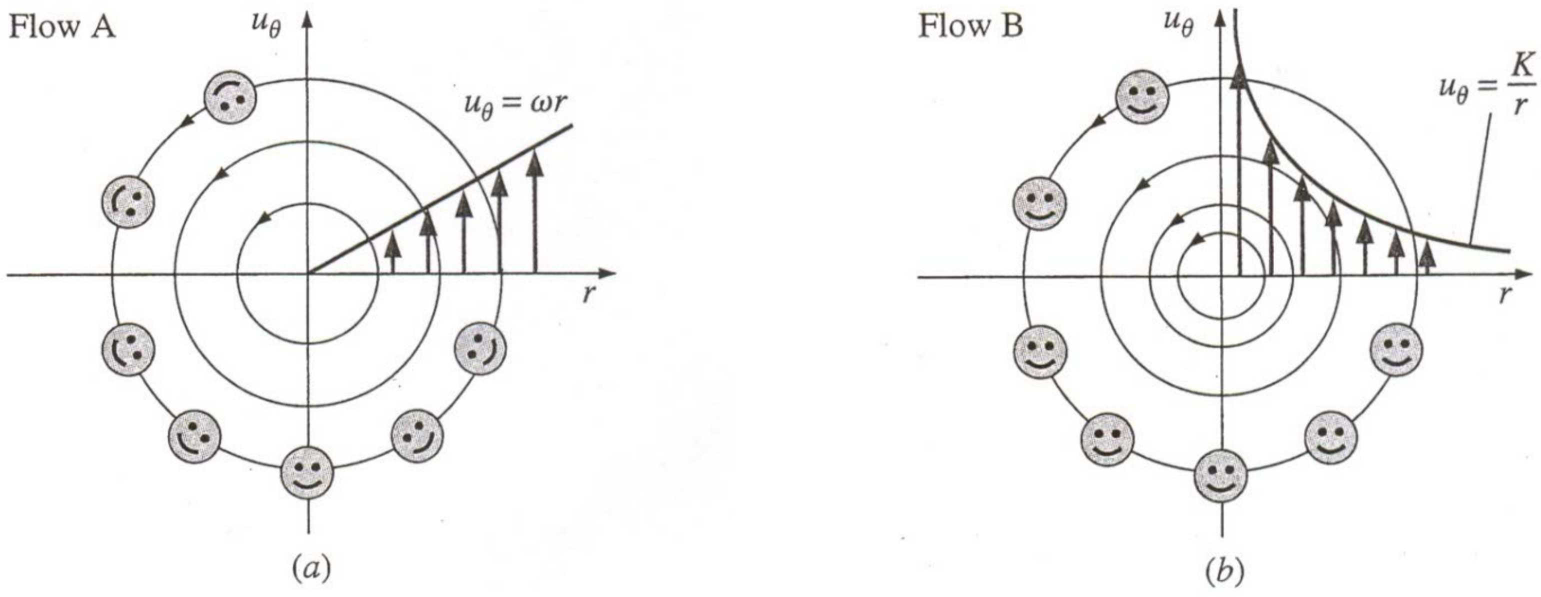
At $r = 0$ of drain hole vortex, either fluid does not occupy the space or fluid is rotational (forced vortex) when drain in the tank bottom is suddenly closed.

→ Rankine combined vortex

→ fluid motion is ultimately dissipated through viscous action



6.8 Vortex Motion



6.8 Vortex Motion

(a) (b)

Homework Assignment

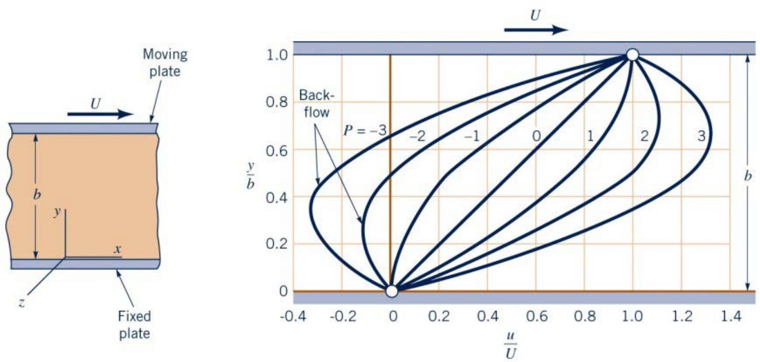
Homework Assignment # 6

Due: 2 weeks from today

1. (6-4) Consider an incompressible two-dimensional flow of a viscous fluid in the xy-plane in which the body force is due to gravity. (a) Prove that the divergence of the vorticity vector is zero. (This expresses the conservation of vorticity, $\nabla \cdot \vec{\zeta} = 0$.) (b) Show that the Navier- Stokes equation for this flow can be written in terms of the vorticity as $\frac{d\vec{\zeta}}{dt} = \nu \nabla^2 \vec{\zeta}$. (This is a "diffusion" equation and indicates that vorticity is diffused into a fluid at a rate which depends on the magnitude of the kinematic viscosity.) Note that $d\vec{\zeta}/dt$ is the substantial derivative defined in Section 2-1.

6.8 Vortex Motion

2. (6-5) Consider a steady, incompressible laminar flow between parallel plates as shown in Fig. 6-4 for the following conditions: a =0.03 m, U =0.3 m/sec, $\mu = 0.476$ N·sec/m^2, $\partial p / \partial x$ =625 N/m^3 (pressure increases in + x-direction). (a) Plot the velocity distribution, $u(z)$, in the z-direction. Use Eq. (6.24) (b) In which direction is the net fluid motion? (c) Plot the distribution of shear stress τ_{zx} in the z-direction.



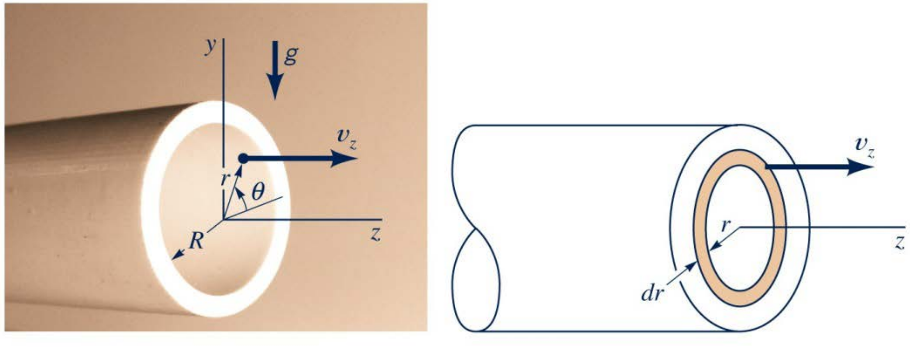
(a) (b)

6.8 Vortex Motion

3. (6-7) An incompressible liquid of density ρ and viscosity μ flows in a thin film down glass plate inclined at an angle α to the horizontal. The thickness, a , of the liquid film normal to the plate is constant, the velocity is everywhere parallel to the plate, and the flow is steady. Neglect viscous shear between the air and the moving liquid at the free surface. Determine the variation in longitudinal velocity in the direction normal to the plate, the shear stress at the plate, and the average velocity of flow.

6.8 Vortex Motion

4. (6-11) Consider steady laminar flow in the horizontal axial direction through the annular space between two concentric circular tubes. The radii of the inner and outer tube are r_1 and r_2, respectively. Derive the expression for the velocity distribution in the direction as a function of viscosity, pressure gradient $\partial p / \partial x$, and tube dimensions.



(a) (b)

6.8 Vortex Motion

5. (6-15) The velocity potential for a steady incompressible flow is given by $\Phi = (-a/2)(x^2 + 2y - z^2)$, where a is an arbitrary constant greater than zero.

(a) Find the equation for the velocity vector $\vec{q} = \vec{i}u + \vec{j}v + \vec{k}w$.

(b) Find the equation for the streamlines in the xz ($y = 0$) plane.

(c) Prove that the continuity equation is satisfied.

6. (6-21) The velocity variation across the radius of a rectangular bend (Fig. 6-22) may be approximated by a free vortex distribution $v_\theta r = const.$ Derive an expression for the pressure difference between the inside and outside of the bend as a function of the discharge Q, the fluid density ρ, and the geometric parameters R and b, assuming frictionless flow.

6.8 Vortex Motion

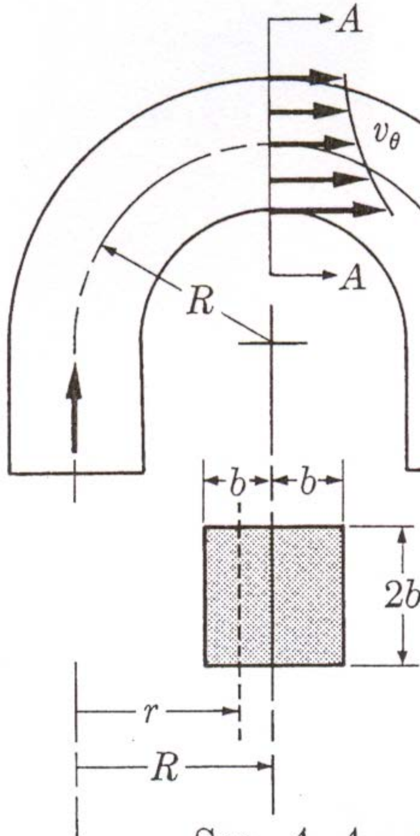


FIGURE 6–22