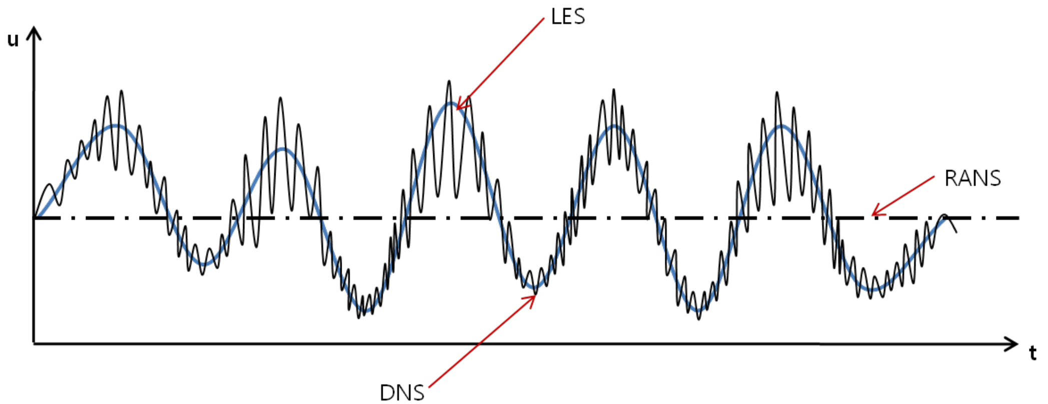
Chapter 10

Turbulence Models and Their Applications



Chapter 10 Turbulence Models and Their Applications

Contents

Objectives

- What is turbulence modeling?
- To derive mean flow equation and specialized equations of motion in natural water bodies
- To study equations of turbulence models

10.1 Introduction

- References

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10.1 Introduction

- Lesieur, M. (1987). Turbulence in Fluids, Netherlands

Turbulence is a dangerous topic which is at the origin of serious fights in scientific meetings since it represents extremely different points of view, all of which have in common their complexity, as well as an inability to solve the problem. It is even difficult to agree on what exactly is the problem to be solved.

10.1 Introduction

10.1.1 The Role of turbulence models

- **Why we need turbulence models?**

- Turbulent flows of practical relevance

→ highly random, unsteady, three-dimensional

→ Turbulent motion (velocity distribution), heat and mass transfer processes are extremely difficult to describe and to predict theoretically.

- **Solution for turbulent flows**

(1) Navier-Stokes equation (DNS)

- Exact equations describing the turbulent motion are known.
- Numerical procedures are available to solve N-S eqs.

10.1 Introduction

- Computer simulations of the full N-S equation are usually limited to flows where periodicity of the flow can be assumed and the boundaries are simple, usually rectangular.

- Size of numerical grids used must be small enough to resolve the smallest significant eddy scale present in the flow, and the simulation must be carried out for a significantly long time that initial conditions have died out and significant features of the flow have evolved.

→ Storage capacity and speed of present-day computers are still not sufficient to allow a solution for any practically relevant turbulent flows.

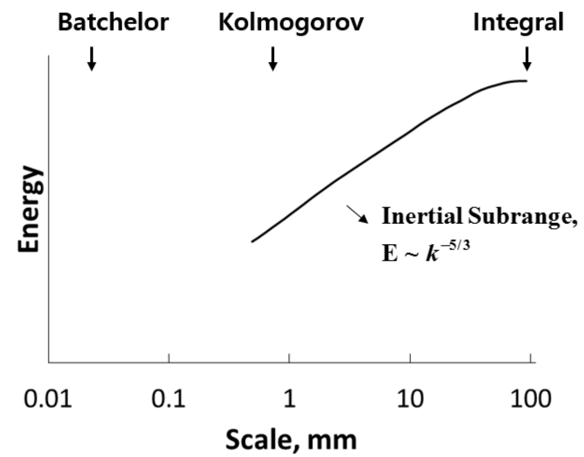
3.1 Introduction

- **Kolmogorov microscale**

- Dissipation length scale of small eddy is

$$\eta \propto \left(\frac{\nu^3}{\varepsilon}\right)^{\frac{1}{4}} \qquad (1)$$

$$\varepsilon \propto \tilde{u}^2 \cdot \frac{\tilde{u}}{l} \propto \frac{\tilde{u}^3}{l} \qquad (2)$$



- l is Integral length scale which is the same as turbulent velocity field.
- Integral scale divided by dissipation length scale is

$$\frac{l}{\eta} = \left(\frac{\tilde{u}l}{\nu}\right)^{3/4} = \mathrm{Re}^{3/4} \qquad (3)$$

- Thus, number of computational grid is proportional $\mathrm{Re}^{3/4}$

$[Ex]\ \mathrm{Re} = 10^5 \rightarrow \text{No. of grid points in 3D} \sim 10^{45/4}$

10.1 Introduction

(2) Reynolds equation (RANS)

- Average N-S equations to remove turbulent fluctuations completely

-Describe the complete effect of turbulence on the average motion by using turbulence model

- Model the small eddies

(3) LES

- numerical resolution of only the large eddies

10.1 Introduction

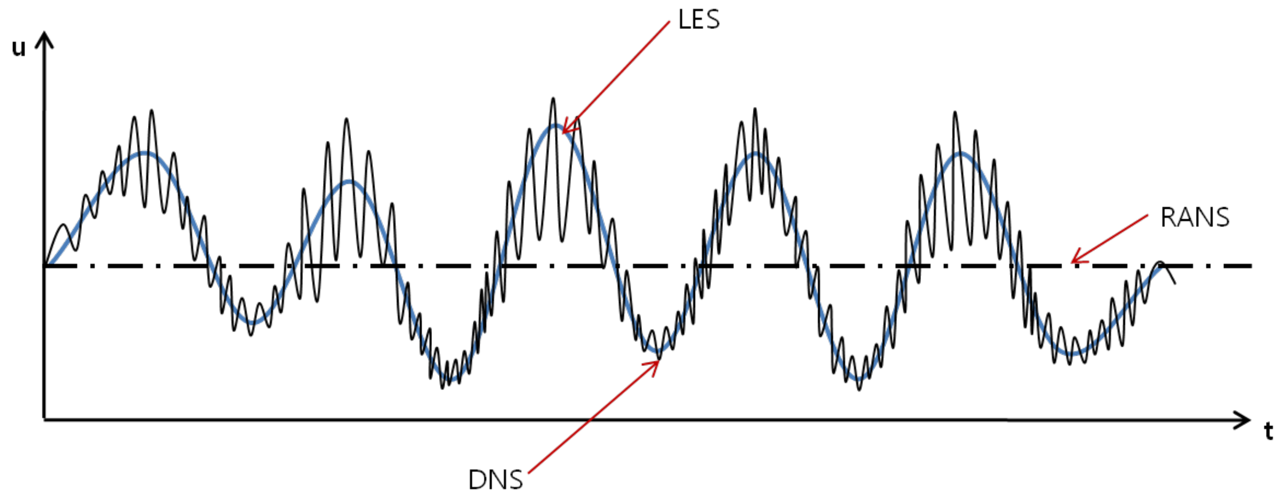
- Turbulence Modeling

DNS: direct numerical simulation of N-S eq.

LES: numerical resolution of only the large eddies (periodic vortex)

RANS: solution of Reynolds-Averaged N-S eq.

→ effects of all turbulent motions are accounted for by the turbulence model



10.1 Introduction

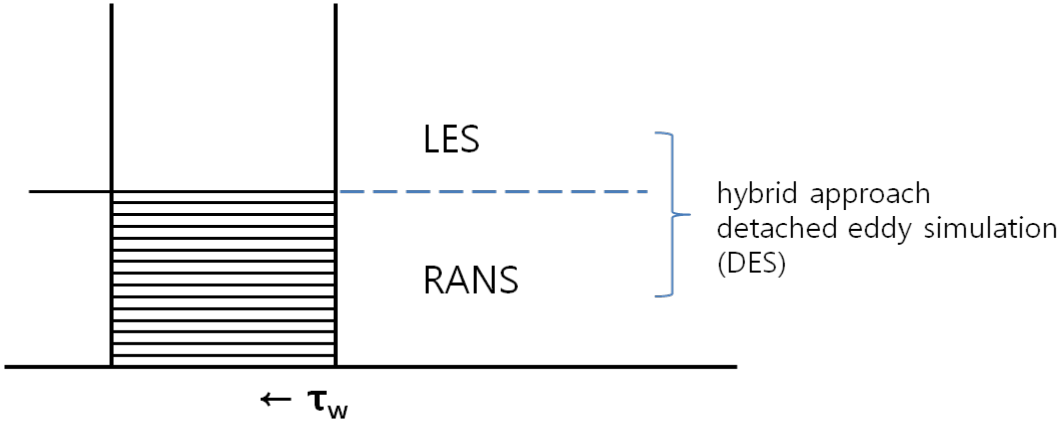
- LES

- small-scale motion is filtered out

$$\overline{u_i} = \int_{\Delta x}\int_{\Delta y}\int_{\Delta z} \frac{u_i dxdydz}{\Delta x \Delta y \Delta z}$$

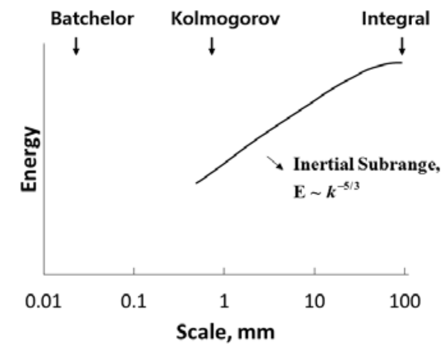
- Hybrid approach

- High Re → wall model is needed



10.1 Introduction

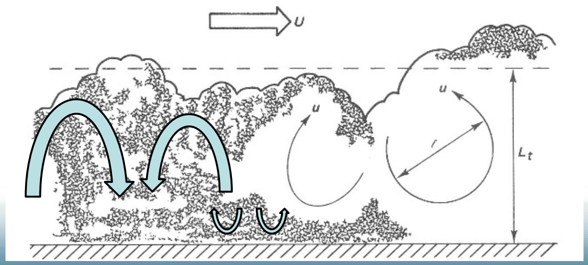
- Scale of turbulence



- eddying motion with a wide spectrum of eddy sizes and a corresponding spectrum of fluctuation frequencies

i) Large-scale eddies:

- contain much of the kinetic energy and little of the vorticity
- eddies tend to be anisotropic
- The forms of the largest eddies (low-frequency fluctuations) are determined by the boundary conditions (size of the flow domain).
- These large eddies gradually break down into smaller eddies.



10.1 Introduction

ii) Small eddies:

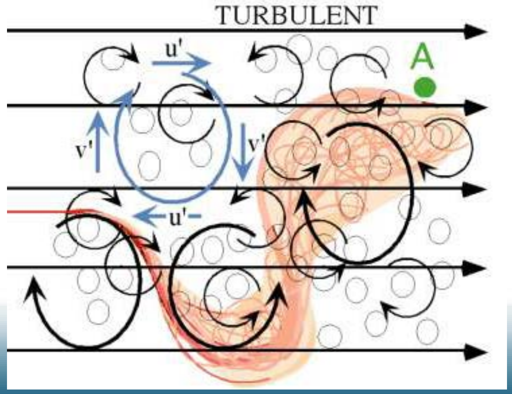
- have little kinetic energy but much vorticity

- The small eddies tend to be isotropic

- The forms of the smallest eddies (highest-frequency fluctuations) are determined by the viscous forces.

- several orders of magnitude smaller

→ In numerical solution, to resolve the small-scale turbulent motion, 10^9 to 10^{12} grid points would be necessary to cover the flow domain in three dimensions.



10.1 Introduction

- Classification of turbulence

i) anisotropic turbulence ~ general turbulence; it varies in intensity in direction

ii) isotropic turbulence ~ smallest turbulence; independent of direction (orientation)

$$\overline{u_i u_j} = \begin{cases} 0, \ i \neq j \\ const. \ , i = j \end{cases}$$

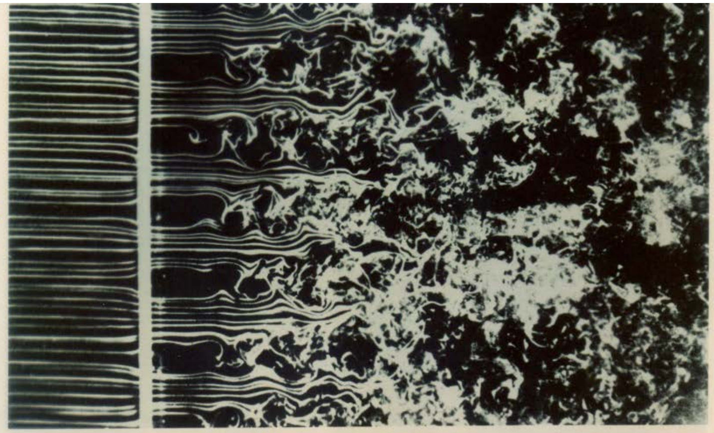
iii) nonhomogeneous turbulence

iv) homogeneous turbulence ~ statistically independent of the location

$$\overline{u_{i_a}^{\ 2}} = \overline{u_{i_b}^{\ 2}}$$

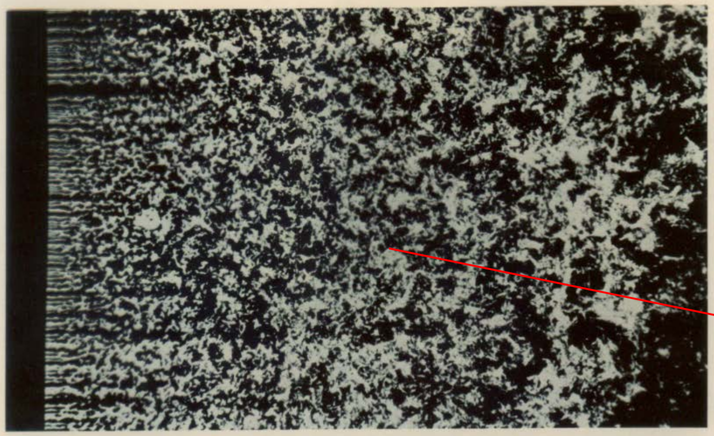
$$\overline{\left(u_i u_j\right)_a} = \overline{\left(u_i u_j\right)_b}$$

10.1 Introduction



152. Generation of turbulence by a grid. Smoke wires show a uniform laminar stream passing through a $\frac{1}{16}$-inch plate with ¼-inch square perforations. The Reynolds number is 1500 based on the 1-inch mesh size. Instability of the shear layers leads to turbulent flow downstream. *Photograph by Thomas Corke and Hassan Nagib*

grid turbulence



153. Homogeneous turbulence behind a grid. Behind a finer grid than above, the merging unstable wakes quickly form a homogeneous field. As it decays downstream, it provides a useful approximation to the idealization of isotropic turbulence. *Photograph by Thomas Corke and Hassan Nagib*

10.1 Introduction

- Turbulence models

~ a set of equations (algebraic or differential) which determine the turbulent transport terms in the mean-flow equations and thus close the system of equations

1) Time-averaging approaches (models)

Model	No. of turbulent transport eqs.	Turbulence quantities transported
Zero equation model	0	None
One equation model	1	k (turbulent kinetic energy)
Two equation model	2	k-ε (turbulent energy, dissipation rate), k-l
Stress/flux model	6	$\overline{u_i u_j}$ components (stress terms)
Algebraic stress model	2	k, ε used to calculate

10.1 Introduction

2) Space-averaged approaches

→ Large Eddy Simulation (LES)

- simulate the larger and more easily-resolvable scales of the motions while accepting the smaller scales will not be properly represented

10.2 Mean Flow Equation and Closure Problem

10.2.1 Reynolds averaged basic equation

- Navier-Stokes eq.

~ Eq. of motion for turbulent motion

~ describes all the details of the turbulent fluctuating motion

~ These details cannot presently be resolved by a numerical calculation procedure.

~ Engineers are not interested in obtaining these details but interested in average quantities.

10.2 Mean Flow Equation and Closure Problem

- Definition of mean quantities by Reynolds

$$U_i = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \tilde{u}_i \, dt \quad (10.1a)$$

$$\Phi = \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} \tilde{\phi} \, dt \quad (10.1b)$$

where $t_2 - t_1$ = averaging time; $\tilde{\phi}$ = scalar quantity (temperature, concentration)

- Averaging time should be long compared with the time scale of the turbulent motion but small compared with that of the mean flow in transient (unsteady) problems.

Example: in stream $t_2 - t_1 \sim 10^1 \sim 10^2$ sec

10.2 Mean Flow Equation and Closure Problem

- Decomposition of instantaneous values

$$\tilde{u}_i = U_i + u_i \qquad (10.2a)$$

$$\tilde{\phi} = \Phi + \phi \qquad (10.2b)$$

↓ mean ↘ fluctuations

Substitute (10.2) into time-dependent equations of continuity and N-S eqs. and average over time as indicated by (10.1)

→ mean flow equations (RANS)

10.2 Mean Flow Equation and Closure Problem

$\frac{\mu}{\rho}\nabla^2 U_i$ (molecular viscosity) dropped

Continuity:

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}=0 \tag{10.3}$$

x-momentum:

$$\frac{\partial U}{\partial t}+\frac{\partial U^2}{\partial x}+\frac{\partial (VU)}{\partial y}+\frac{\partial (WU)}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial x}+fV-\frac{\partial \overline{u^2}}{\partial x}-\frac{\partial \overline{uv}}{\partial y}-\frac{\partial \overline{uw}}{\partial z} \tag{10.4}$$

y-momentum:

$$\frac{\partial V}{\partial t}+\frac{\partial UV}{\partial x}+\frac{\partial (V^2)}{\partial y}+\frac{\partial (WV)}{\partial z} = -\frac{1}{\rho}\frac{\partial P}{\partial y}-fU-\frac{\partial \overline{vu}}{\partial x}-\frac{\partial \overline{v^2}}{\partial y}-\frac{\partial \overline{vw}}{\partial z} \tag{10.5}$$

10.2 Mean Flow Equation and Closure Problem

z-momentum

$$\frac{\partial W}{\partial t}+\frac{\partial (UW)}{\partial x}+\frac{\partial (VW)}{\partial y}+\frac{\partial (W^2)}{\partial z} \quad (10.6)$$

$$=-\frac{1}{\rho}\frac{\partial P}{\partial z}-g-\frac{\partial \overline{wu}}{\partial x}-\frac{\partial \overline{wv}}{\partial y}-\frac{\partial \overline{w^2}}{\partial z}$$

Scalar transport:

$$\frac{\partial \Phi}{\partial t}+\frac{\partial (U\Phi)}{\partial x}+\frac{\partial (V\Phi)}{\partial y}+\frac{\partial (W\Phi)}{\partial z} \quad (10.7)$$

$$=S_\Phi-\frac{\partial \overline{u\phi}}{\partial x}-\frac{\partial \overline{v\phi}}{\partial y}-\frac{\partial \overline{w\phi}}{\partial z}$$

$D\nabla^2 U_i$ (molecular diffusion) dropped

in which P = mean static pressure

f = Coriolis parameter

ρ = fluid density

S_Φ = volumetric source/sink term of scalar quantity

10.2 Mean Flow Equation and Closure Problem

Eqs. (10.3)~(10.7) do not form a closed set.

- Non-linearity of the original N-S eq. and scalar transport eq.

$$\left(\frac{\partial \overline{u^2}}{\partial x}, \frac{\partial \overline{uv}}{\partial y}, \frac{\partial \overline{uw}}{\partial z}, \cdots ; \frac{\partial \overline{u\phi}}{\partial x}, \frac{\partial \overline{v\phi}}{\partial x}, \frac{\partial \overline{w\phi}}{\partial x} \right)$$

→ introduce unknown correlations between fluctuating velocities and between velocity and scalar fluctuations in the averaging processes

$$(\overline{u^2}, \overline{v^2}, \overline{uv}, \cdots\cdots ; \overline{u\phi} \ etc.,)$$

$\overline{\rho u^2} \ etc.$ = rate of transport of momentum = turbulent Reynolds stresses

$\overline{\rho u\phi} \ etc.$ = rate of transport of heat or mass = turbulent heat or mass fluxes

10.2 Mean Flow Equation and Closure Problem

- In Eqs. (10.3)~(10.7), viscous stresses and molecular heat or mass fluxes are neglected because they are much smaller than their turbulent counterparts except in the viscous sublayer very near walls.

- Eqs. (10.3)~(10.7) can be solved for average dependent variables when the turbulence correlation can be determined in some way.

→ task of the **turbulence models**

10.2 Mean Flow Equation and Closure Problem

- Level of a turbulence model

~ depends on the relative importance of the turbulent transport terms

For the turbulent jet motion, simulation of turbulence is important.

For the horizontal motion in large shallow water bodies, refined turbulence modeling is not important because the inertial term in the momentum equations are balanced mainly by the pressure gradient and/or buoyancy terms.

→ The simulation of turbulence in heat and mass transport models is always important because the scalar transport equation does not contain any pressure gradient and/or buoyancy terms.

10.3 Specialized Model Equations

10.3.1 Three-dimensional lake circulation and transport models → Quasi-3D model

- In most shallow water situations and especially in calculating wind-driven lake circulation as well as continental shelf and open coast transport, vertical momentum equation can be reduced by the hydrostatic pressure approximation.

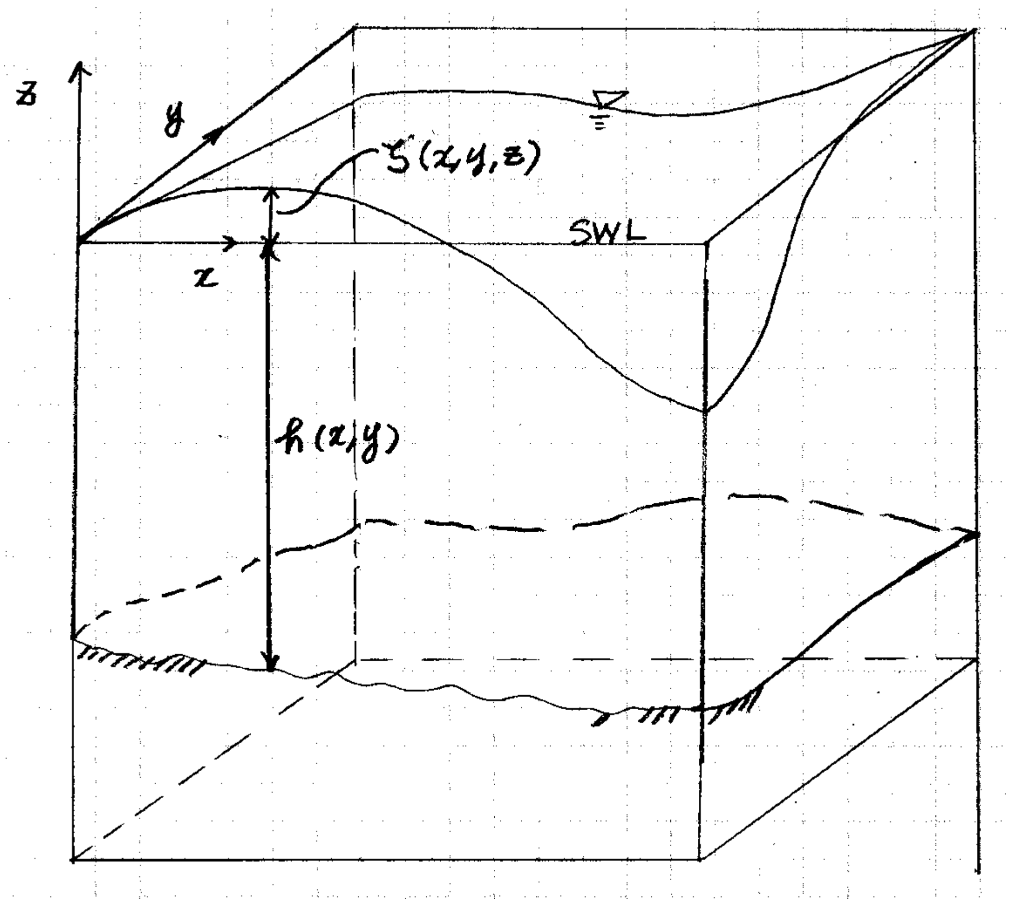
$$\frac{\partial p}{\partial z} = -\rho g \qquad \text{(a)}$$

Simplifies the calculation of the pressure field

Only horizontal two-dimensional pressure distribution must be calculated from the differential equations

The vertical variation of pressure follows Eq. (a).

10.3 Specialized Model Equations



10.3 Specialized Model Equations

▪ Two ways of determining the horizontal variation of pressure

→ Two ways of surface approximation

1) Assume atmospheric pressure at the water surface

→ calculate surface elevation ζ with kinematic boundary condition at the surface

$$\frac{\partial \zeta}{\partial t} + U\frac{\partial \zeta}{\partial x} + V\frac{\partial \zeta}{\partial y} - W = 0 \qquad (10.8)$$

With this kinematic condition, the continuity equation can be integrated over the depth H to yield an equation governing the surface elevation ζ.

10.3 Specialized Model Equations

2) Use rigid-lid approximation

- assume that the surface is covered by a frictionless lid
- allows no surface deformations but permits variations of the surface pressure

→ properly accounts for the pressure-gradient terms in the momentum equations, but an error is made in the continuity equations.

→ is valid when the relative surface elevation ζ /h is small

→ suppresses surface waves and therefore permits longer time steps in a numerical solutions

- Bennett (1974) , J. Physical Oceanography, 4(3), 400-414
- Haq and Lick (1975), J. Geophysical Res, 180, 431-437

10.3 Specialized Model Equations

10.3.2 Two-dimensional depth-averaged models

- For shallow water situations

~ vertical variation of flow quantities is small

~ horizontal distribution of vertically averaged quantities is determined

$$\bar{U} = \frac{1}{H}\int_{-h}^{\zeta} U\, dz; \quad U = \bar{U} + U' \tag{10.9a}$$

$$\bar{\Phi} = \frac{1}{H}\int_{-h}^{\zeta} \Phi\, dz; \quad \Phi = \bar{\Phi} + \Phi' \tag{10.9b}$$

in which H = total water depth $= h + \zeta$

h = location of bed below still water level

ζ = surface elevation

10.3 Specialized Model Equations

Average Eqs. (10.3)-(10.7) over depth

continuity: $$\frac{\partial \zeta}{\partial t}+\frac{\partial (H\bar{U})}{\partial x}+\frac{\partial (H\bar{V})}{\partial y}=0 \qquad (10.10)$$

x-momentum: $$\frac{\partial (H\bar{U})}{\partial t}+\frac{\partial (H\bar{U}^2)}{\partial x}+\frac{\partial (H\bar{V}\bar{U})}{\partial y}=-gH\frac{\partial \zeta}{\partial x}$$
$$+\frac{1}{\rho}\frac{\partial (H\overline{\tau_{xx}})}{\partial x}+\frac{1}{\rho}\frac{\partial (H\overline{\tau_{xy}})}{\partial y}+\frac{\tau_{sx}-\tau_{bx}}{\rho}$$
$$+\frac{1}{\rho}\frac{\partial}{\partial x}\int_{-h}^{\zeta}\rho(U-\bar{U})^2 dz+\frac{1}{\rho}\frac{\partial}{\partial y}\int_{-h}^{\zeta}\rho(U-\bar{U})(V-\bar{V})dz \qquad (10-11)$$

Turbulent shear stress

dispersion stress

10.3 Specialized Model Equations

y-momentum:
$$\frac{\partial(H\bar{V})}{\partial t}+\frac{\partial(H\overline{UV})}{\partial x}+\frac{\partial(H\bar{V}^2)}{\partial y}=-gH\frac{\partial\zeta}{\partial y}$$

$$+\frac{1}{\rho}\frac{\partial(H\overline{\tau_{yx}})}{\partial x}+\frac{1}{\rho}\frac{\partial(H\overline{\tau_{yy}})}{\partial y}+\frac{\tau_{sy}-\tau_{by}}{\rho}$$

$$+\frac{1}{\rho}\frac{\partial}{\partial x}\int_{-h}^{\zeta}\rho(U-\bar{U})(V-\bar{V})dz+\frac{1}{\rho}\frac{\partial}{\partial y}\int_{-h}^{\zeta}\rho(V-\bar{V})^2dz \qquad (10.12)$$

Scalar transport:

Turbulent diffusion

$$\frac{\partial(H\bar{\Phi})}{\partial t}+\frac{\partial(H\bar{U}\bar{\Phi})}{\partial x}+\frac{\partial(H\bar{V}\bar{\Phi})}{\partial y}=\frac{1}{\rho}\frac{\partial(H\bar{J}_x)}{\partial x}+\frac{1}{\rho}\frac{\partial(H\bar{J}_y)}{\partial y}$$

$$+\frac{q_s}{\rho}+\frac{1}{\rho}\frac{\partial}{\partial x}\int_{-h}^{\zeta}\rho(U-\bar{U})(\Phi-\bar{\Phi})dz+\frac{1}{\rho}\frac{\partial}{\partial y}\int_{-h}^{\zeta}\rho(V-\bar{V})(\Phi-\bar{\Phi})dz$$

Shear flow dispersion (10.13)

10.3 Specialized Model Equations

where $\overline{\tau}_{ij}$= depth-averaged turbulent stress ($-\rho\overline{uv}$) acting in x_i-direction on a face perpendicular to x_j ; τ_b = bottom shear stress; τ_s = surface shear stress; $\overline{J}_i$ = depth-averaged turbulent flux of $\Phi(-\rho\overline{u\phi}\ or\ -\rho\overline{v\phi})$ in direction x_i ; q_s = heat flux through surface

①Buoyancy effects

~ cannot be represented in a depth-averaged model because the hydrodynamic model, (10.10) ~ (10.12), is not coupled to the scalar transport model, (10.13).

10.3 Specialized Model Equations

② Turbulent stresses and diffusion terms

▪ Vertical turbulent transport has been eliminated by the depth-averaging and appear only as bottom stresses, τ_b as well surface stresses, τ_s and as surface flux, q_s.

▪ Horizontal momentum transport by the turbulent motion

~ represented by $\overline{\tau}_{ij}$

~ These terms are often neglected in large water body calculations.

~ A turbulence model is needed when terms are important.

▪ Horizontal mass or heat transport by the turbulent motion

~ represented by $\overline{J}_i$

~ A turbulence model is always needed.

10.3 Specialized Model Equations

③ Dispersion terms

~ have same physical effects as turbulent terms but do not represent turbulent transport

~ due to vertical non-uniformities (variations) of various quantities (velocity, concentration)

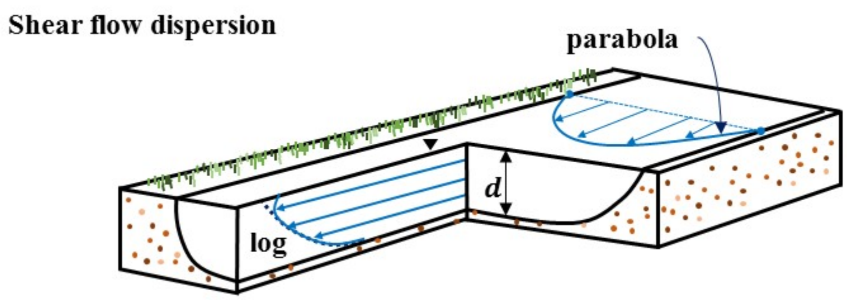
~ consequence of the depth-averaging process

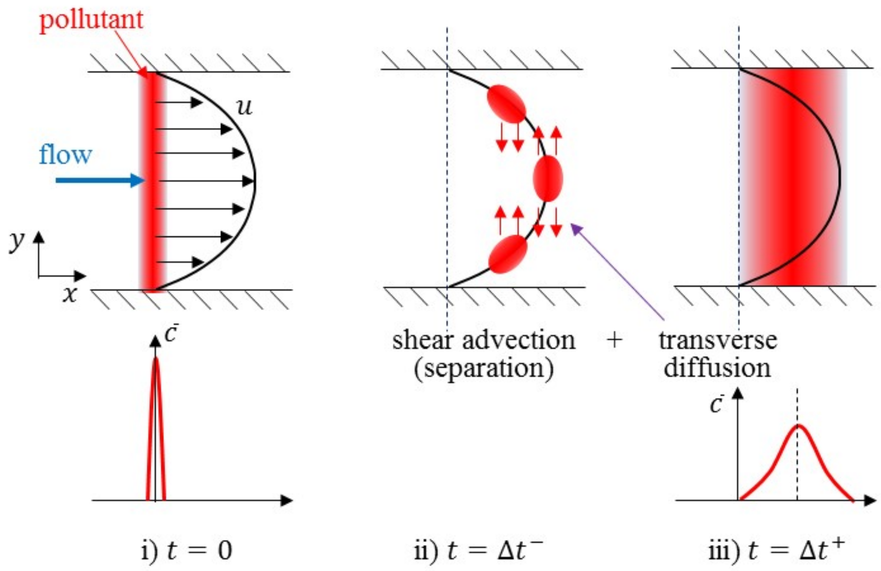
~ are very important in unsteady condition and require accurate modeling (Fischer et al., 1979)

[Re 1] **Dispersion stress model**

For open flows in which vertical variations of the velocity components are significant, such as modeling of the secondary currents in channels, models should be incorporated in order to represent the dispersion stress terms.

10.3 Specialized Model Equations





10.3 Specialized Model Equations

i) Moment of momentum approach

~ use additional equations of moment of momentum equations

~ should solve additional transport equations

$$\frac{\partial \hat{u}_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\frac{q_j \hat{u}_i}{h}\right) + \hat{u}_k \frac{\partial}{\partial x_k}\left(\frac{q_i}{h}\right) = \frac{3}{2}\left[\frac{4\tau_{ij}}{h\rho}\frac{\partial z_m}{\partial x_j} - \frac{4\tau_{iz}}{h\rho} + \frac{2}{h\rho}\tau_{bi}\right]$$

where $\hat{u}_i$ = velocities at the water surface in excess of mean velocity in the x-, y-directions

10.3 Specialized Model Equations

ii) Dispersion stress approach

~ Dispersion stress terms associated with the integration of the products of the fluctuating velocity components are directly calculated by incorporating vertical profiles of both longitudinal and transverse velocities

~ For the vertical profiles of both longitudinal and transverse velocities, several equations can be adopted (Rozovskii, 1961; Kikkawa et al., 1976; de Vriend, 1977; Odgaard, 1986).

Use de Vriend equation, then, the first term (S_{11}) indicates the integration of the products of the discrepancy between the mean and the vertically varying velocity distribution in *x*-direction

10.3 Specialized Model Equations

$$S_{11} = \frac{1}{h}\int_{H}^{H+h} \left(u_1(z) - u_1\right)^2 dz = \int_0^1 \left(u_1(\zeta) - u_1\right)^2 d\zeta$$

$$= u_1^2 \left(\frac{\sqrt{g}}{\kappa C}\right)^2 - 2hu_1 U_1 \frac{\sqrt{g}}{\kappa C} FF_1 + h^2 U_1^2 FF_2$$

where

$$FF_1 = \int_0^1 (1 + \ln\zeta)\, f_s(\zeta)\, d\zeta$$

$$FF_2 = \int_0^1 f_s^2(\zeta)\, d\zeta$$

$$f_s(\zeta) = 2F_1(\zeta) + \frac{\sqrt{g}}{\kappa C} F_2(\zeta) - 2\left(1 - \frac{\sqrt{g}}{\kappa C}\right) f_m(\zeta)$$

$$f_m(\zeta) = 1 + \frac{\sqrt{g}}{\kappa C}(1 + \ln\zeta)$$

$$F_1(\zeta) = \int_0^1 \frac{\ln\zeta}{\zeta - 1} d\zeta$$

$$F_2(\zeta) = \int_0^1 \frac{\ln^2\zeta}{\zeta - 1} d\zeta$$

10.3 Specialized Model Equations

The second term (S_{12}) indicates the integration of the products of the discrepancy in *x*-, and *y*-directions

$$S_{12} = S_{21} = \int_0^1 \left(u_1(\zeta) - u_1\right)\left(u_2(\zeta) - u_2\right) d\zeta$$

$$= u_1 u_2 \left(\frac{\sqrt{g}}{\kappa C}\right)^2 - h\left(u_1 U_1 + u_2 U_2\right)\frac{\sqrt{g}}{\kappa C} FF_1 + h^2 U_1 U_2 FF_2$$

The third term (S_{22}) indicates the integration of the products of the discrepancy *y*-direction

$$S_{22} = \int_0^1 \left(u_2(\zeta) - u_2\right)^2 d\zeta = u_2^2 \left(\frac{\sqrt{g}}{\kappa C}\right)^2 - 2hu_2 U_2 \frac{\sqrt{g}}{\kappa C} FF_1 + h^2 U_2^2 FF_2$$

10.3 Specialized Model Equations

iii) Gradient model → find existing theory

In analogy to eddy viscosity concept (Boussinesq, 1877), assume that the dispersion stresses are proportional to the mean velocity gradients

$$\overline{U'V'} = \frac{1}{H}\int_{-h}^{\zeta}(U-\bar{U})(V-\bar{V})dz = \nu_d \frac{\partial \bar{U}}{\partial y}$$

$$\overline{V'U'} = \frac{1}{H}\int_{-h}^{\zeta}(V-\bar{V})(U-\bar{U})dz = \nu_d \frac{\partial \bar{V}}{\partial y}$$

where ν_d = dispersion viscosity coefficient

10.3 Specialized Model Equations

[Re 2] **Shear flow dispersion**

In direct analogy to the turbulent diffusion, mass transport by dispersion is assumed to be proportional to the gradient of the transported quantity (**Gradient model**).

$$\overline{U_i'\Phi'} = \frac{1}{H}\int_{-h}^{\zeta}(U_i - \bar{U}_i)(\Phi - \bar{\Phi})dz = \Gamma_d \frac{\partial \bar{\Phi}}{\partial x_i}$$

where Γ_d = dispersive diffusivity of heat or mass

→ dispersion mixing coefficient

10.3 Specialized Model Equations

10.3.3 Two-dimensional vertical plane and width-averaged models

Examples:

- long-wave-affected mixing of water masses with different densities

- salt wedges in seiche

- tide-affected estuaries

- separation regions behind obstacles, sizable vertical motion

Define <u>width-averaged quantities</u>

$$\overline{\overline{U}} = \frac{1}{B(x,z)} \int_{y_1(x,z)}^{y_2(x,z)} U \, dy \tag{10.14a}$$

$$\overline{\overline{\Phi}} = \frac{1}{B(x,z)} \int_{y_1(x,z)}^{y_2(x,z)} \Phi \, dy \tag{10.14b}$$

in which *B* = channel width (local width of the flow)

10.3 Specialized Model Equations

(1) Models for the vertical structure are obtained by width-averaging the original three dimensional eqs.

continuity:

$$\frac{\partial}{\partial x}(B\overline{\overline{U}}) + \frac{\partial}{\partial z}(B\overline{\overline{W}}) = 0 \qquad (10.15)$$

x-momentum:

$$\frac{\partial}{\partial t}(B\overline{\overline{U}}) + \frac{\partial}{\partial x}(B\overline{\overline{U^2}}) + \frac{\partial}{\partial z}(B\overline{\overline{WU}}) = -gB\frac{\partial \zeta}{\partial x} - \frac{B}{\rho_0}\frac{\partial p_d}{\partial x}$$

$$+\frac{\tau_{wx}}{\rho_0} + \frac{1}{\rho_0}\frac{\partial}{\partial x}(B\overline{\overline{\tau}}_{xx}) + \frac{1}{\rho_0}\frac{\partial}{\partial z}(B\overline{\overline{\tau}}_{xz})$$

$$\underbrace{+\frac{1}{\rho_0}\frac{\partial}{\partial x}\int_{y_1}^{y_2}\rho(U-\overline{\overline{U}})^2 dy + \frac{1}{\rho_0}\frac{\partial}{\partial z}\int_{y_1}^{y_2}(U-\overline{\overline{U}})(W-\overline{\overline{W}})dy}_{dispersion\ stress}$$

$$(10.16)$$

10.3 Specialized Model Equations

z-momentum:

$$\frac{\partial}{\partial t}(B\overline{\overline{W}})+\frac{\partial}{\partial x}(B\overline{\overline{UW}})+\frac{\partial}{\partial z}(B\overline{\overline{W^2}})=-\frac{B}{\rho_0}\frac{\partial p_d}{\partial z}$$

$$+\frac{\overline{\overline{\rho}}-\rho_0}{\rho_0}\zeta B+\frac{\tau_{wz}}{\rho_0}+\frac{1}{\rho_0}\frac{\partial}{\partial x}(B\overline{\overline{\tau}}_{xz})+\frac{1}{\rho_0}\frac{\partial}{\partial z}(B\overline{\overline{\tau}}_{zz})$$

$$\underbrace{+\frac{1}{\rho_0}\frac{\partial}{\partial x}\int_{y_1}^{y_2}\rho(U-\overline{\overline{U}})(W-\overline{\overline{W}})dy+\frac{1}{\rho_0}\frac{\partial}{\partial z}\int_{y_1}^{y_2}\rho(W-\overline{\overline{W}})^2dy}_{dispersion\ stress} \qquad (10.17)$$

scalar transport :

$$\frac{\partial(B\overline{\overline{\Phi}})}{\partial t}+\frac{\partial(B\overline{\overline{U\Phi}})}{\partial x}+\frac{\partial(B\overline{\overline{W\Phi}})}{\partial z}$$

$$=\frac{Bq_s}{\rho_0}+\frac{1}{\rho_0}\frac{\partial(B\overline{\overline{J}}_x)}{\partial x}+\frac{1}{\rho_0}\frac{\partial(B\overline{\overline{J}}_x)}{\partial z}$$

$$\underbrace{+\frac{1}{\rho_0}\frac{\partial}{\partial x}\int_{y_1}^{y_2}\rho(U-\overline{\overline{U}})(\Phi-\overline{\overline{\Phi}})dy+\frac{1}{\rho_0}\frac{\partial}{\partial z}\int_{y_1}^{y_2}\rho(W-\overline{\overline{W}})(\Phi-\overline{\overline{\Phi}})dy}_{dispersion\ mixing} \qquad (10.18)$$

10.3 Specialized Model Equations

Where ρ_0 = reference density

τ_{wx}, τ_{wz} = side shear stresses

p_d =dynamic pressure

~ pressure due to motion and buoyancy forces

(2) kinematic free surface condition

$$\frac{\partial \zeta}{\partial t} + \overline{\overline{U}} \frac{\partial \zeta}{\partial x} - \overline{\overline{W}} = 0 \qquad (10.19)$$

(3) dispersion terms

~ due to lateral non-uniformities of the flow quantities

10.3 Specialized Model Equations

(4) Further simplification

Replace *z*-momentum Eq. by hydrostatic pressure assumption

$$\frac{\partial p_d}{\partial z} = (\bar{\bar{\rho}} - \rho_0) g \qquad (10.20)$$

Replace $\frac{\partial p_d}{\partial x}$ in *x*-momentum Eq. as

$$\frac{\partial p_d}{\partial x} = g \frac{\partial}{\partial x} \int_z^{\zeta} (\bar{\bar{\rho}} - \rho_0) dz \qquad (10.21)$$

Integrate continuity Eq. (10.15) over the depth and combine with Eq. (10.19)

$$\frac{\partial \zeta}{\partial t} + \frac{1}{B_s} \frac{\partial}{\partial x} \int_{-h}^{\zeta} B \bar{\bar{U}} dz = 0 \qquad (10.22)$$