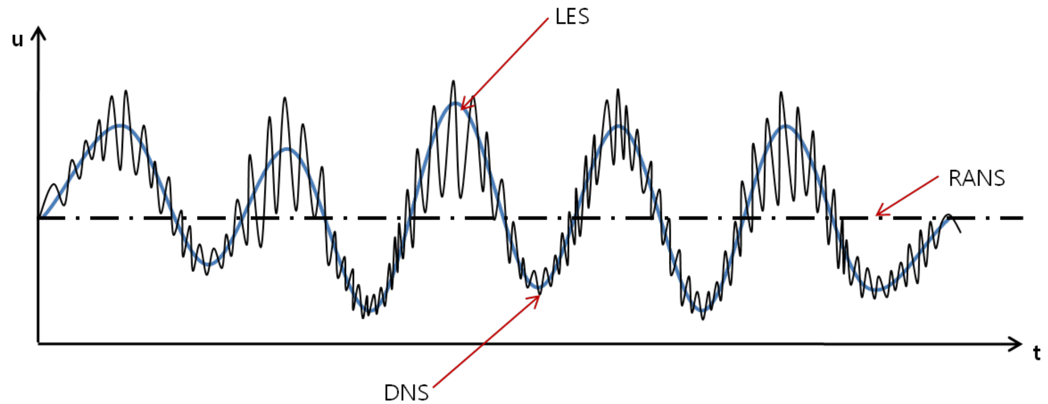
Chapter 10

Turbulence Models and Their Applications



Chapter 10 Turbulence Models and Their Applications

Contents

Objectives

- What is turbulence modeling?
- To derive mean flow equation and specialized equations of motion in natural water bodies
- To study equations of turbulence models

10.4 Turbulence-Closure Models

○ Turbulence model

~ represent the turbulence correlations $\overline{u^2}, \overline{uv}, \overline{u\phi}$ etc. in the mean-flow equations in a way that these equations are closed by relating the turbulence correlations to the averaged dependent variables (*U, V, W*)

○ **Hypotheses** must be introduced for the behavior of these correlations which are based on empirical information.

→ Turbulence models always contain empirical constants and functions.

→ Turbulence models do not describe the details of the turbulent fluctuations (*u, v, w*) but only the average effects of these terms on the mean quantities.

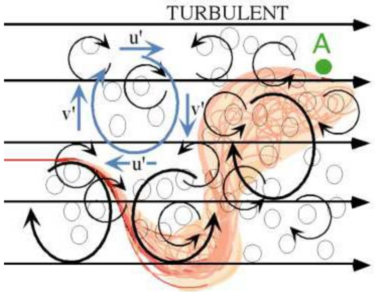
10.4 Turbulence-Closure Models

○ Parameterization of turbulence

~ core of turbulence modeling

~ local state of turbulence and turbulence correlations are assumed to be characterized by only a few parameters.

→ Two important scales are velocity scale and length scale.



○ Three steps of parameterization

1) choose parameters: $v(k), l(\varepsilon)$

2) establish relation between turbulence correlations and chosen parameters: $\nu_t = c_\mu \dfrac{k^2}{\varepsilon}$

3) determine distribution of these parameters over the flow field: $v(x, y, z, t)$

10.4 Turbulence-Closure Models

[Re] Friction coefficient and mixing coefficient

For 1D flow models, parameterization of turbulence and its effects has been achieved by the use of friction coefficients (Chow, 1956) or mixing coefficients (Fischer et el., 1979).

→ In 1D calculations, the flow is assumed to be fully mixed by the turbulence over any cross section so that the only further effect that turbulence can have is to exert wall friction, which can be accounted for adequately by the use of friction coefficients.

But for multi-dimensional flow models, turbulence has been parameterized by constant or mixing-length-controlled eddy viscosities and diffusivities.

10.4 Turbulence-Closure Models

10.4.1 Basic concepts

(1) Eddy viscosity concept

(1) Boussinesq (1877) introduced **eddy viscosity**, ν_t assuming that, in analogy to the viscous stresses in laminar flow, the turbulent stresses are proportional to the mean velocity gradients.

$$\boxed{-\overline{u_i u_j} = \nu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij}} \qquad (10.23)$$

where k = turbulent kinetic energy per unit mass (normal stress)

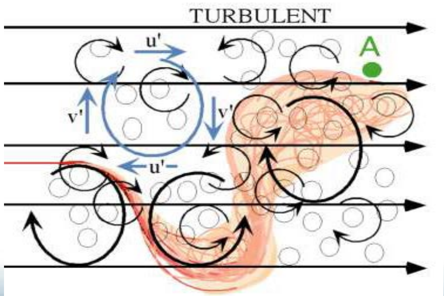
δ_{ij}= Kronecker delta

δ_{ij} = 1 for $i = j$ and δ_{ij} = 0 for $i \neq j$

$$k = \frac{1}{2}(\overline{u^2} + \overline{v^2} + \overline{w^2}) \qquad (10.24)$$

10.4 Turbulence-Closure Models

- This eddy viscosity concept is based on the close analogy between laminar and turbulent stresses, and has often been criticized as physically unsound.
- This concept has often been found to work well in practice because ν_t can be determined to good approximation in many flows.
- Eq. (10.23) alone does not constitute a turbulence model.
- It provides the frame-work for constructing the turbulence model.
- The turbulence model is to determine the distribution of ν_t.



10.4 Turbulence-Closure Models

- Eddy viscosity, ν_t

~ not a fluid property, and depends on state of the turbulence

~ may vary considerably over the flow field

~ is proportional to a velocity scale $\hat{V}$, and a length scale *L*

$$\nu_t \propto \hat{V}L \qquad (10.25)$$

→ it is actually the distribution of the velocity and length scales that can be approximated reasonably well in many flows.

10.4 Turbulence-Closure Models

(2) Eddy diffusivity concept

In direct analogy to the turbulent momentum transport, the turbulent heat or mass transport is assumed to be proportional to the gradient of the transported quantity,

$$\boxed{-\overline{u_i \phi} = \Gamma_t \frac{\partial \Phi}{\partial x_i}} \qquad (10.26)$$

where Γ_t = eddy (turbulent) diffusivity of heat or mass

10.4 Turbulence-Closure Models

- Eddy diffusivity, Γ_t

~ is not a fluid property, like the eddy viscosity, and depends on state of the turbulence.

~ depends in general on the direction of the heat or mass flux → <u>anisotropic</u>

- Relation between eddy viscosity and eddy diffusivity

→ use turbulent <u>Prandtl (heat) or Schmidt number (mass),</u> σ_t

(10.27)

where σ_t ~ is assumed to be constant, is usually <u>less than unity</u>

10.4 Turbulence-Closure Models

10.4.2 Types of turbulence models

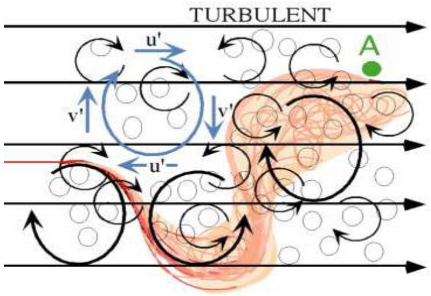
(1) Classification based on the use of eddy viscosity concept

- Classification of turbulence model would be according to whether the models use the eddy viscosity concept.

1) Eddy viscosity model

2) Non- eddy viscosity model: Bradshaw et al.'s model

Reynolds-stress equations



(2) Classification based on the use of transport equations

1) No transport model

- These models do not involve transport equations for turbulence quantities

- These models assume that the <u>turbulence is dissipated by viscous action at the point where it is generated by shear</u>

10.4 Turbulence-Closure Models

2) Transport model

- These models employ transport equations for quantities (k, l, ε) characterizing the turbulence in order to account for the transport of turbulence in space and time.

- These models are adequate in cases where the status of turbulence at a point is influenced by the turbulence generation somewhere else in the flow or by the generation at previous times (history effects).

- These equations, similar to the mass/heat transport equation, contain terms representing both advective transport by mean motion and the diffusive transport by the turbulent motion

10.4 Turbulence-Closure Models

- Classification based on the number of transport equations

It is customary to classify turbulence models according to the number of transport equations used for turbulence parameters.

(i) Zero-Equation Models

- Constant eddy viscosity (diffusivity) model

- Mixing-length model

- Free-shear-layer model

(ii) One-Equation Models

- *k* equation model

- Bradshaw et al.'s model: non-eddy viscosity model

10.4 Turbulence-Closure Models

(iii) Two-Equation Models

- k-ε model

- k-l model

(iv) Turbulent Stress/Flux-Equation Models

- <u>non-eddy viscosity model</u>

• Reynolds-stress equations (Chou, 1945)

- 6 transport equation for momentum transport and 3 transport equation for scalar transport

- employ transport equations for the individual stresses $\overline{u_i u_j}$

10.4 Turbulence-Closure Models

- Algebraic stress/flux models

- Use algebraic relation for $\overline{u_i u_j}$

$$\overline{u_i u_j} = k\left\{\frac{2}{3}\delta_{ij} + \frac{(1-\alpha)\left(\frac{P_{ij}}{\varepsilon}-\frac{2}{3}\delta_{ij}\frac{P}{\varepsilon}\right)+(1-c_3)\left(\frac{G_{ij}}{\varepsilon}-\frac{2}{3}\delta_{ij}\frac{G}{\varepsilon}\right)}{c_1+\left\{\frac{P+G}{\varepsilon}\right\}+1}\right\}$$

- Need to use transport equation of *k*

- Useful tools between the isotropic-eddy viscosity models and stress/flux equation models but have been little tested so far

RANS

Boussinesq' s eddy viscosity equation

$$-\overline{u_i u_j} = \nu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij}$$

$$\nu_t = l_m^{\,2} \left| \frac{\partial U}{\partial z} \right|$$

$$\nu_t = C\delta \left| U_{\max} - U_{\min} \right|$$

$$\nu_t = c_\mu{}' \sqrt{k} L$$

$$\nu_t = c_\mu \frac{k^2}{\varepsilon}$$

Mixing-length models

- $l_m = c\delta$
- $l_m = f(R)$
- $l_m = ky$

Prandtl' s Free-shear-layer model

C = 0.01 Mixing layer
0.014 Plane jet
0.011 Round jet
0.019 Radial jet
0.026 Plane wake

One-equation model

- Transport equation of k
- $L = \kappa \frac{\Psi}{\frac{\partial \Psi}{\partial z}}$

Two-equation model

- Transport equation of k
- Transport equation of ε

Fig. 10-1 Summary of Turbulence Models Based on Eddy Viscosity Concept

RANS

One-equation model

Bradshaw et al.

- **Transport equation of** $\frac{\overline{uv}}{a_1}$
- $L/\delta = f\left(\frac{y}{\delta}\right)$

Turbulent stress/flux-equation models

- Transport equation of $\overline{u_i u_j}$

– 6 transport eqs for momentum transport
– 3 transport eqs for scalar transport

Algebraic stress/flux models

- **Algebraic equation of** $\overline{u_i u_j}$

$$\overline{u_i u_j} = k\left\{\frac{2}{3}\delta_{ij} + \frac{(1-\alpha)\left(\frac{P_{ij}}{\varepsilon} - \frac{2}{3}\delta_{ij}\frac{P}{\varepsilon}\right) + (1-c_3)\left(\frac{G_{ij}}{\varepsilon} - \frac{2}{3}\delta_{ij}\frac{G}{\varepsilon}\right)}{c_1 + \left\{\frac{P+G}{\varepsilon}\right\} + 1}\right\}$$

- Transport equation of k

Fig. 10-2 Summary of Turbulence Models not Employing Eddy-Viscosity Concept

10.4 Turbulence-Closure Models

10.4.3 Zero-equation models

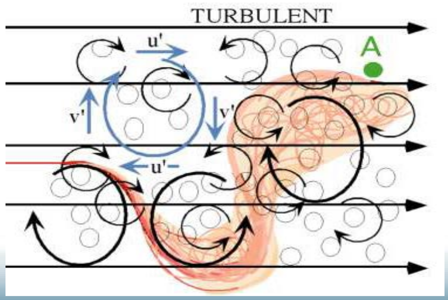
~ do not involve transport equations for turbulence quantities

~ assume that the turbulence is dissipated by viscous action at the point where it is generated by shear

~ there is no transport of turbulence over the flow field

~ employ the eddy viscosity concept

~ specify the eddy viscosity from experiments, by trial and error, through empirical formulae, by relating it to the mean-velocity distribution



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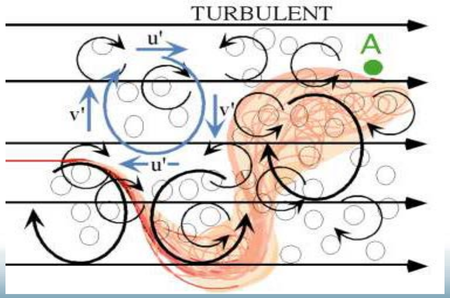


10.4 Turbulence-Closure Models

(1) Constant eddy viscosity (diffusivity) model

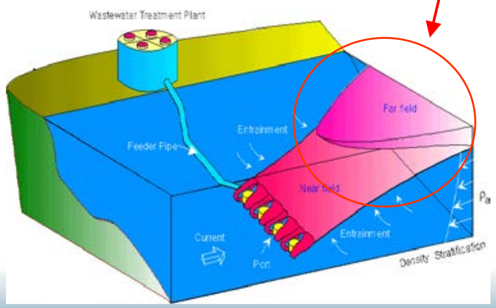
~ the simplest turbulence model

~ used for large water bodies in which the turbulence terms in the momentum equations are unimportant

~ use constant eddy viscosity (diffusivity) over the whole flow field

~ The constant eddy diffusivity model is appropriate only for far-field situations where the turbulence is governed by the natural water body and not by local man-made disturbances such as water intake or discharges.

$$-\overline{u_i u_j} = \nu_t \left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right) - \frac{2}{3} k \delta_{ij} \qquad (10.23)$$



10.4 Turbulence-Closure Models

- **Depth-variable viscosity/diffusivity**

· Open channel flow: ν_t has a nearly parabolic distribution with depth

$$\nu_t = \kappa d u^* \left(\frac{z}{d}\right)\left[1 - \frac{z}{d}\right]$$

· Plane jet: ν_t increases with the one-half power of the distance from the origin

- **Depth-averaged viscosity/diffusivity**

- Constant eddy viscosity (diffusivity) concept has its greatest importance in depth average calculation where only horizontal transport is considered.

10.4 Turbulence-Closure Models

- The depth-averaged 2D model

- Vertical momentum transport is not important.

- The vertical transport of momentum is represented by the bottom shear.

- When turbulences are mainly bed-generated, as in the channel flow, the depth-mean diffusivity for the horizontal transport is given as

$$\overline{\Gamma} = C\,hu^{*}$$

$$u^{*} = \sqrt{\frac{\tau_0}{\rho}}$$

where h = water depth; u^* = friction velocity;

C = empirical constant ~ 0.135 for wide laboratory channels

10.4 Turbulence-Closure Models

[Re] Mixing coefficients for 3D transport model

$$\frac{\partial c}{\partial t} + u\frac{\partial c}{\partial x} + v\frac{\partial c}{\partial y} + w\frac{\partial c}{\partial z} = \frac{\partial}{\partial x}(\varepsilon_x \frac{\partial c}{\partial x}) + \frac{\partial}{\partial y}(\varepsilon_y \frac{\partial c}{\partial y}) + \frac{\partial}{\partial z}(\varepsilon_z \frac{\partial c}{\partial z})$$

<u>Turbulent</u> diffusion coefficients

$$\varepsilon_x = \varepsilon_l = 0.15du^*$$

$$\varepsilon_y = \varepsilon_t = 0.15du^*$$

$$\varepsilon_z = \varepsilon_v = 0.067du^*$$

10.4 Turbulence-Closure Models

- Mixing coefficients for 2D model

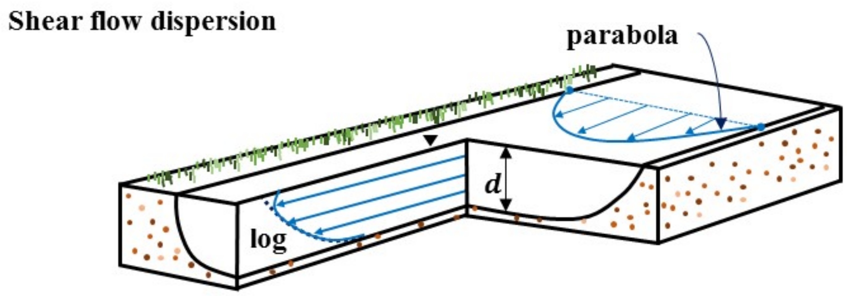
Depth-averaged 2D transport model is

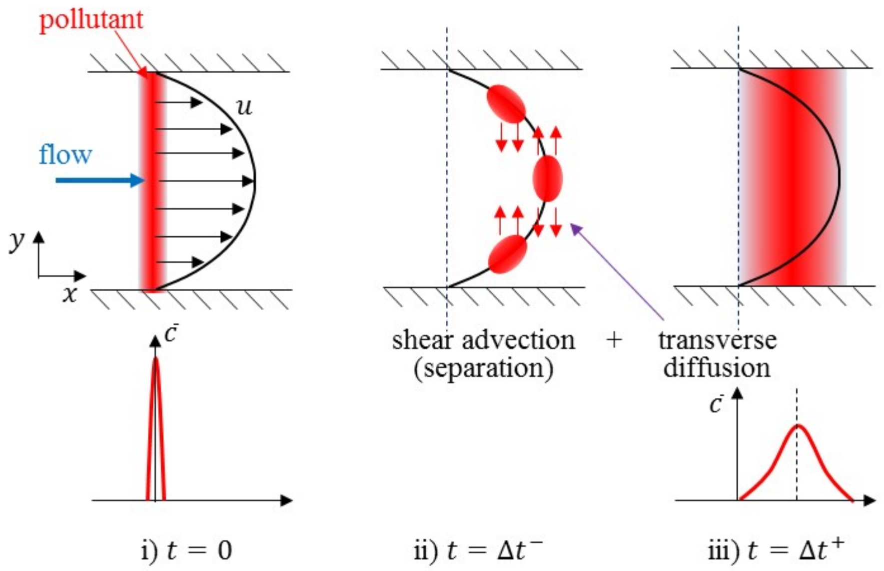
$$\frac{\partial \overline{c}}{\partial t} + \overline{u}\frac{\partial \overline{c}}{\partial x} + \overline{v}\frac{\partial \overline{c}}{\partial z} = D_L \frac{\partial^2 \overline{c}}{\partial x^2} + D_T \frac{\partial^2 \overline{c}}{\partial z^2}$$

The depth-mean diffusivities account for both turbulent transport and the dispersive transport due to vertical non-uniformities of velocity.

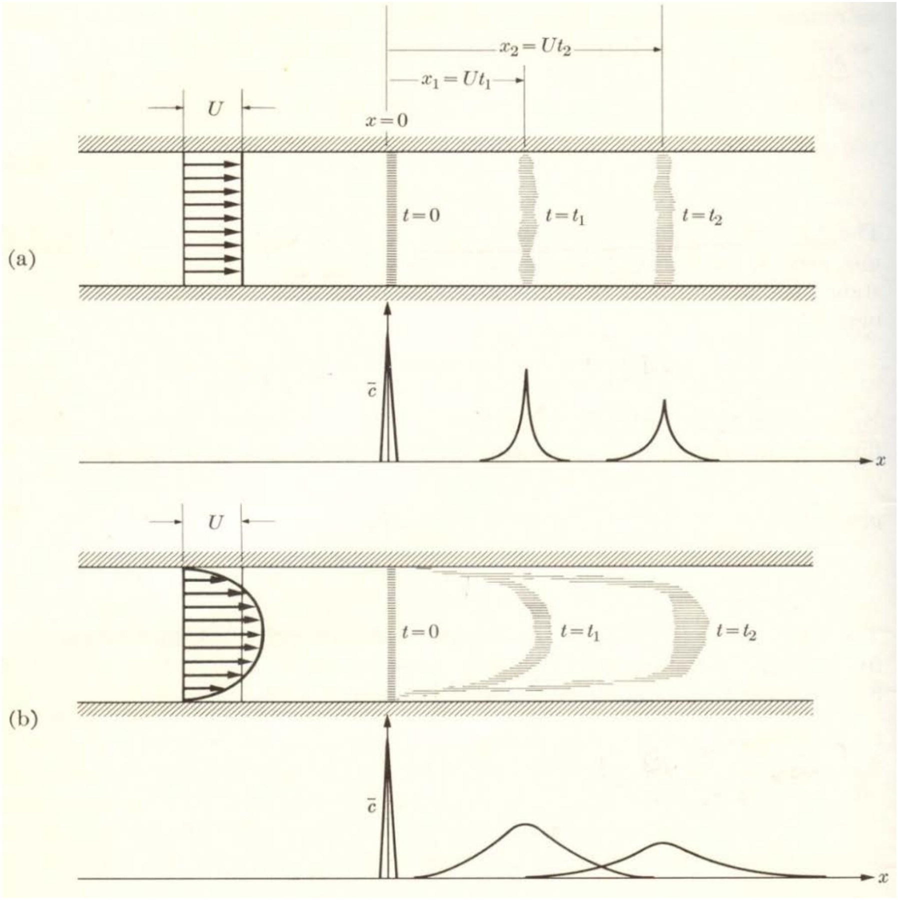
→ Mixing coefficients = dispersion coefficient + turbulent diffusion coefficients

10.4 Turbulence-Closure Models





10.4 Turbulence-Closure Models



Turbulent diffusion in uniform velocity flow vs.
Shear dispersion due to non-uniform velocity distribution (Daily and Harlemann, 1966)

10.4 Turbulence-Closure Models

- Longitudinal mixing coefficient $$D_L = D_l + \varepsilon_\ell$$

- Elder's formula is based on logarithmic velocity distribution (1959)

$$D_l = 5.93du^* \approx 40\varepsilon_l$$

$$\varepsilon_l = 0.15du^*$$

- Observed longitudinal dispersion coefficient is much larger

$$D_L = 60du^* \approx 400\varepsilon_l$$

10.4 Turbulence-Closure Models

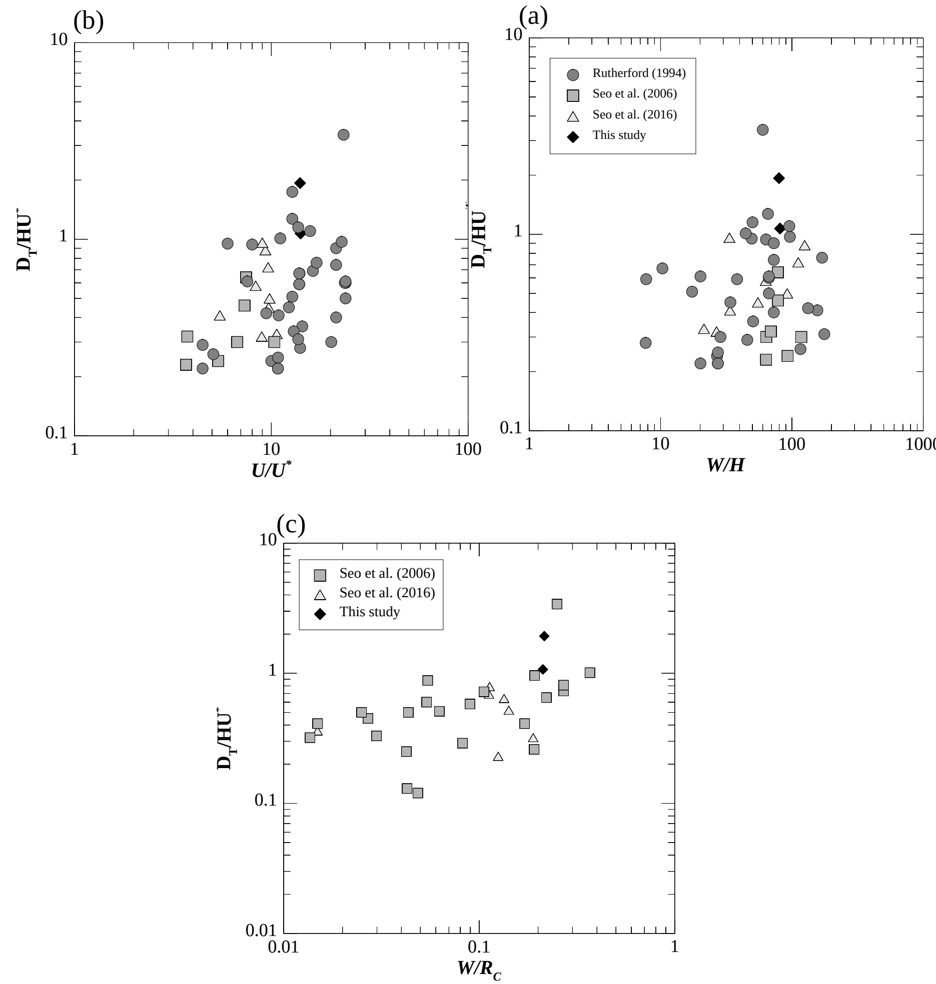
- Transverse mixing coefficient

$$D_T = D_t + \varepsilon_t$$

$$\varepsilon_t = 0.15du^*$$

$$\frac{D_t}{du^*} = 0.3 \sim 3.0 = (2 \sim 20)\,\varepsilon_t$$

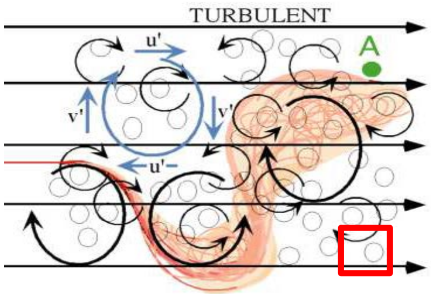
$$\frac{D_t}{du^*} = 0.029\left(\frac{\bar{u}}{u^*}\right)^{0.463}\left(\frac{W}{d}\right)^{0.299} S_n^{\ 0.733}$$



10.4 Turbulence-Closure Models

▪ Mixing coefficients in numerical model

· In numerical calculations of large water bodies, additional processes are represented by the diffusivity.



i) Sub-grid advection

Owing to computer limitations, the numerical grid of the numerical calculations cannot be made so fine as to obtain grid-independent solutions.

→ All advective motions smaller than the mesh size, such as in small recirculation zones, cannot be resolved. Thus, their contribution to the transport must be accounted for by the diffusivity (numerical dispersion).

10.4 Turbulence-Closure Models

ii) Numerical diffusion (Truncation error)

The approximation of the differential equations by difference equations introduces errors which act to smooth out variations of the dependent variables and thus effectively increase the diffusivity.

→ This numerical diffusion is larger for coarser grids.

· An effective diffusivity accounts for turbulent transport, numerical diffusion, sub-grid scale motions, and dispersion (in the case of depth-average calculations).

→ The choice of a suitable mixing coefficient (D_{MT}) is usually not a turbulence model problem but a matter of numerical model calibration.

For 2D model,

$$D_{MT} = D_t + \varepsilon_t + \varepsilon_{sgm} - \varepsilon_{nd}$$

10.4 Turbulence-Closure Models

(2) Mixing-length model

- Application:

For near-field problems involving discharge jets, wakes, and the vicinity of banks and structures, assumption of a constant eddy viscosity is not sufficient.

→ distribution of ν_t over the flow field should be determined

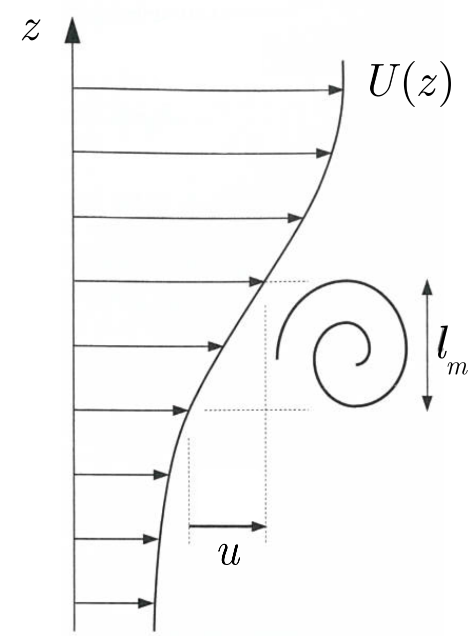
Mixing length, l_m is defined as the cross-stream distance traveled by a fluid particle before it gives up its momentum and loses its identity.

10.4 Turbulence-Closure Models

- Prandtl's mixing-length hypothesis (Prandtl, 1925)

- Prandtl assumed that eddy viscosity ν_t is proportional to a mean representation of the <u>fluctuating velocity u</u> and a <u>mixing-length l_m</u>.

$$\nu_t \propto \hat{V} l_m \qquad \text{(A)}$$



Considering shear layers with <u>only one</u> significant turbulent stress ($\overline{uv}$) and velocity gradient , $\frac{\partial U}{\partial z}$ he postulated

$$\hat{V} = u = l_m \frac{\partial U}{\partial z} \qquad \text{(B)}$$

[Re] Taylor's mixing-length concept (1915)

10.4 Turbulence-Closure Models

Combine (A) and (B)

$$\boxed{\nu_t = l_m^{\ 2}\left|\frac{\partial U}{\partial z}\right|} \tag{10.28}$$

- The eddy viscosity is related directly to the local mean velocity gradient.
- Therefore, the mixing length hypothesis involves a single parameter that needs <u>empirical specification</u>; the mixing length l_m .
- Combine (10.28) with (10.23)

$$-\overline{uv} = \nu_t \frac{\partial U}{\partial z} = l_m^{\ 2}\left|\frac{\partial U}{\partial z}\right|\frac{\partial U}{\partial z}$$

10.4 Turbulence-Closure Models

- Mixing length

i) Boundary-layer flows along walls

① Near-wall region

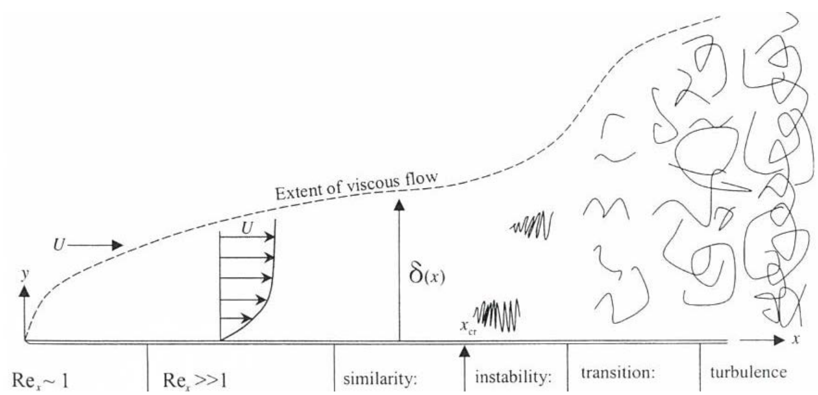
$$l_m = \kappa z$$

where κ = von Karman constant (0.4)

② Outer region

$$l_m \propto \delta$$

where δ = boundary layer thickness

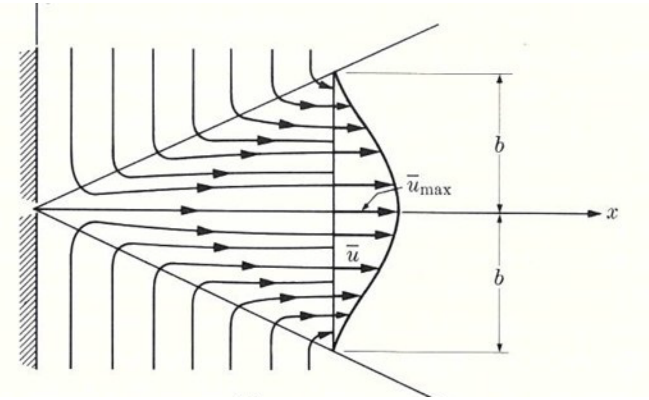


10.4 Turbulence-Closure Models

ii) Free shear flows: mixing layers, jets, wakes

$$l_m \propto b$$

where b = local shear-layer width



	Plane mixing layer	Plane jet	Round jet	Radial jet	Plane wake
$\frac{l_m}{b}$	0.07	0.09	0.075	0.125	0.16

10.4 Turbulence-Closure Models

- Effect of Buoyancy

~ Buoyancy forces acting on stratified fluid layers have a strong effect on the vertical turbulent transport of momentum and heat or mass

→ eddy viscosity relations for vertical transport must be modified by introducing a Richardson number correction

Munk-Anderson (1948) relation

$$\nu_{tz} = (\nu_{tz})_0 (1 + 10R_i)^{-0.5} \tag{10.29a}$$

$$\Gamma_{tz} = (\Gamma_{tz})_0 (1 + 3.3R_i)^{-1.5} \tag{10.29b}$$

10.4 Turbulence-Closure Models

$$\frac{l_m}{l_{m_0}} = 1 - \beta_1 R_i \;, \; R_i > 0 \text{ (stable stratification)} \tag{10.30a}$$

$$l_m / l_{m_0} = \left(1 - \beta_2 R_i\right)^{-1/4} \;, \; R_i < 0 \text{ (unstable stratification)} \tag{10.30b}$$

where $\beta_1 \approx 7, \quad \beta_2 \approx 14$

Subscript 0 refers to values during unstratified conditions ($R_i = 0$)

Define gradient local <u>Richardson number</u> R_i as

$$R_i = -\frac{g}{\rho} \frac{\dfrac{\partial \rho}{\partial z}}{\left(\dfrac{\partial U}{\partial z}\right)^2} \tag{10.31}$$

~ ratio of gravity to inertial forces

10.4 Turbulence-Closure Models

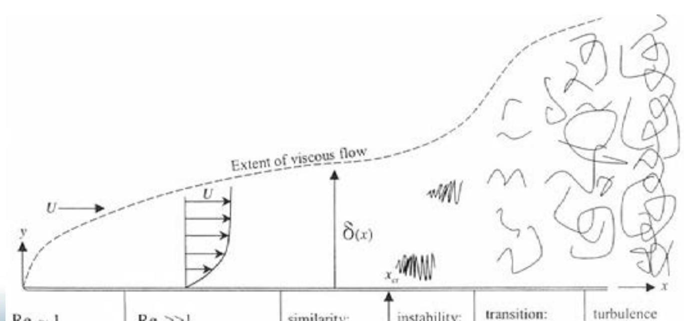
- Limitation of the mixing length model

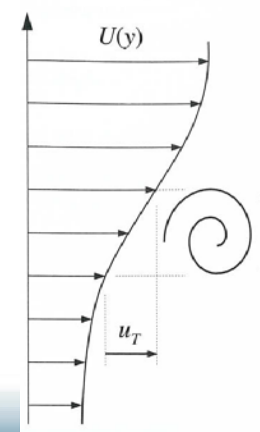
i) The mixing length model has been applied mainly to <u>two-dimensional shear-flows</u> with <u>only one significant velocity gradient</u>. $-\overline{uv} = l_m^{\ 2} \left| \frac{\partial U}{\partial z} \right| \frac{\partial U}{\partial z}$

ii) Mixing-length <u>distribution is empirical</u> and rather problem-dependent.

→ model lacks universality

ii) Close link of eddy viscosity (diffusivity) with velocity gradient, i.e. $\nu_t = 0$ when $\frac{\partial U_i}{\partial x_i} = 0$, implies that this model is <u>based on the assumption of local equilibrium of turbulence</u>.





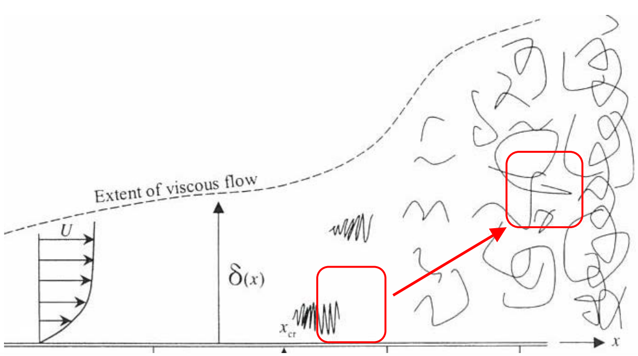
10.4 Turbulence-Closure Models

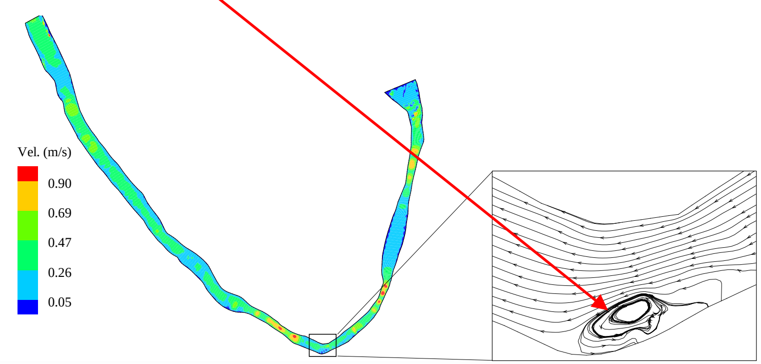
- Local equilibrium of turbulence

\- Turbulence is locally dissipated by viscous action at the same rate as it is produced by shear.

\- Transport and history effects are neglected (turbulence generation at previous times).

\- Thus, this model is not suitable when these effects are important as is the case in rapidly developing flows, recirculating flows and also in unsteady flows.





10.4 Turbulence-Closure Models

- Mixing length model for general flows is given as

$$\nu_t = l_m^{\,2}\left[\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)\frac{\partial U_i}{\partial x_j}\right]^{\frac{1}{2}} \qquad (10.32)$$

- It is very difficult to specify the distribution of l_m in complex flow

- In general duct flows (Buleev, 1962)

$$l_m = \kappa \frac{1}{\pi}\int_D \frac{1}{\delta}\, d\Omega$$

where δ = distance of the point at which l_m is to be determined from wall along direction Ω; D = integration domain (= cross section of the duct)

10.4 Turbulence-Closure Models

- Mixing-length hypothesis for heat and mass transfer

The mixing-length hypothesis is also used in heat and mass transfer calculations.

$$\boxed{\Gamma_t = \frac{\nu_t}{\sigma_t} = \frac{1}{\sigma_t} l_m^{\,2} \left| \frac{\partial U}{\partial z} \right|} \qquad (10.33)$$

where σ_t = turbulent Prandtl (Schmidt) number

$$= \begin{cases} 0.9 \text{ in near-wall flows} \\ 0.5 \text{ in plane jets and mixing layers} \\ 0.7 \text{ in round jets} \end{cases}$$

10.4 Turbulence-Closure Models

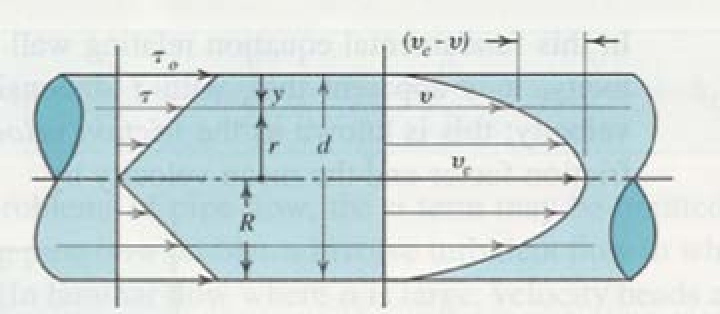
- Shortcomings of mixing-length model for heat and mass transport

i) ν_t and Γ_t vanish whenever the <u>velocity gradient is zero</u>.

[Ex] For pipes and channels,

In reality, ν_t ⓐ centerline $\approx 0.8(\nu_t)_{\max}$

However, $\dfrac{\partial U}{\partial z} = 0$ ⓐ centerline $\rightarrow \nu_t = \Gamma_t = 0$



ii) The mixing-length model implies that turbulence is in a <u>state of local equilibrium</u>.

→ Thus, this model is unable to account for <u>transport by turbulent motion</u>.

10.4 Turbulence-Closure Models

(3) Prandtl's free-shear-layer model

Prandtl (1942) proposed a simpler model applicable only to free shear layers (mixing layers, jets, wakes).

$$l_m \propto \delta$$

$$\hat{V} \propto \left| U_{\max} - U_{\min} \right| \qquad (10.34)$$

$$\boxed{\nu_t = C\delta \left| U_{\max} - U_{\min} \right|}$$

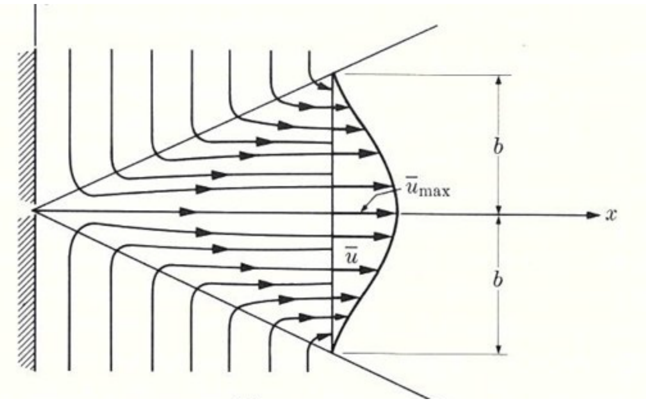


Table 10.1 Values of empirical constant *C*

Plane mixing layers	Plane jet	Round jet	Radial jet	Plane wake
0.01	0.014	0.01	0.019	0.026