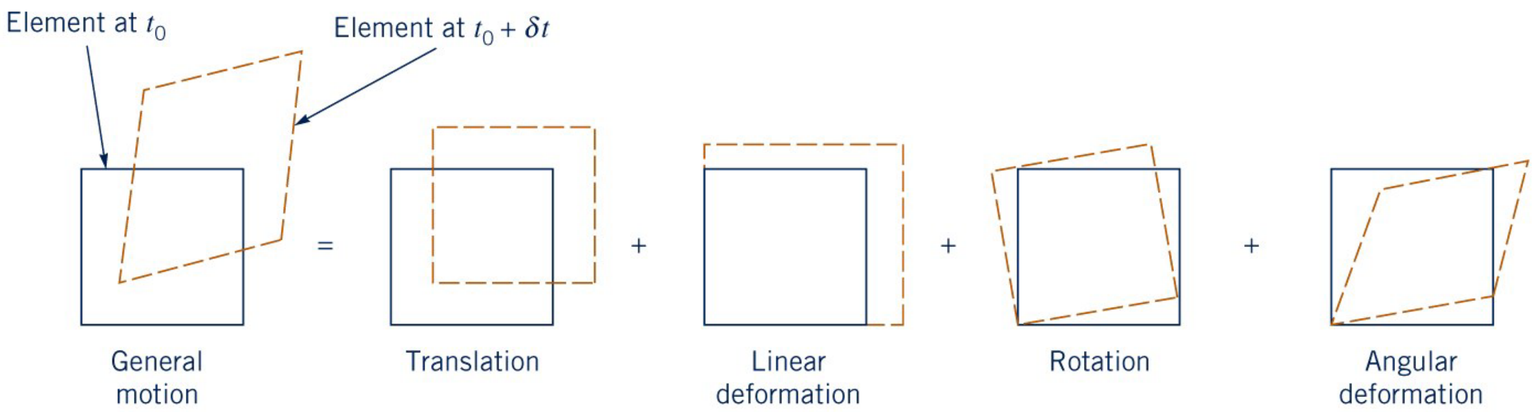
Chapter 5

Stress–Strain Relation



Chapter 5 Stress–Strain Relation

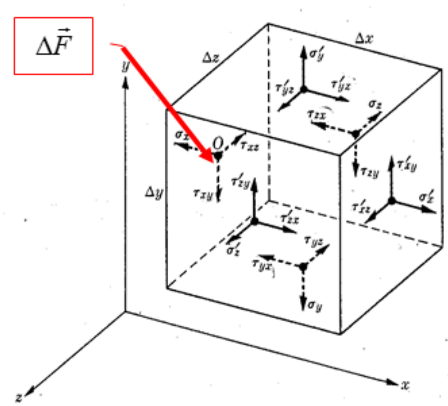
Contents

Objectives

- Understand tensor systems of stress and strain
- Study difference between displacement and deformation
- Study solid mechanics to deduce stress-rate of strain relations for fluid

5.1 General Stress–Strain System

Parallelepiped, cube → infinitesimal C.V.



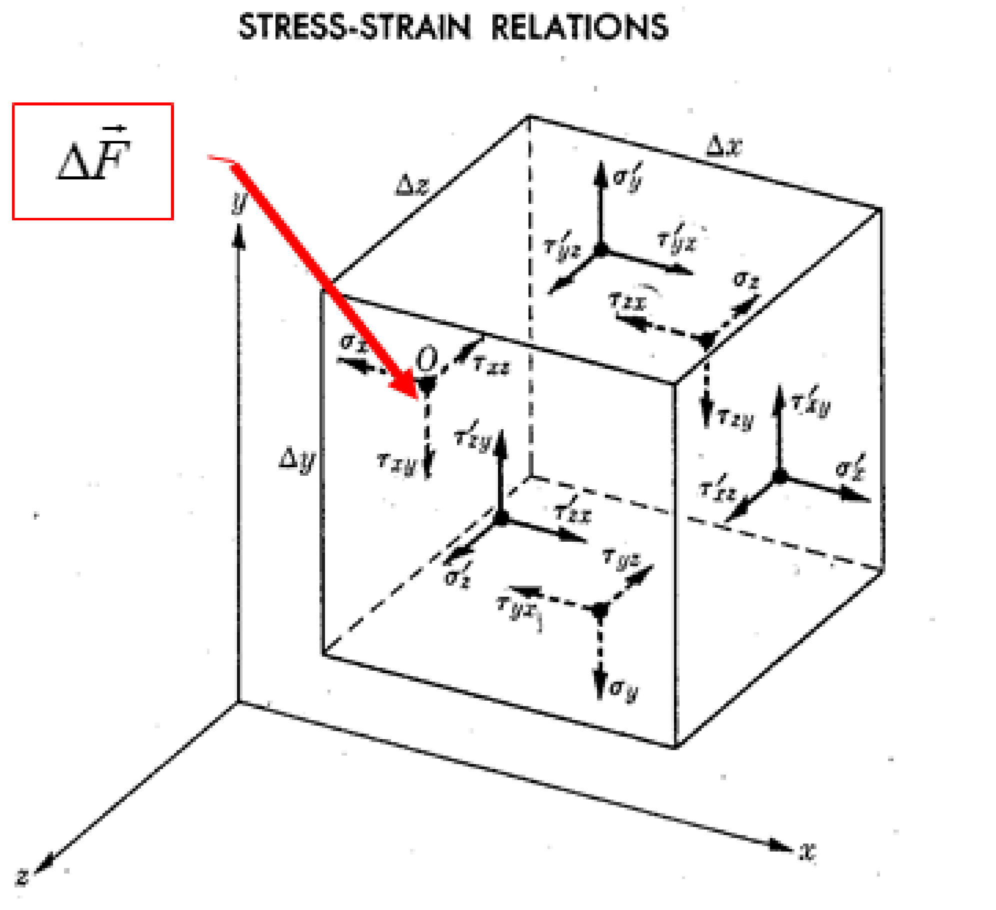
5.1.1 Surface Stress

Surface stresses:
- normal stress – σ_{xx}
- shear stress – τ_{xy}

5.1 General Stress–Strain System

Surface force

$\Delta\vec{F}$



5.1 General Stress–Strain System

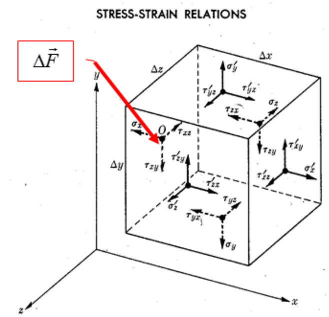
$$\sigma_{xx} = \sigma_x = \lim_{\Delta A_x \to 0} \frac{\Delta F_x}{\Delta A_x} \qquad \tau_{xy} = \lim_{\Delta A_x \to 0} \frac{\Delta F_y}{\Delta A_x} \qquad \tau_{xz} = \lim_{\Delta A_x \to 0} \frac{\Delta F_z}{\Delta A_x}$$

$$(\Delta A_x = \Delta y \Delta z)$$

Same surface but different direction

$$\tau_{yx} = \lim_{\Delta A_y \to 0} \frac{\Delta F_x}{\Delta A_y} \qquad \sigma_{yy} = \sigma_y = \lim_{\Delta A_y \to 0} \frac{\Delta F_y}{\Delta A_y} \qquad \tau_{yz} = \lim_{\Delta A_y \to 0} \frac{\Delta F_z}{\Delta A_y}$$

$$(\Delta A_y = \Delta x \Delta z)$$



$$\tau_{zx} = \lim_{\Delta A_z \to 0} \frac{\Delta F_x}{\Delta A_z} \qquad \tau_{zy} = \lim_{\Delta A_z \to 0} \frac{\Delta F_y}{\Delta A_z} \qquad \sigma_{zz} = \sigma_z = \lim_{\Delta A_z \to 0} \frac{\Delta F_z}{\Delta A_z}$$

$$(\Delta A_z = \Delta x \Delta y)$$

Same direction but different surface

where ΔF_x , ΔF_y, ΔF_z = component of force vector $\Delta \vec{F}$

5.1 General Stress–Strain System

- subscripts

σ_x : subscript indicates the direction of stress

τ_{xy} : 1st - direction of the normal to the face on which τ acts

2nd - direction in which τ acts

- general stress system: stress tensor

~ 9 scalar components

$$\begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

5.1 General Stress–Strain System

[Re] Tensor

~ an ordered array of entities which is invariant under coordinate transformation; includes scalars & vectors

~ 3^n

0th order – 1 component, scalar (mass, length, pressure, energy)

1st order – 3 components, vector (velocity, force, acceleration)

2nd order – 9 components (stress, rate of strain, turbulent diffusion)

At three other surfaces,

$$\sigma_x' = \sigma_x + \frac{\partial \sigma_x}{\partial x}\Delta x$$

5.1 General Stress–Strain System

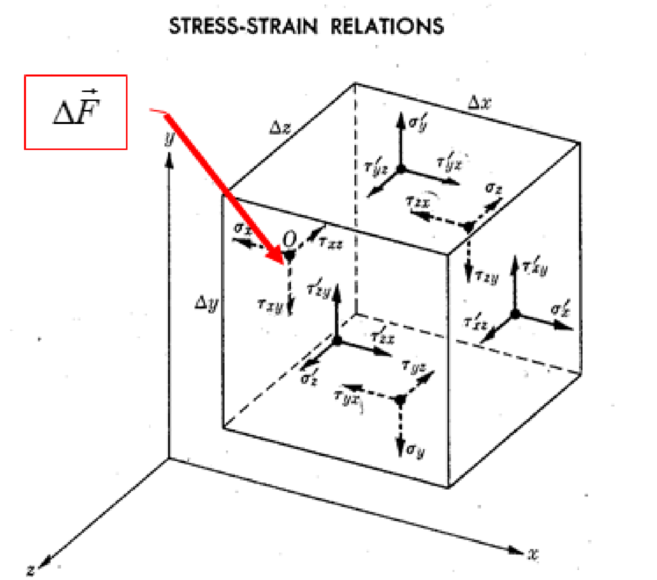
$$\sigma_y{}' = \sigma_y + \frac{\partial \sigma_y}{\partial y}\Delta y$$

$$\sigma_z{}' = \sigma_z + \frac{\partial \sigma_z}{\partial z}\Delta z$$

$$\tau_{xy}{}' = \tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\Delta x$$

$$\tau_{yx}{}' = \tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\Delta y$$

$$\tau_{zx}{}' = \tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\Delta z \qquad (5.1)$$



5.1 General Stress–Strain System

◈ Shear stress is symmetric.

→ Shear stress pairs with subscripts differing in order are equal.

→ $\tau_{xy} = \tau_{yx}$

[Proof]

In static equilibrium, sum of all moments and sum of all forces equal zero for the element.

$$\begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix}$$

First, apply Newton's 2nd law

$$\sum F = m\frac{du}{dt}$$

5.1 General Stress–Strain System

Then, consider torque (angular momentum), T

$$\sum T = \frac{d}{dt}(rmu) = \frac{d}{dt}(r^2 m\omega) = \frac{d}{dt}(I\omega) = I\frac{d\omega}{dt}$$

where I = moment of inertia = $r^2 m$

r = radius of gyration

$\frac{d\omega}{dt}$ = angular acceleration

Thus,

$$\sum T = mr^2 \frac{d\omega}{dt} \qquad \text{(A)}$$

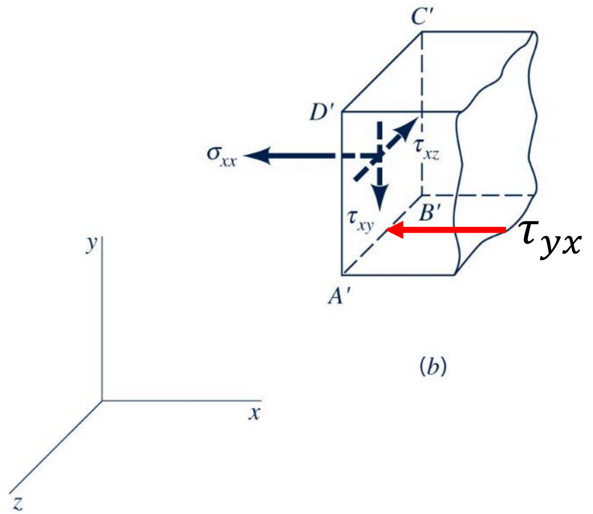
5.1 General Stress–Strain System

Now, take a moment about a centroid axis in the *z*-direction

$$LHS = \sum T = \left(\Delta y \Delta z \tau_{xy}\right)\frac{\Delta x}{2} - \left(\tau_{yx}\Delta x \Delta z\right)\frac{\Delta y}{2} = \frac{\Delta x \Delta y \Delta z}{2}\left(\tau_{xy} - \tau_{yx}\right)$$

$$RHS = \rho dvol r^2 \frac{d\omega}{dt} = \Delta x \Delta y \Delta z \rho r^2 \frac{d\omega}{dt}$$

$$\therefore \ \left(\tau_{xy} - \tau_{yx}\right)\Delta x \Delta y \Delta z = 2\Delta x \Delta y \Delta z \rho r^2 \frac{d\omega}{dt}$$



5.1 General Stress–Strain System

After canceling terms, this gives

$$\tau_{xy} - \tau_{yx} = 2\rho r^2 \frac{d\omega}{dt}$$

$$\lim_{\Delta x, \Delta y, \Delta z \to 0} r^2 \to 0$$

$$\tau_{xy} - \tau_{yx} = 0$$

$$\therefore \quad \tau_{xy} = \tau_{yx}$$

5.1 General Stress–Strain System

[Homework Assignment-Special work]

Due: 1 week from today

1. Fabricate your own “Stress Cube” using paper box.

5.1 General Stress–Strain System

5.1.2 Strain components

Motion and deformation of fluid element

Motion
- translation (displacement) - 단순이동
- rotation - 회전

Deformation
- linear (normal) deformation - 선형변형
- angular (shear) deformation – 각변형(전단변형)

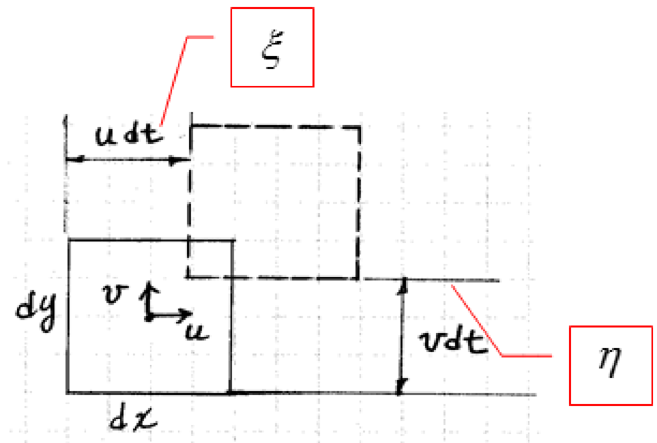
5.1 General Stress–Strain System

(1) Motion: no change in shape

1) Translation: ξ, η

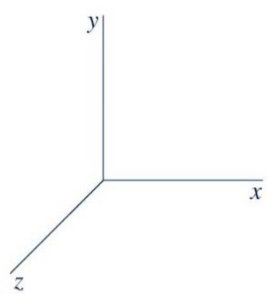
$$\xi = u\ dt,\ \ u = \frac{d\xi}{dt}$$

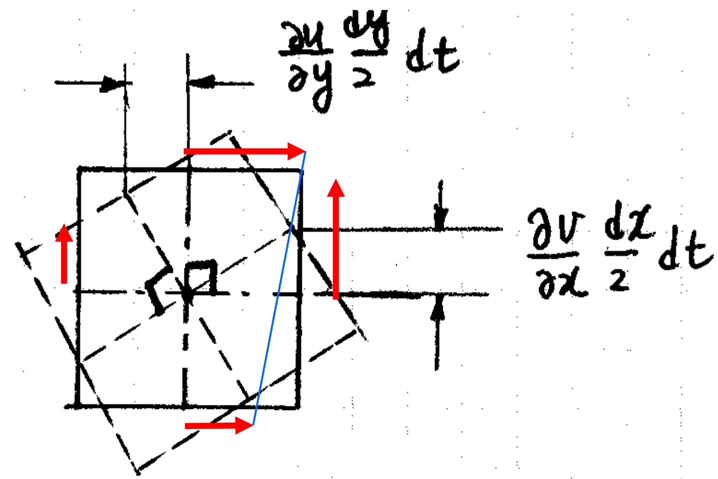
$$\eta = v\ dt,\ \ v = \frac{d\eta}{dt}$$



2) Rotation ← Shear flow

$$\omega_x = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$





5.1 General Stress–Strain System

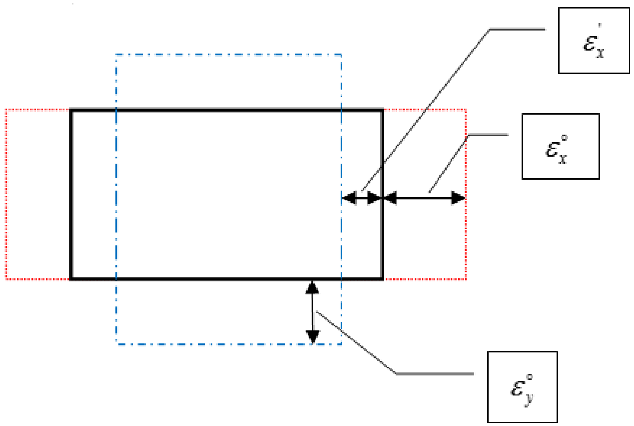
(2) Deformation: change in shape

1) Linear deformation – normal strain

$$\varepsilon_x = \frac{\partial \xi}{\partial x}$$

Non-dimensional

$$\varepsilon_y = \frac{\partial \eta}{\partial y}$$



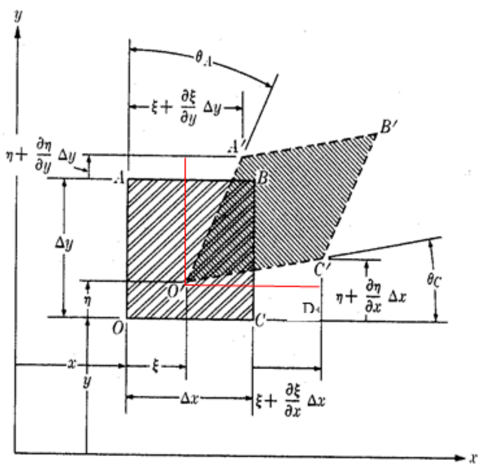
i) For compressible fluid, changes in temperature or pressure cause change in volume.

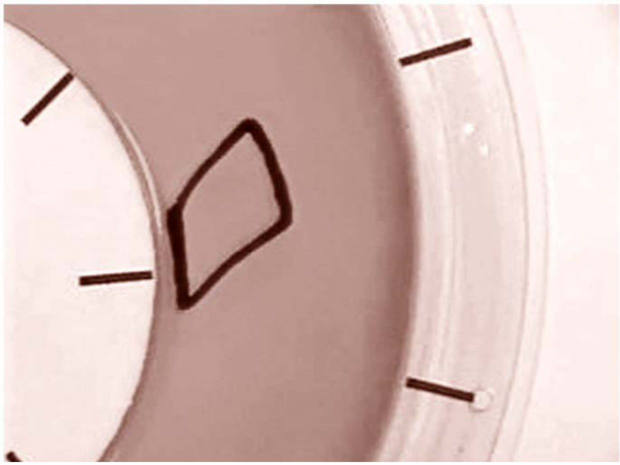
ii) For incompressible fluid, if length in 2-D increases, then length in another 1-D decreases in order to make total volume unchanged.

5.1 General Stress–Strain System

2) Angular deformation– shear strain

$$\gamma_{xy} = \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}$$

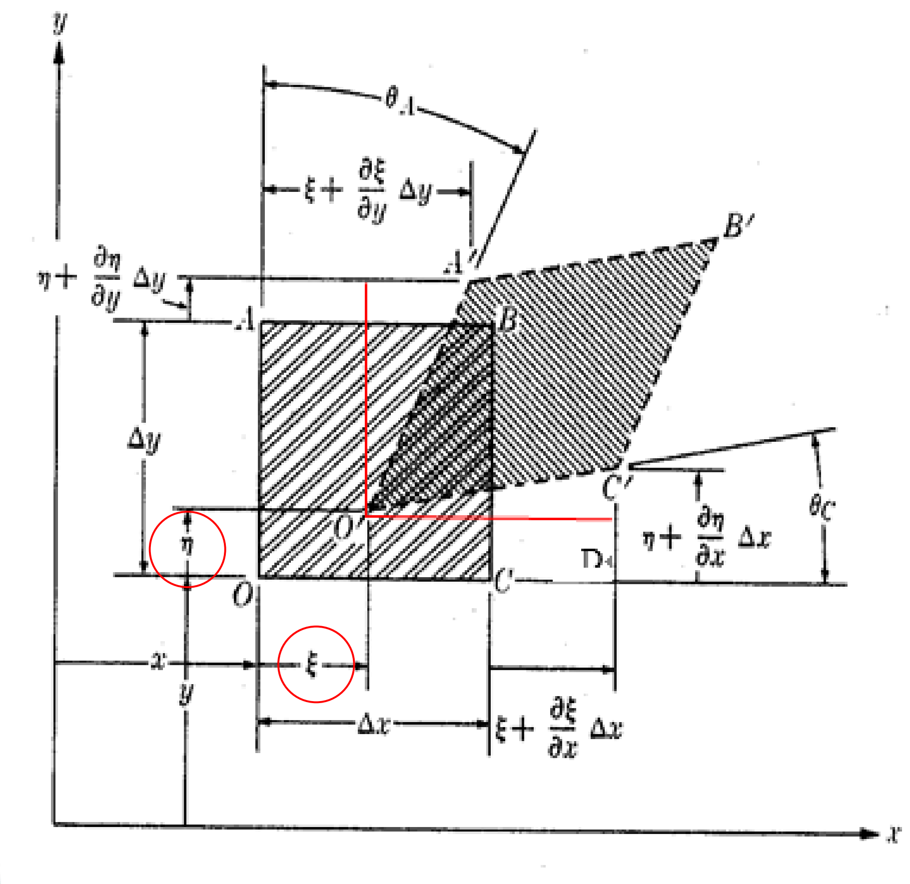




○ Strain
- normal strain: ε ← linear deformation
- shear strain: γ ← angular deformation

5.1 General Stress–Strain System

Consider a small element *OABC*



5.1 General Stress–Strain System

i) Displacement (translation): ξ, η, ζ

$$O(x, y, z) \rightarrow O'(x+\xi,\ y+\eta,\ z+\zeta)$$

$$C(x+\Delta x, y+\Delta y, z+\Delta z) \rightarrow$$

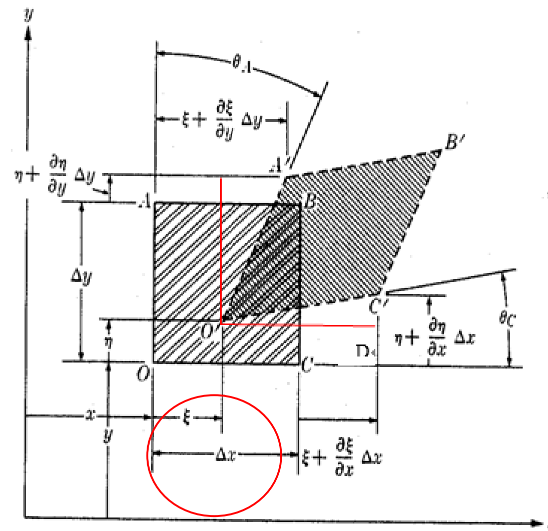
$$C'\left(x+\Delta x+\xi+\frac{\partial \xi}{\partial x}\Delta x,\ y+\eta+\frac{\partial \eta}{\partial x}\Delta x,\ z+\zeta+\frac{\partial \zeta}{\partial x}\Delta x \right)$$

ii) Deformation: due to system of external forces

$$OABC \rightarrow O'A'B'C'$$

1) Normal strain, ε

$$\varepsilon = \frac{\text{change in length}}{\text{original length}}$$



5.1 General Stress–Strain System

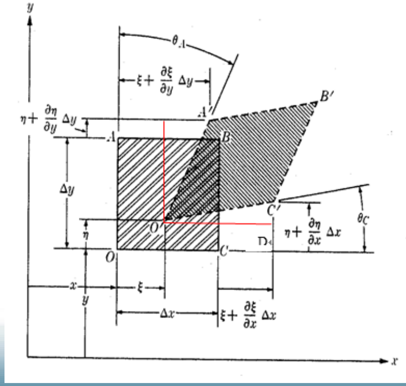
$O'C'$ OC

$$\varepsilon_x = \lim_{\Delta x \to 0} \frac{O'C' - OC}{OC} = \lim_{\Delta x \to 0} \frac{\left\{ \left(x + \Delta x + \xi + \frac{\partial \xi}{\partial x} \Delta x \right) - (x + \xi) \right\} - \Delta x}{\Delta x} = \frac{\partial \xi}{\partial x}$$

$$\varepsilon_y = \lim_{\Delta y \to 0} \frac{O'A' - OA}{OA} = \lim_{\Delta y \to 0} \frac{\left\{ \left(y + \Delta y + \eta + \frac{\partial \eta}{\partial y} \Delta y \right) - (y + \eta) \right\} - \Delta y}{\Delta y} = \frac{\partial \eta}{\partial y}$$

$$\varepsilon_z = \frac{\partial \zeta}{\partial z}$$

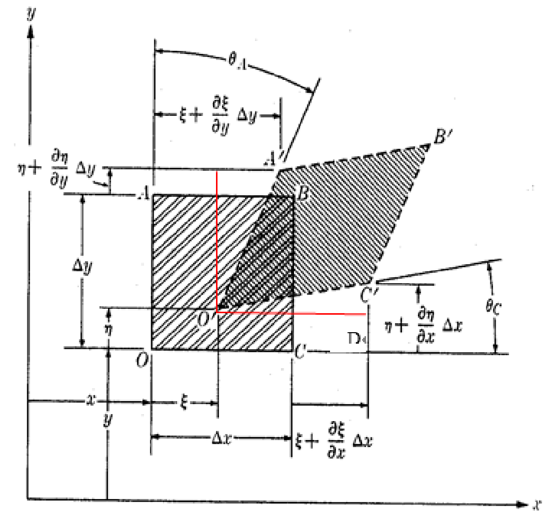
~ ε is positive when <u>element elongates</u> under deformation



5.1 General Stress–Strain System

2) Shear strain, γ

~ change in angle between two originally perpendicular elements



For xy –plane

$$\gamma_{xy} = \lim_{\Delta x,\Delta y \to 0} \left(\theta_c + \theta_A\right) \cong \lim_{\Delta x,\Delta y \to 0} \left(\tan\theta_c + \tan\theta_A\right)$$

C'D A'E

$$= \lim_{\Delta x,\Delta y \to 0} \left\{ \frac{\dfrac{\partial \eta}{\partial x}\Delta x}{\Delta x + \dfrac{\partial \xi}{\partial x}\Delta x} + \frac{\dfrac{\partial \xi}{\partial y}\Delta y}{\Delta y + \dfrac{\partial \eta}{\partial y}\Delta y} \right\} = \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y}$$

O'D O'E

$$\left(\therefore\ \Delta x \frac{\partial \xi}{\partial x} < \Delta x \right)$$

5.1 General Stress–Strain System

$$\gamma_{xy} = \frac{\partial \eta}{\partial x} + \frac{\partial \xi}{\partial y} \tag{5.2}$$

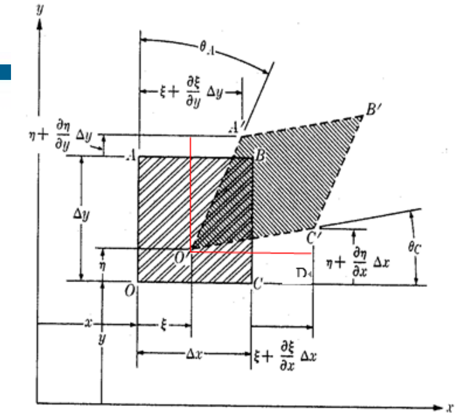
$$\gamma_{yz} = \frac{\partial \zeta}{\partial y} + \frac{\partial \eta}{\partial z} \tag{5.3}$$

$$\gamma_{zx} = \frac{\partial \xi}{\partial z} + \frac{\partial \zeta}{\partial x} \tag{5.4}$$

(2) displacement vector $\vec{\delta}$

$$\vec{\delta} = \xi \vec{i} + \eta \vec{j} + \zeta \vec{k}$$

5.1 General Stress–Strain System



(3) Volume dilation (dilatation)

$$e = \frac{\text{change of volume of deformed element}}{\text{original volume}}$$

$O'C'$ $\quad$ $O'A'$

$$e = \frac{d(\Delta V)}{\Delta V} = \frac{\left(\Delta x + \frac{\partial \xi}{\partial x}\Delta x\right)\left(\Delta y + \frac{\partial \eta}{\partial y}\Delta y\right)\left(\Delta z + \frac{\partial \zeta}{\partial z}\Delta z\right) - \Delta x \Delta y \Delta z}{\Delta x \Delta y \Delta z}$$

$$\cong \frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = \varepsilon_x + \varepsilon_y + \varepsilon_z \qquad (5.5)$$

$$e = \varepsilon_x + \varepsilon_y + \varepsilon_z \qquad (5.6)$$

$$e = \frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial y} + \frac{\partial \zeta}{\partial z} = \nabla \cdot \vec{\delta} \qquad (5.7)$$

--- divergence