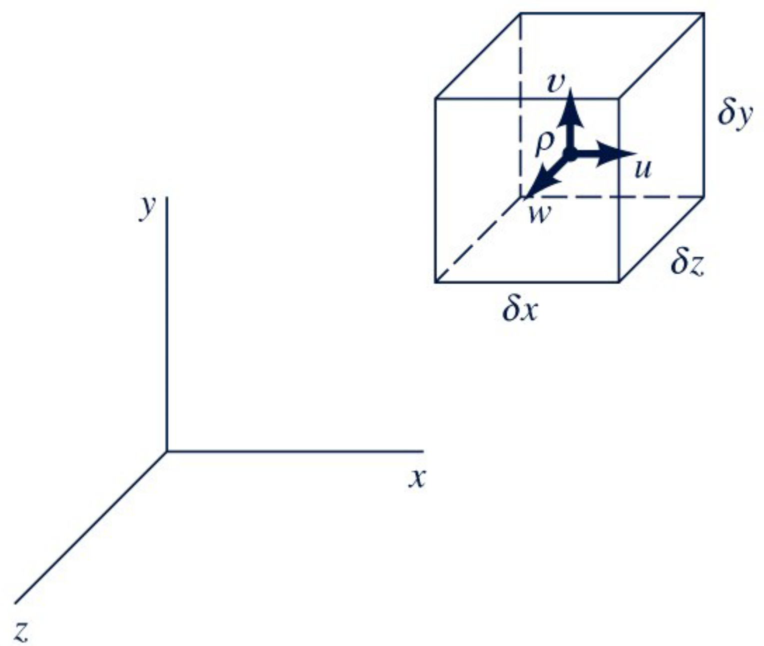
Chapter 6 Equations of Continuity and Motion

Session 6-2 Equation of motion



Chapter 6 Equations of Continuity and Motion

Contents

6.5 Examples of Laminar Motion

- N-S equations are important in viscous flow problems.

♦ Laminar motion

~ orderly state of flow in which macroscopic fluid particles move in layers

~ viscosity effect is dominant

~ no-slip condition @ boundary wall

~ apply concept of the Newtonian viscosity

~ low Re

[Ex]

1. Laminar flow between two parallel plates → Couette flow

2. Laminar flow through a tube (pipe) of constant diameter → Poiseuelle flow

6.5 Examples of Laminar Motion

[Re] Reynolds number = inertial force / viscous force = destabilizing force / stabilizing force

- Viscous force

~ dissipative

~ have a stabilizing or damping effect on the motion

~ use Reynolds number

[Cf] Turbulent flow

~ unstable flow

~ instantaneous velocity is no longer unidirectional

~ destabilizing force > stabilizing force

~ high Re

6.5 Examples of Laminar Motion

6.5.1 Laminar flow between two parallel plates

Consider the two-dimensional, steady, laminar flow between parallel plates in which either of two surfaces is moving at constant velocity and there is also an external pressure gradient.

♦ Assumptions:

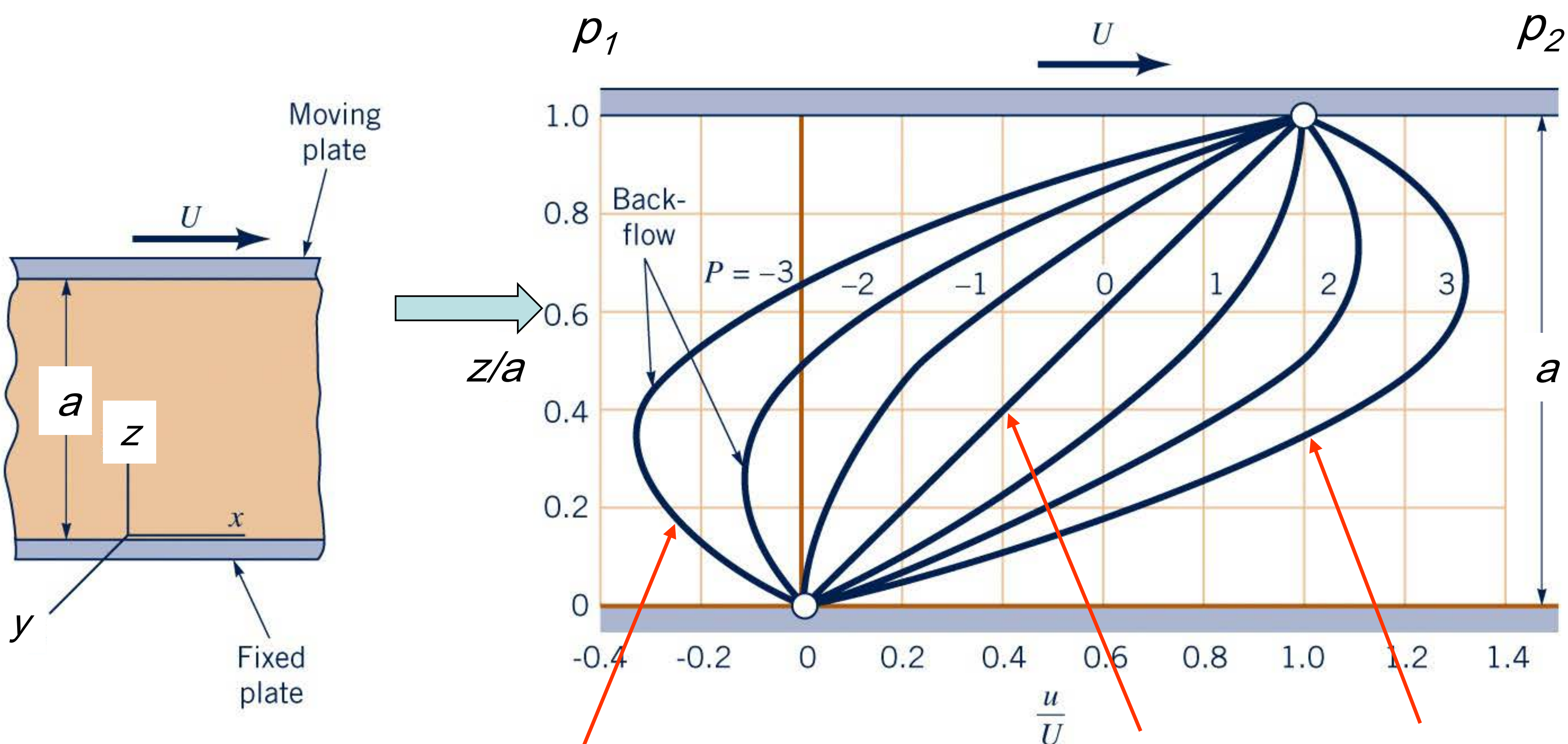
2-D flow (x, z) $\rightarrow$ $v = 0 \;;\; \dfrac{\partial(\;\;)}{\partial y} = 0$

steady flow $\rightarrow$ $\dfrac{\partial(\;\;)}{\partial t} = 0$

parallel flow $\rightarrow$ $w = 0 \;;\; \dfrac{\partial w}{\partial(\;\;)} = 0$

z-axis coincides with *h* $\rightarrow$ $\dfrac{\partial h}{\partial x} = \dfrac{\partial h}{\partial y} = 0 \;;\; \dfrac{\partial h}{\partial z} = 1$

6.5 Examples of Laminar Motion



(a) (b)

$$\frac{\partial p}{\partial x} > 0 \qquad \frac{\partial p}{\partial x} = 0 \qquad \frac{\partial p}{\partial x} < 0$$

6.5 Examples of Laminar Motion

♦ External pressure gradient

$$p_1 > p_2$$

i) $\dfrac{\partial p}{\partial x} < 0 \rightarrow$ pressure gradient <u>assists</u> the viscously induced motion to overcome the shear force at the lower surface

ii) $\dfrac{\partial p}{\partial x} > 0 \rightarrow$ pressure gradient <u>resists</u> the motion which is induced by the motion of the upper surface

$$p_1 < p_2$$

6.5 Examples of Laminar Motion

Continuity eq. for two-dimensional, parallel flow:

$$\frac{\partial u}{\partial x} + \cancel{\frac{\partial w}{\partial z}} = 0$$

$$\rightarrow \begin{cases} \dfrac{\partial^2 u}{\partial x^2} = 0 \\ u = f\left(z\right) \text{ only} \end{cases}$$

6.5 Examples of Laminar Motion

N-S Eq.:

Steady flow

Continuity eq. for incompressible fluid

2D flow

parallel flow

$$x - dir.: \cancel{\frac{\partial u}{\partial t}} + u\cancel{\frac{\partial u}{\partial x}} + v\cancel{\frac{\partial u}{\partial y}} + w\cancel{\frac{\partial u}{\partial z}}$$

$$= -g\cancel{\frac{\partial h}{\partial x}} - \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left[\cancel{\frac{\partial^2 u}{\partial x^2}} + \cancel{\frac{\partial^2 u}{\partial y^2}} + \frac{\partial^2 u}{\partial z^2}\right]$$

$$\therefore \quad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial z^2}\right) \qquad (6.31a)$$

6.5 Examples of Laminar Motion

$$z-dir.:\ \cancel{\frac{\partial w}{\partial t}}+u\cancel{\frac{\partial w}{\partial x}}+v\cancel{\frac{\partial w}{\partial y}}+w\cancel{\frac{\partial w}{\partial z}}$$

$$=-g\frac{\partial h}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{\mu}{\rho}\left[\cancel{\frac{\partial^2 w}{\partial x^2}}+\cancel{\frac{\partial^2 w}{\partial y^2}}+\cancel{\frac{\partial^2 w}{\partial z^2}}\right]$$

$$\therefore\quad 0=-g-\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad (6.31b)$$

(6.31b): $\dfrac{\partial p}{\partial z}=-\rho g=-\gamma$

$$\therefore\quad p=-\gamma z+f\left(x\right) \qquad (6.32)$$

6.5 Examples of Laminar Motion

→ hydrostatic pressure distribution normal to flow

→ For any orientation of z-axis. in case of a parallel flow, pressure is distributed hydrostatically in a direction normal to the flow.

(6.31a): $\dfrac{\partial p}{\partial x} \rightarrow \dfrac{dp}{dx}$ ~ independent of z

$$\therefore \quad \frac{dp}{dx} = \mu \frac{\partial^2 u}{\partial z^2} \qquad \text{(A)}$$

Pressure drop

Energy loss due to viscosity

6.5 Examples of Laminar Motion

Integrate (A) twice w.r.t. z to <u>derive *u(z)*</u> $\mu \dfrac{\partial^2 u}{\partial z^2} = \dfrac{dp}{dx}$

$$\iint \frac{dp}{dx} dzdz = \iint \mu \frac{\partial^2 u}{\partial z^2} dzdz$$

$$\int \frac{dp}{dx} z\, dz = \int \mu \frac{\partial u}{\partial z} dz + \int C_1\, dz$$

$$\frac{dp}{dx}\frac{z^2}{2} = \mu u + C_1 z + C_2 \qquad (6.33)$$

Use the boundary conditions,

i) $z = 0, \quad u = 0 \rightarrow \dfrac{dp}{dx} \times 0 = \mu(0) + C_2 \qquad \therefore\ C_2 = 0$

6.5 Examples of Laminar Motion

ii) $z = a, \quad u = U \rightarrow \dfrac{dp}{dx}\dfrac{a^2}{2} = \mu U + C_1 a$

$$\therefore \quad C_1 = \frac{1}{a}\left(\frac{dp}{dx}\frac{a^2}{2} - \mu U\right)$$

$\therefore$ (6.33) becomes

$$\frac{dp}{dx}\frac{z^2}{2} = \mu u + \frac{1}{a}\left(\frac{dp}{dx}\frac{a^2}{2} - \mu U\right)z$$

$$\therefore \quad \mu u = \frac{z}{a}\mu U - \frac{dp}{dx}\left(\frac{az}{2} - \frac{z^2}{2}\right)$$

6.5 Examples of Laminar Motion

$$u(z) = u = \frac{U}{a}z - \frac{a}{2\mu}\frac{dp}{dx}\left(1 - \frac{z}{a}\right)z \qquad (6.34)$$

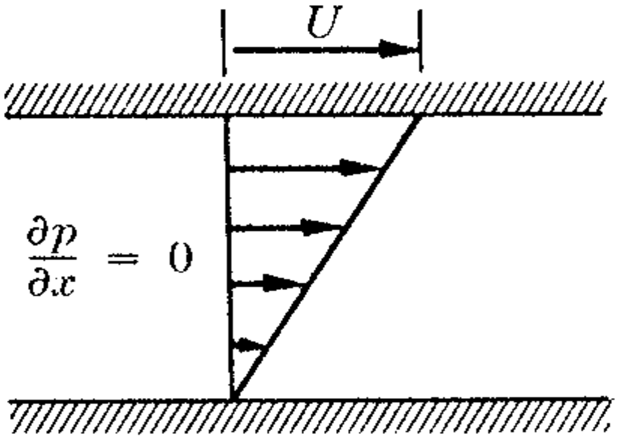
Velocity driven

Pressure driven

i) If $\frac{dp}{dx} = 0 \quad \rightarrow$ Couette flow (plane Couette flow)

$$u = \frac{U}{a}z \qquad (6.35)$$

$\rightarrow$ driving mechanism = U (velocity)



6.5 Examples of Laminar Motion

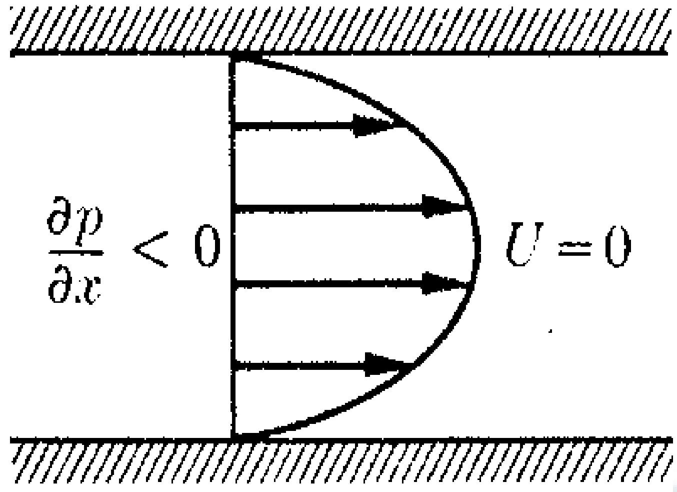
ii) If $U = 0 \rightarrow$ 2-D Poiseuille flow (plane Poiseuille flow)

$$u = \frac{1}{2\mu}\frac{dp}{dx}(z-a)z \quad \sim \text{parabolic} \qquad (6.36)$$

→ driving mechanism = external pressure gradient, $\frac{dp}{dx}$

$u_{\max}$ @ $z = \frac{a}{2}$

$$u_{\max} = -\frac{a^2}{8\mu}\frac{dp}{dx} \qquad (6.37)$$



6.5 Examples of Laminar Motion

V = average velocity

$$= \frac{Q}{A} = \frac{2}{3} u_{\max} = -\frac{a^2}{12\mu} \frac{dp}{dx} \qquad (6.38)$$

[Re] detail

$$Q = \int_0^a u\, dz = \int_0^a \frac{1}{2\mu} \frac{dp}{dx} \left(z^2 - az \right) dz = -\frac{1}{12\mu} \frac{dp}{dx} a^3$$

$$A = a \times 1 \qquad \therefore\ V = \frac{Q}{A} = -\frac{a^2}{12\mu} \frac{dp}{dx} = \frac{2}{3} u_{\max}$$

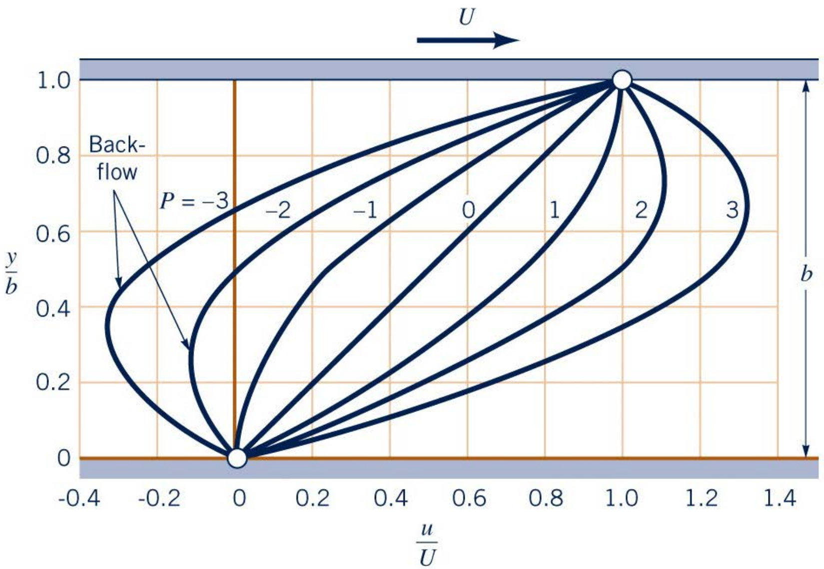
6.5 Examples of Laminar Motion

[Re] Dimensionless form

$$\frac{u}{U}=\frac{z}{a}-\frac{a^2}{2\mu U}\frac{dp}{dx}\frac{z}{a}\left(1-\frac{z}{a}\right)$$

$$P=-\frac{a^2}{2\mu U}\frac{dp}{dx}$$

$$\frac{u}{U}=\frac{z}{a}+P\frac{z}{a}\left(1-\frac{z}{a}\right)$$



(b)

6.5 Examples of Laminar Motion

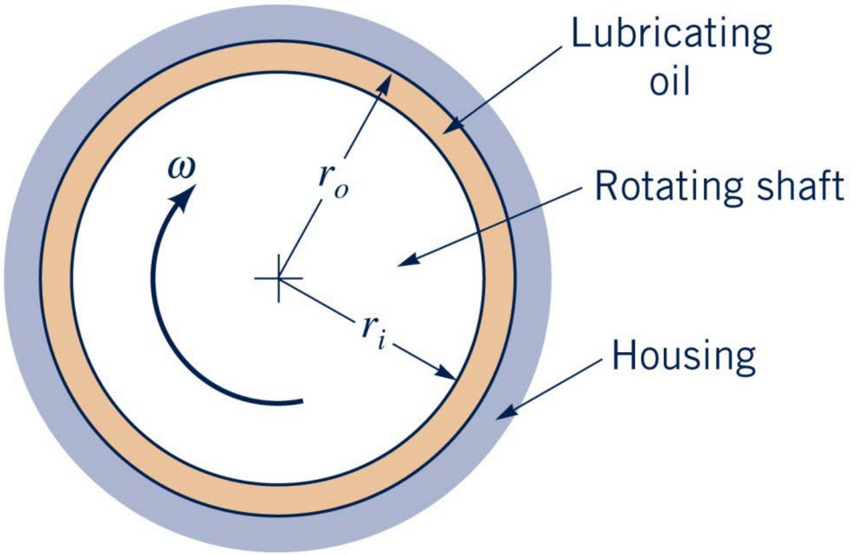
[Cf] Couette flow in the narrow gap of a journal bearing

Flow between closely spaced concentric cylinders in which one cylinder is fixed and the other cylinder rotates with a constant angular velocity, ω

$$U = r_i \omega$$

$$a = r_o - r_i$$

$$\tau \approx \mu \frac{U}{a}$$



6.5 Examples of Laminar Motion

6.5.2 Laminar flow in a circular tube of constant diameter

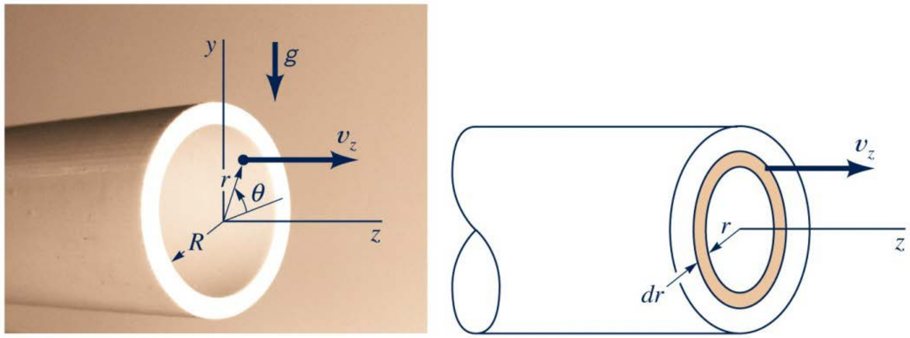
$$\boxed{\frac{\partial p}{\partial x} < 0}$$

→ Hagen-Poiseuille flow

→ Poiseuille flow: steady laminar flow due to pressure drop along a tube

Assumptions:

– use cylindrical coordinates



(a) (b)

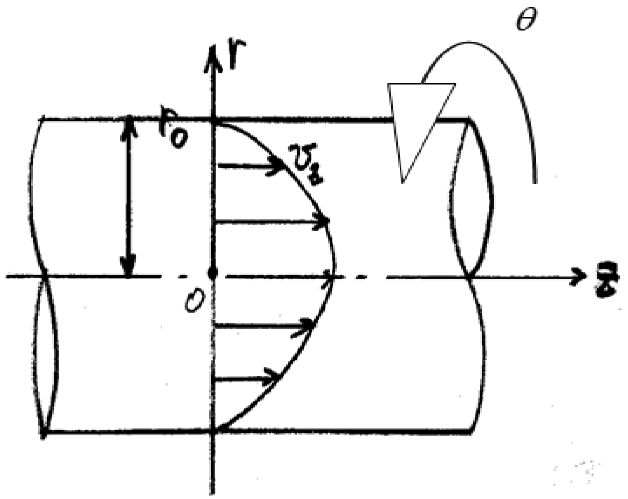
6.5 Examples of Laminar Motion

parallel flow $\rightarrow$ $v_r = 0$, $v_\theta = 0$ — $v_z \neq 0$

Continuity eq. $\rightarrow$ $\dfrac{\partial v_z}{\partial z} = 0$ — $\dfrac{\partial v_z}{\partial r} \neq 0$

paraboloid $\rightarrow$ $\dfrac{\partial v_z}{\partial \theta} = 0$

steady flow $\rightarrow$ $\dfrac{\partial v_z}{\partial t} = 0$



6.5 Examples of Laminar Motion

Eq. (6.29c) becomes

$$0 = -\frac{\partial p}{\partial z} + \rho g_z + \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r} \right) \quad \text{(A)}$$

By the way,

$$-\frac{\partial p}{\partial z} + \rho g_z = -\frac{\partial}{\partial z}(p + \gamma h) = -\frac{d}{dz}(p + \gamma h)$$

$$\left[\rho g_z = -\rho g \frac{\partial h}{\partial z} \right]$$

independent of *r*

$r -$ comp. Eq. $\rightarrow$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + g_r$$

$$\rightarrow \frac{\partial}{\partial r}(p + \gamma r) = 0$$

6.5 Examples of Laminar Motion

Then (A) becomes

$$\frac{d}{dz}\left(p+\gamma h\right)=\mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)$$

$$\frac{1}{\mu}\frac{d}{dz}\left(p+\gamma h\right)r=\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) \qquad \text{(B)}$$

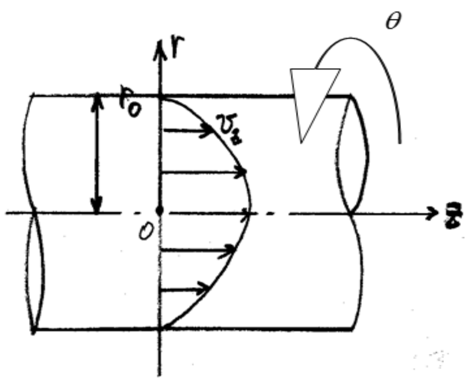
Integrate (B) twice w.r.t. r to derive $v_z(r)$

$$\frac{1}{\mu}\frac{d}{dz}\left(p+\gamma h\right)\frac{r^2}{2}=r\frac{\partial v_z}{\partial r}+C_1 \qquad \text{(C)}$$

6.5 Examples of Laminar Motion

$$\frac{1}{2\mu}\frac{d}{dz}(p+\gamma h)r=\frac{\partial v_z}{\partial r}+\frac{C_1}{r}$$

$$\frac{1}{2\mu}\frac{d}{dz}(p+\gamma h)\frac{r^2}{2}=v_z+C_1\ln r+C_2 \qquad \text{(D)}$$



Using BCs

$$r=0\ ,\ v_z=v_{z_{\max}} \quad \rightarrow \quad \text{(C)}: \quad C_1=0$$

$$r=r_0\ ,\ v_z=0 \quad \rightarrow \quad \text{(D)}: \quad C_2=\frac{1}{2\mu}\frac{d}{dz}(p+\gamma h)\frac{r_0^{\ 2}}{2} \qquad \text{(D1)}$$

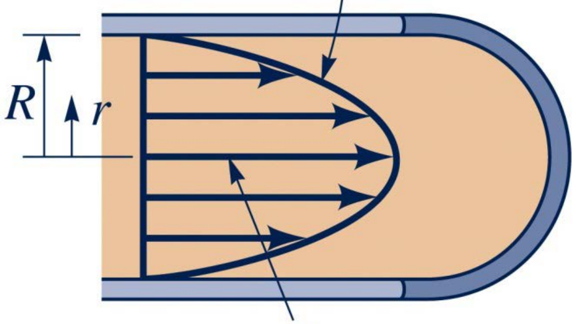
6.5 Examples of Laminar Motion

Then, substitute (D1) into (D) to obtain v_z

piezometric pressure

$$\therefore \quad v_z = \frac{1}{4\mu}\left[-\frac{d}{dz}(p+\gamma h)\right]\left(r_0^{\,2} - r^2\right) \qquad (6.39)$$

$$v_z = -\frac{d}{dz}(p+\gamma h)\frac{r_0^{\,2}}{4\mu}\left[1-\left(\frac{r}{r_0}\right)^2\right]$$



→ equation of a paraboloid of revolution

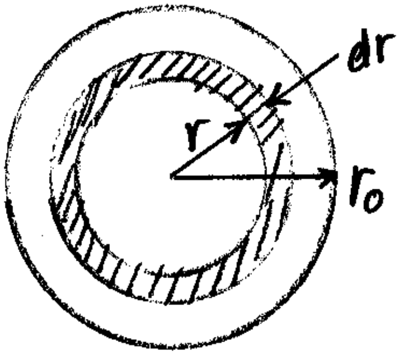
6.5 Examples of Laminar Motion

(1) maximum velocity, $v_{z_{\max}}$

$$v_{z_{\max}} \quad @ \quad r = 0 \tag{6.40}$$

$$v_{z_{\max}} = -\frac{d}{dz}(p + \gamma h)\frac{r_0^{\,2}}{4\mu} \tag{6.41}$$

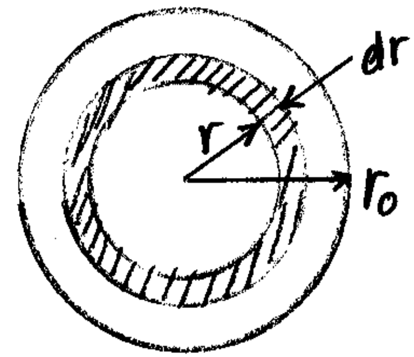
(2) mean velocity, V_z



$$dQ = v_z dA$$

$$= \frac{1}{4\mu}\left[-\frac{d}{dz}(p + \gamma h)\right]\left(r_0^{\,2} - r^2\right)2\pi r dr$$

6.5 Examples of Laminar Motion



$$Q=\int_0^{r_0}\frac{1}{4\mu}\left[-\frac{d}{dz}(p+\gamma h)\right]\left(r_0^{\ 2}-r^2\right)2\pi r dr$$

$$=\frac{\pi}{2\mu}\left[-\frac{d}{dz}(p+\gamma h)\right]\left[r_0^{\ 2}\frac{r^2}{2}-\frac{r^4}{4}\right]_r^{r_0}$$

$$=\frac{\pi r_0^{\ 4}}{8\mu}\left[-\frac{d}{dz}(p+\gamma h)\right]$$

$$\therefore\quad V_z=\frac{Q}{A}=\frac{Q}{\pi r_0^{\ 2}}=\frac{r_0^{\ 2}}{8\mu}\left[-\frac{d}{dz}(p+\gamma h)\right]=\frac{v_{z_{\max}}}{2} \qquad \text{(E)}$$

[Cf] For 2 - D Poiseuille flow $V=\frac{2}{3}u_{\max}$

6.5 Examples of Laminar Motion

(3) Head loss per unit length of pipe

Total head = piezometric head + velocity head

Here, velocity head is constant.

Thus, total head loss = piezometric head change

$$\frac{h_f}{L} \equiv \frac{\Delta p}{\gamma}$$

$$\frac{h_f}{L} \equiv \frac{1}{\gamma}\left[-\frac{d}{dz}(p+\gamma h)\right] = \underbrace{\frac{8\mu V_z}{\gamma r_0^{\,2}}}_{\text{(E)}} = \frac{32\mu V_z}{\gamma D^2} \qquad (6.42)$$

where $D = 2r_0 = \text{diameter}$

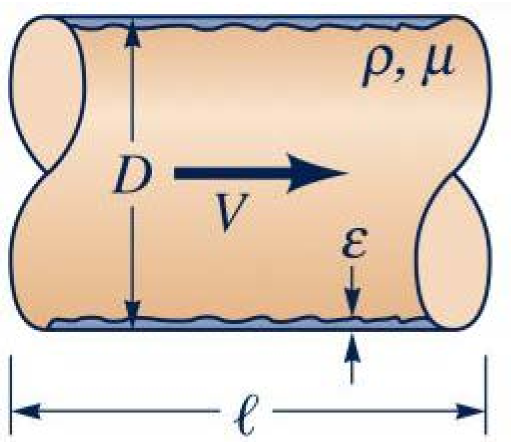
6.5 Examples of Laminar Motion

[Re] Consider Darcy-Weisbach Eq.

$$\frac{h_f}{L} = f\,\frac{1}{D}\frac{V_z^{\,2}}{2g} \qquad \text{(F)}$$

h_f = head loss due to friction

f = friction factor



Combine (6.42) and (F)

$$\frac{32\mu V_z}{\gamma D^2} = f\,\frac{1}{D}\frac{V_z^{\,2}}{2g} \qquad (6.43)$$

6.5 Examples of Laminar Motion

$$f = \frac{64}{V_z}\frac{\nu}{D} = \frac{64}{V_z D/\nu} = \frac{64}{\mathrm{Re}}$$ → For laminar flow (6.44)

(4) Shear stress

$$\tau_{zr} = \mu\left(\cancel{\frac{\partial v_r}{\partial z}} + \frac{\partial v_z}{\partial r}\right) = \mu\frac{\partial v_z}{\partial r} \qquad \text{(G)}$$

Differentiate (6.39) w.r.t. r

$$\frac{\partial v_z}{\partial r} = \frac{d}{dz}(p+\gamma h)\frac{1}{2\mu}r \qquad \text{(H)}$$

6.5 Examples of Laminar Motion

Combine (G) and (H)

Linear profile

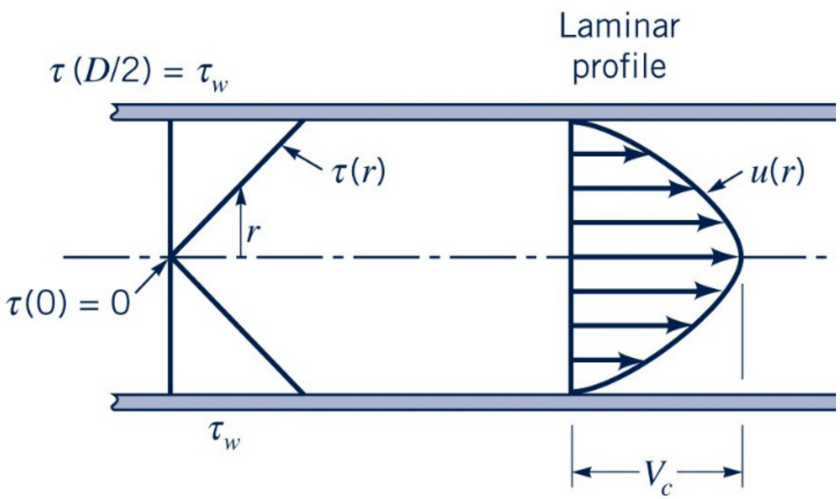
$$\boxed{\tau_{zr} = \frac{1}{2}\frac{d}{dz}(p+\gamma h)r} \qquad (6.45)$$

At center and walls

$$r = 0\,, \quad \tau_{zr} = 0$$

$$r = r_0\,, \quad \tau_{zr_{\max}} = \frac{1}{2}\frac{d}{dz}(p+\gamma h)r_0$$

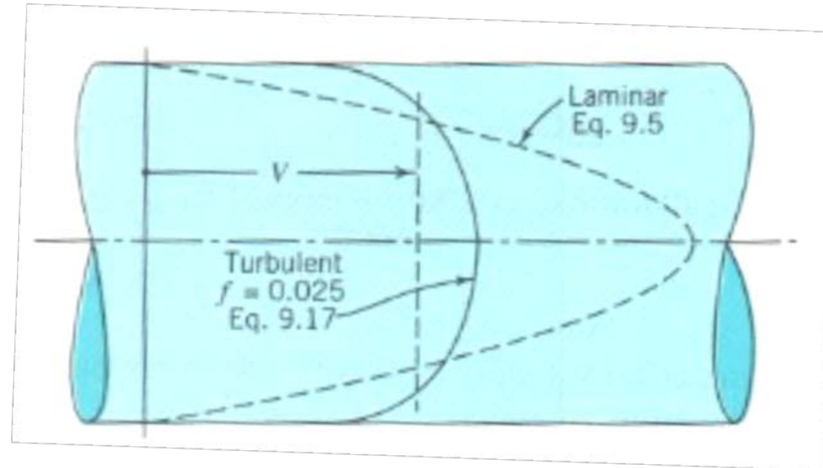
$$\tau_{zr} = \tau_{zr_{\max}}\frac{r}{r_0}$$



Laminar flow :

$$v = v_c\left(1 - \frac{r^2}{R^2}\right)$$

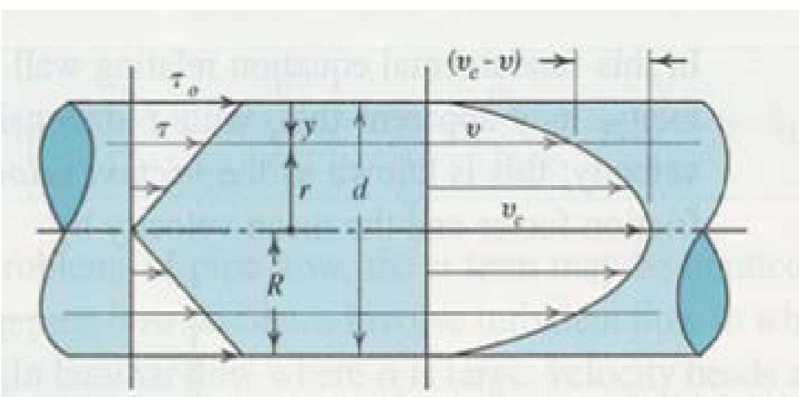
$$\tau = \tau_0\left(1 - \frac{y}{R}\right) = \frac{\tau_0 r}{R}$$



Turbulent flow:

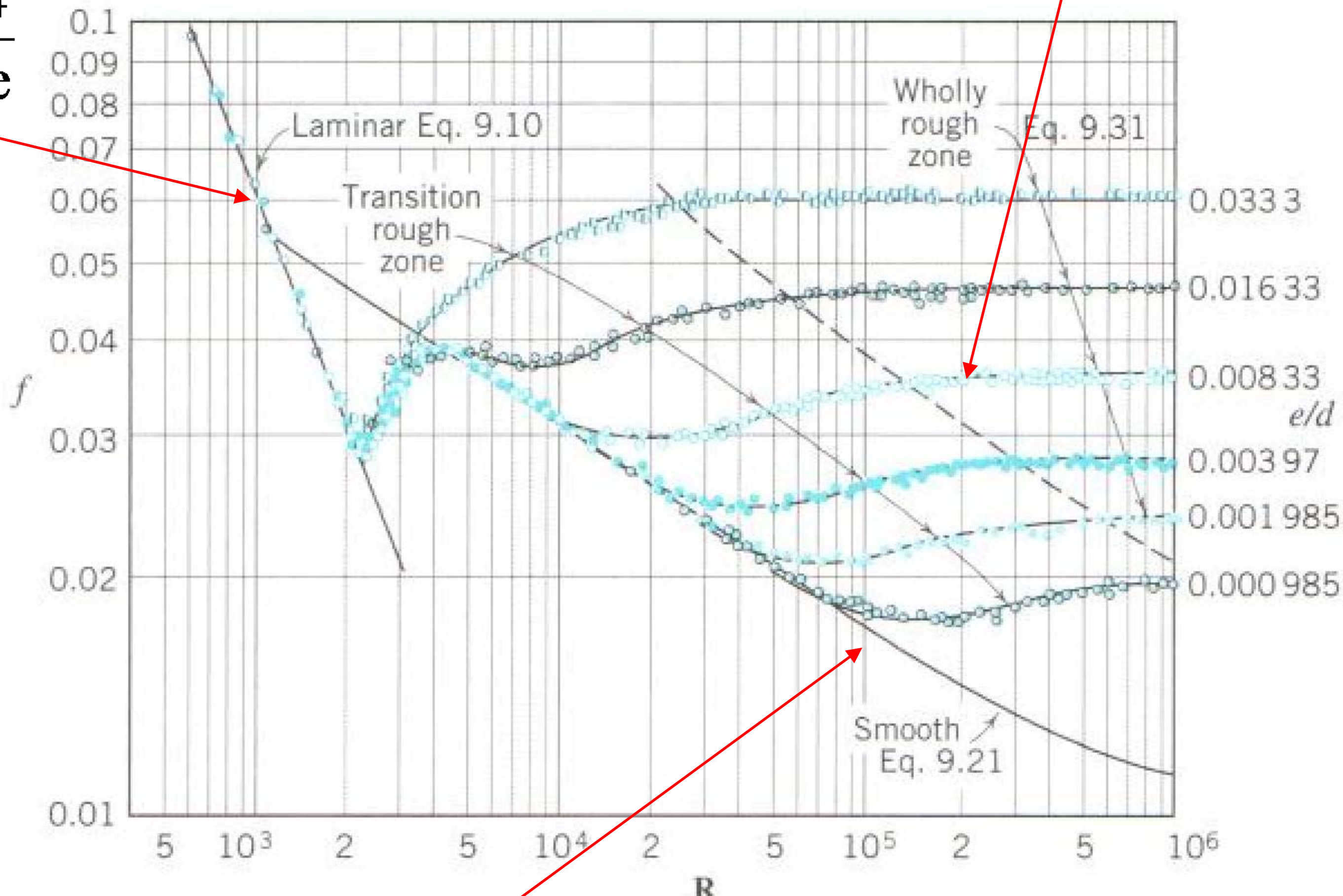
$$\frac{v_c - v}{v_*} = -\frac{1}{\kappa}\ln\frac{y}{R}$$

$$\tau = \tau_0\left(1 - \frac{y}{R}\right) = \frac{\tau_0 r}{R}$$



$$\frac{1}{\sqrt{f}} = 2.0\log\frac{d}{e} + 1.14$$

$$f = \frac{64\mu}{Vd\rho} = \frac{64}{\mathrm{Re}}$$



$$\frac{1}{\sqrt{f}} = 2.0\log\left(\mathrm{Re}\sqrt{f}\right) - 0.8$$

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