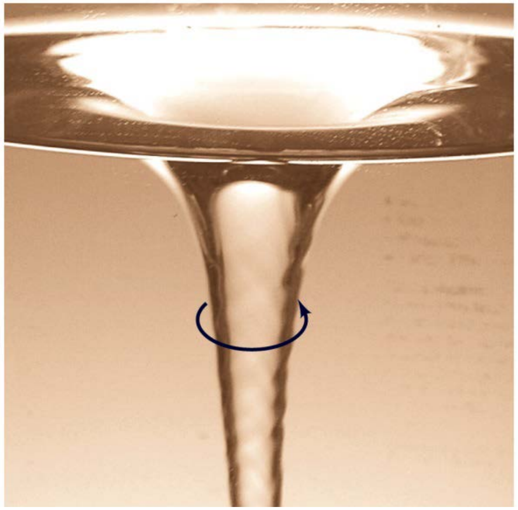
Chapter 6 Equations of Continuity and Motion

Session 6-3 Motions of viscous and inviscid fluids



Chapter 6 Equations of Continuity and Motion

Contents

6.6 Irrotational Motion

○ In Ch. 4, 1st law of thermodynamics

→ 1D Energy eq.

→ Bernoulli eq. for steady flow of an incompressible fluid with zero friction (ideal fluid)

○ In Ch. 6,

Newton's 2nd law → Momentum eq. → Eq. of motion (6.4) → Bernoulli eq.

↑ Integration assuming irrotational flow (6.3)

○ Irrotational flow = Potential flow

6.6 Irrotational Motion

6.6.1 Velocity potential and stream function

If $\phi(x, y, z, t)$ is any scalar quantity having continuous first and second derivatives, then by a fundamental vector identity

$$\rightarrow curl(grad\ \phi) \equiv \nabla \times (\nabla \phi) \equiv 0 \qquad (6.46)$$

[Detail] vector identity

$$\nabla \phi = grad\ \phi = \frac{\partial \phi}{\partial x}\vec{i} + \frac{\partial \phi}{\partial y}\vec{j} + \frac{\partial \phi}{\partial z}\vec{k}$$

6.6 Irrotational Motion

$$curl(grad\ \phi) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial \phi}{\partial x} & \dfrac{\partial \phi}{\partial y} & \dfrac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial \phi^2}{\partial y \partial z} - \frac{\partial \phi^2}{\partial y \partial z}\right) + \vec{j}\left(\frac{\partial \phi^2}{\partial z \partial x} - \frac{\partial \phi^2}{\partial z \partial x}\right) + \vec{k}\left(\frac{\partial \phi^2}{\partial x \partial y} - \frac{\partial \phi^2}{\partial x \partial y}\right) \Rightarrow 0$$

6.6 Irrotational Motion

By the way, for irrotational flow

Eq.(6.17) : $\nabla \times \vec{q} = 0$ (A)

Thus, from (6.46) and (A), we can say that for irrotational flow there must exist a scalar function ϕ whose gradient is equal to the velocity vector $\vec{q}$.

$$grad\ \phi = \vec{q} \qquad \text{(B)}$$

Now, let's define the positive direction of flow is the direction in which ϕ is decreasing, then

$$\vec{q} = -grad\ \phi(x, y, z, t) = -\nabla \phi \qquad (6.47)$$

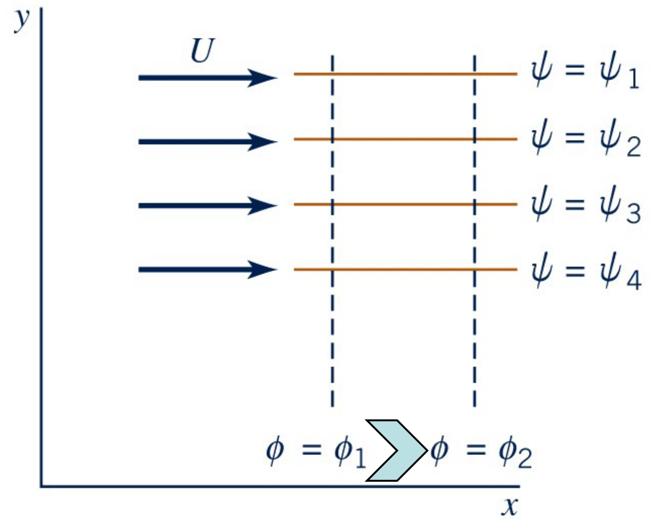
6.6 Irrotational Motion

where ϕ = velocity potential

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}, \quad w = -\frac{\partial \phi}{\partial z} \qquad (6.47a)$$

→ Velocity potential exists only for irrotational flows; however stream function is not subject to this restriction.

→ irrotational flow = potential flow for both compressible and incompressible fluids



6.6 Irrotational Motion

(1) Continuity equation for incompressible fluid

Eq. (6.5): $\nabla \cdot \vec{q} = 0$ (C)

Substitute (6.47) into (C)

$$\therefore \nabla \cdot \left(-\nabla \phi \right) = -\nabla^2 \phi = 0 \quad \rightarrow \text{Laplace Eq.} \tag{6.48}$$

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \leftarrow \text{Cartesian coordinates} \tag{6.49}$$

$$\nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \leftarrow \text{Cylindrical coordinates} \tag{6.50}$$

6.6 Irrotational Motion

[Detail] velocity potential in cylindrical coordinates

$$v_r = -\frac{\partial \phi}{\partial r}, \;\; v_\theta = -\frac{\partial \phi}{r \partial \theta}, \;\; v_z = -\frac{\partial \phi}{\partial z}$$

(2) For 2-D incompressible irrotational motion

- Velocity potential

$$u = -\frac{\partial \phi}{\partial x}$$

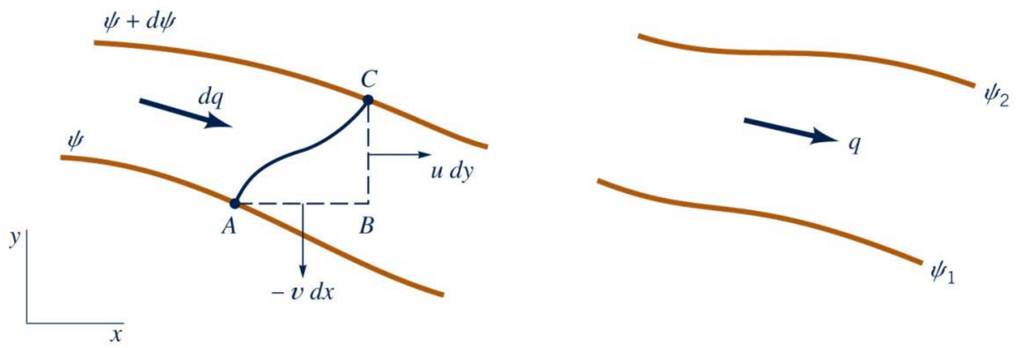
$$v = -\frac{\partial \phi}{\partial y} \qquad (6.51)$$

6.6 Irrotational Motion

- Stream function: Eq. (6.8)

$$u = -\frac{\partial \psi}{\partial y}$$

$$v = \frac{\partial \psi}{\partial x} \qquad (6.52)$$



$$\therefore \left.\begin{aligned} \frac{\partial \psi}{\partial y} &= \frac{\partial \phi}{\partial x} \\ \frac{\partial \psi}{\partial x} &= -\frac{\partial \phi}{\partial y} \end{aligned}\right\} \rightarrow \text{Cauchy-Riemann equation} \qquad (6.53)$$

6.6 Irrotational Motion

Now, substitute stream function, (6.8) into irrotational flow, (6.17)

Eq. (6.17) : $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \leftarrow \left[rotation = 0 \quad \nabla \times \vec{q} = 0 \right]$

$$\therefore \; -\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial^2 \psi}{\partial x^2} \rightarrow \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \rightarrow \text{Laplace eq.} \qquad \text{(D)}$$

Also, for 2-D incompressible flow, velocity potential satisfies the Laplace eq.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \text{(E)}$$

6.6 Irrotational Motion

→ Both ϕ and ψ satisfy the Laplace eq. for 2-D incompressible irrotational motion.

→ ϕ and ψ may be interchanged.

→ Lines of constant ϕ and ψ must form an orthogonal mesh system

→ Flow net

- Flow net analysis

Along a streamline, $\psi =$ constant.

Eq. for a streamline, Eq. (2.10)

$$\left.\frac{dy}{dx}\right|_{\psi = const.} = \frac{v}{u} \tag{6.54}$$

6.6 Irrotational Motion

BTW along lines of constant velocity potential

$$\rightarrow d\phi = 0$$

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = 0 \qquad \text{(F)}$$

Substitute Eq. (6.47a)

$$\left.\frac{dy}{dx}\right|_{\phi = const.} = -\frac{\partial \phi / \partial x}{\partial \phi / \partial y} = -\frac{u}{v} \qquad (6.55)$$

6.6 Irrotational Motion

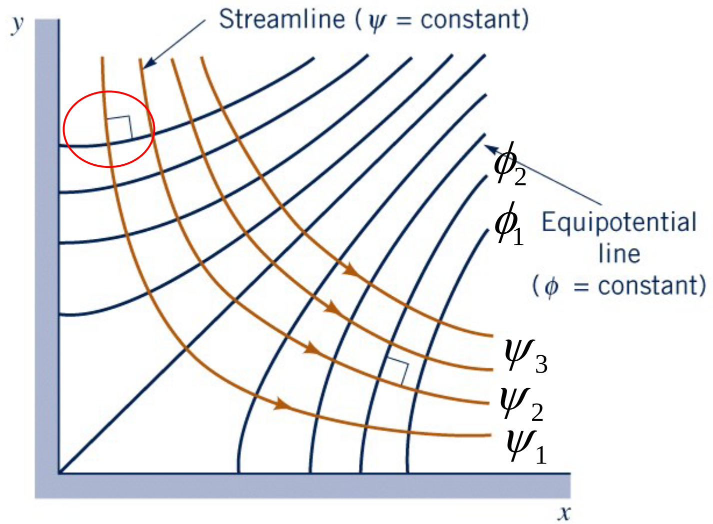
From Eqs. (6.54) and (6.55)

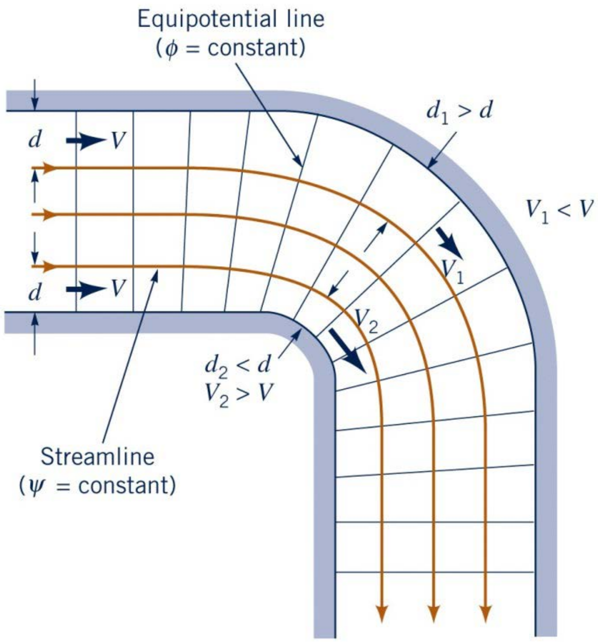
$$\left.\frac{dy}{dx}\right|_{\psi=const.} = -\left.\frac{dx}{dy}\right|_{\phi=const.} \qquad (6.56)$$

→ Slopes are the negative reciprocal of each other.

→ Flow net analysis (graphical method) can be used when a solution of the Laplace equation is difficult for complex boundaries.

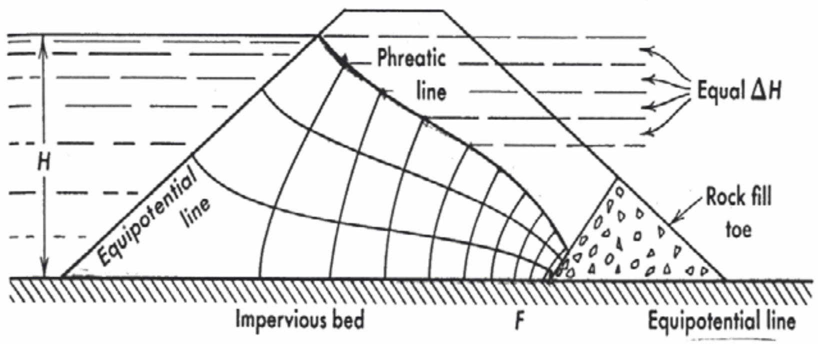
6.6 Irrotational Motion

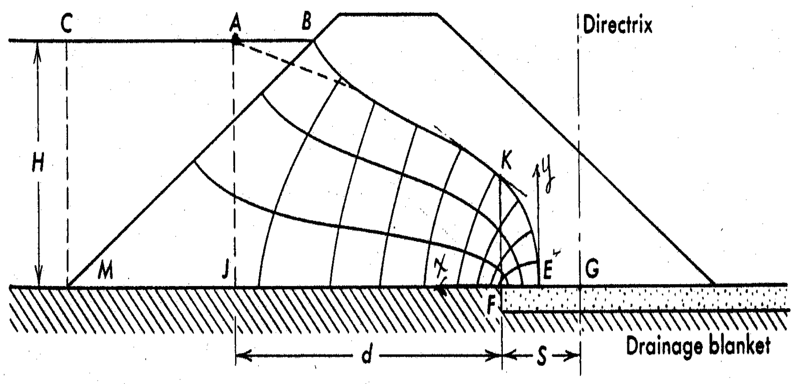


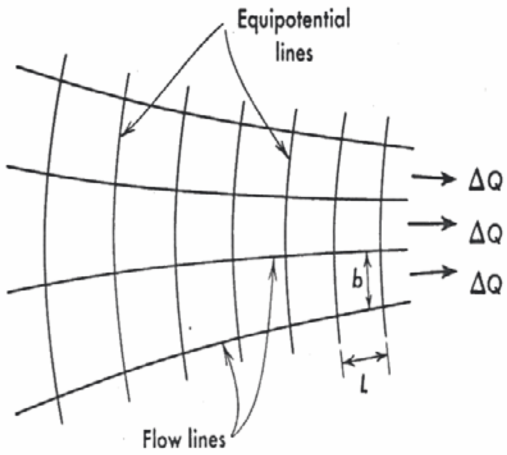


6.6 Irrotational Motion

- Seepage of earth dam







$$Q = \sum \Delta Q = n_f K \Delta H = \frac{n_f}{n_d} KH$$

n_f = number of flowlines;

n_d = number of equipotential lines;

K = permeability coefficient (m/s)

6.6 Irrotational Motion

Potential flows

1. Uniform flow

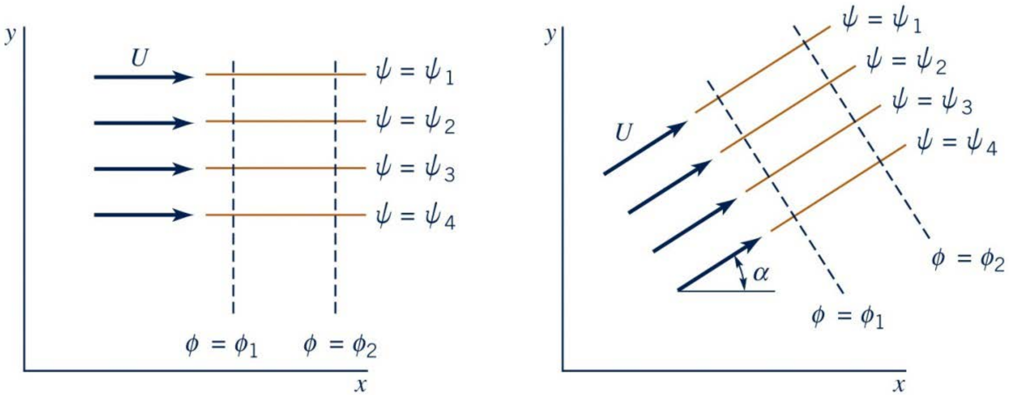
→ streamlines are all straight and parallel, and the magnitude of the velocity is constant

$$\frac{\partial \phi}{\partial x} = -U, \quad \frac{\partial \phi}{\partial y} = 0$$

$$\phi = -Ux + C$$

$$\frac{\partial \psi}{\partial y} = -U, \quad \frac{\partial \psi}{\partial x} = 0$$

$$\psi = -Uy + C'$$



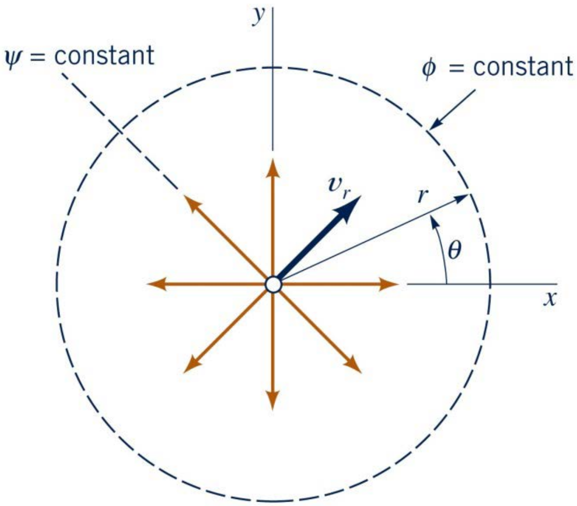
6.6 Irrotational Motion

2. Source and Sink

- Fluid flowing radially outward from a line through the origin perpendicular to the *x*-*y* plane
- Let *m* be the volume rate of flow emanating from the line (per unit length)

$$(2\pi r)v_r = m$$

$$v_r = \frac{m}{2\pi r}$$



The streamlines are radial lines,

and equipotential lines are concentric circles.

6.6 Irrotational Motion

If m is positive, the flow is radially outward → source

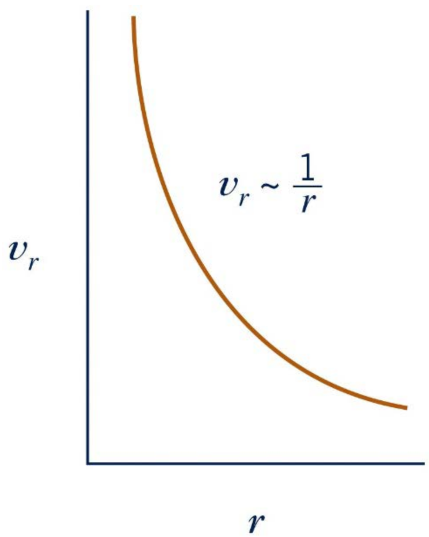
If m is negative, the flow is radially inward → sink

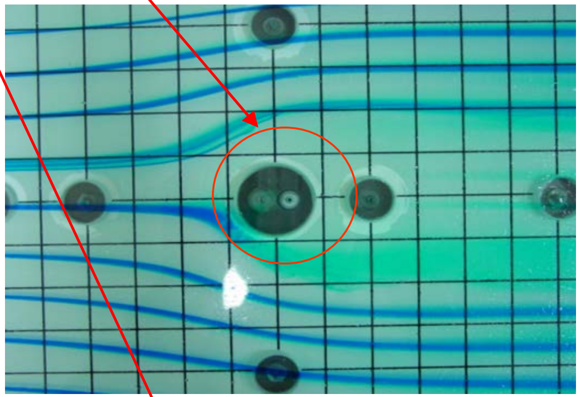
$$\frac{\partial \phi}{\partial r} = \frac{m}{2\pi r}, \quad \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0$$

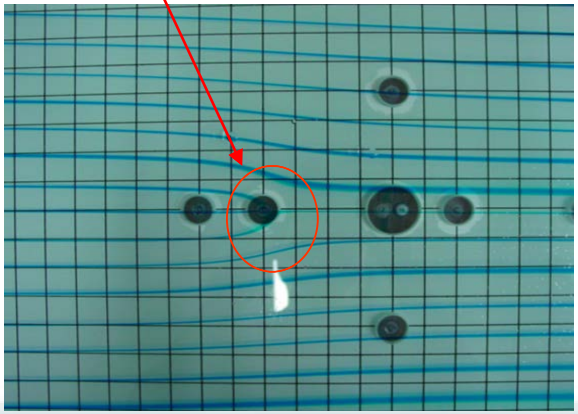
$$\boxed{\phi = \frac{m}{2\pi} \ln r}$$

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{m}{2\pi r}$$

$$\boxed{\psi = \frac{m}{2\pi} \theta}$$





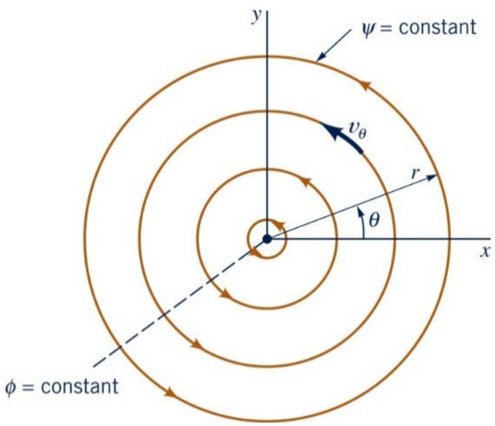


6.6 Irrotational Motion

3. Vortex (Sec. 6.8)

Flow field in which the streamlines are concentric circles

In cylindrical coordinate

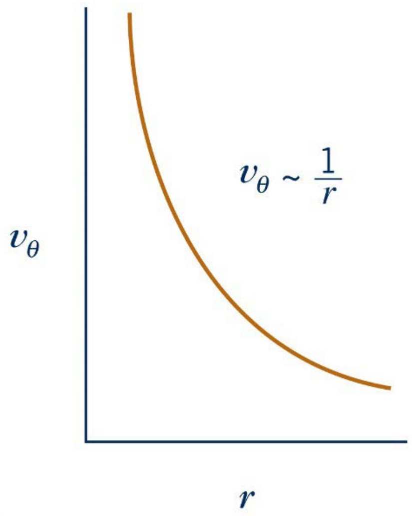


$$\phi = K\theta$$

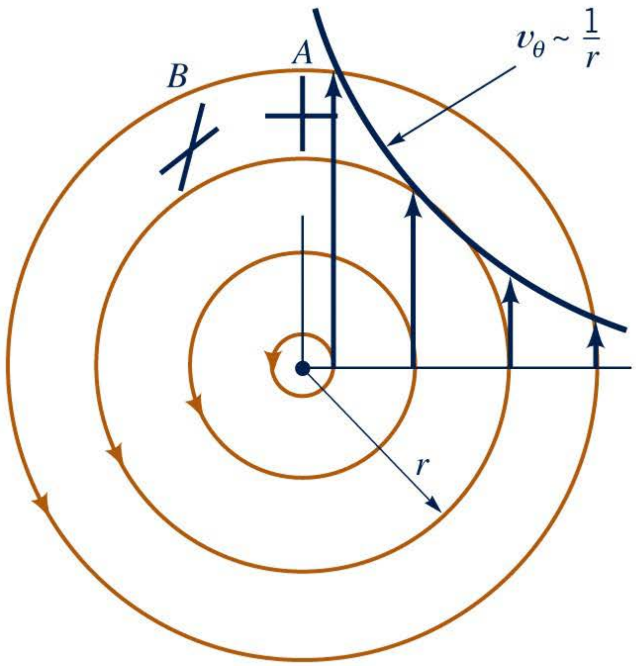
$$\psi = -K \ln r$$

$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = \frac{K}{r}$$

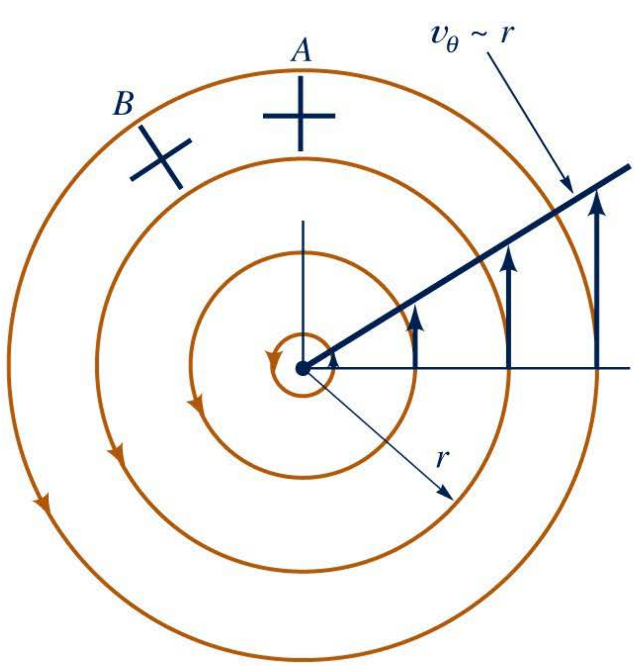
The tangential velocity varies inversely with distance from the origin. → free vortex



6.6 Irrotational Motion



(a)



(b)

Free vortex
→ irrotational flow

Forced vortex
→ rotational flow

6.6 Irrotational Motion

[Appendix II] Potential flow problem

Find velocity potential ϕ

- Find $\psi \rightarrow$ Find flow pattern
- Find velocity
- Find kinetic energy

Find velocity, Find kinetic energy → Find pressure, force (Bernoulli eq.)

6.6 Irrotational Motion

6.6.2 The Bernoulli equation for irrotational incompressible fluids

(1) Find the solution of N-S equation for irrotational incompressible fluids

Substitute Eq. (6.17) into Eq. (6.28)

Eq. (6.17) : $\nabla \times \vec{q} = 0$

$$\left.\begin{aligned} \frac{\partial w}{\partial y} &= \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} &= \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} &= \frac{\partial u}{\partial y} \end{aligned}\right\}$$

irrotational flow

6.6 Irrotational Motion

Eq. (6.28): Navier-Stokes eq. (x -comp.) for irrotational incompressible fluid

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-g\frac{\partial h}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\left[\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right]$$

$$u\frac{\partial u}{\partial x}\rightarrow\frac{1}{2}\frac{\partial u^2}{\partial x}\qquad v\frac{\partial u}{\partial y}\rightarrow v\frac{\partial v}{\partial x}=\frac{1}{2}\frac{\partial v^2}{\partial x}\qquad w\frac{\partial u}{\partial z}\rightarrow w\frac{\partial w}{\partial x}=\frac{1}{2}\frac{\partial w^2}{\partial x}$$

$$\frac{\partial^2 u}{\partial y^2}\rightarrow\frac{\partial^2 v}{\partial y\partial x}\qquad\frac{\partial^2 u}{\partial z^2}\rightarrow\frac{\partial^2 w}{\partial z\partial x}$$

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\frac{u^2}{2}+\frac{v^2}{2}+\frac{w^2}{2}\right)=-g\frac{\partial h}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)\qquad(6.57)$$

6.6 Irrotational Motion

Substitute $q^2 = u^2 + v^2 + w^2$ and continuity eq. for incompressible fluid into Eq. (6.57)

Continuity eq., Eq. (6.5): $\nabla \cdot \vec{q} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$

Then, viscous force term can be dropped.

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q^2}{2}\right) = -g\frac{\partial h}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

→ x - Eq.

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left[\frac{q^2}{2} + gh + \frac{p}{\rho}\right] = 0$$

6.6 Irrotational Motion

$$y-Eq. \qquad \frac{\partial v}{\partial t}+\frac{\partial}{\partial y}\left[\frac{q^2}{2}+gh+\frac{p}{\rho}\right]=0 \qquad (6.58)$$

$$z-Eq. \qquad \frac{\partial w}{\partial t}+\frac{\partial}{\partial z}\left[\frac{q^2}{2}+gh+\frac{p}{\rho}\right]=0 \qquad (6.59)$$

Introduce velocity potential ϕ

$$u=-\frac{\partial \phi}{\partial x}, v=-\frac{\partial \phi}{\partial y}, w=-\frac{\partial \phi}{\partial z}$$

$$\frac{\partial u}{\partial t}=-\frac{\partial^2 \phi}{\partial t \partial x}, \quad \frac{\partial v}{\partial t}=-\frac{\partial^2 \phi}{\partial t \partial y}, \quad \frac{\partial w}{\partial t}=-\frac{\partial^2 \phi}{\partial t \partial z} \qquad \text{(A)}$$

6.6 Irrotational Motion

Substituting (A) into (6.59) yields

$$\frac{\partial}{\partial x}\left[-\frac{\partial\phi}{\partial t}+\frac{q^2}{2}+gh+\frac{p}{\rho}\right]=0 \qquad x-Eq.$$

$$\frac{\partial}{\partial y}\left[-\frac{\partial\phi}{\partial t}+\frac{q^2}{2}+gh+\frac{p}{\rho}\right]=0 \qquad y-Eq.$$

$$\frac{\partial}{\partial z}\left[-\frac{\partial\phi}{\partial t}+\frac{q^2}{2}+gh+\frac{p}{\rho}\right]=0 \qquad z-Eq. \qquad \text{(B)}$$

6.6 Irrotational Motion

Integrating (B) leads to Bernoulli eq.

$$-\frac{\partial \phi}{\partial t}+\frac{q^2}{2}+gh+\frac{p}{\rho}=F(t) \qquad (6.60)$$

~ valid throughout the entire field of irrotational motion

For a steady flow; $\frac{\partial \phi}{\partial t}=0; F(t)=C$

$$\boxed{\frac{q^2}{2}+gh+\frac{p}{\rho}=const.} \qquad (6.61)$$

→ Bernoulli eq. for a steady, irrotational flow of an incompressible fluid

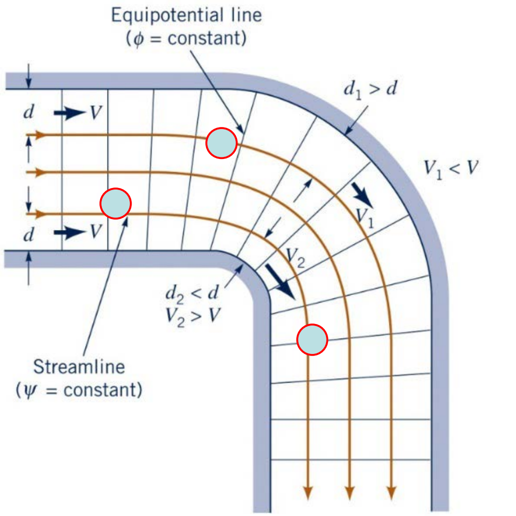
6.6 Irrotational Motion

Dividing (6.61) by g (acceleration of gravity)

gives the head terms

$$\frac{q^2}{2g} + h + \frac{p}{\gamma} = const.$$

$$\frac{q_1^2}{2g} + h_1 + \frac{p_1}{\gamma} = \frac{q_2^2}{2g} + h_2 + \frac{p_2}{\gamma} = H \qquad (6.62)$$



H = total head at a point; constant for entire flow field of irrotational motion

(for both along and normal to any streamline)

→ point form of 1- D Bernoulli Eq.

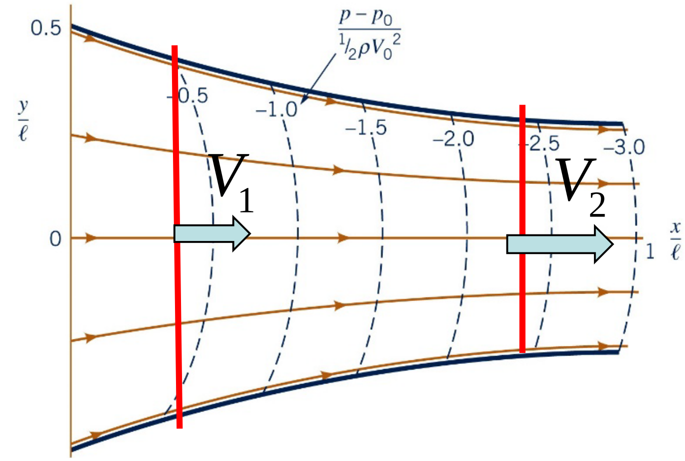
p, H, q = values at particular point → point values in flow field

6.6 Irrotational Motion

[Cf] Eq. (4.26)

$$\frac{p_1}{\gamma} + h_1 + \frac{V_1^2}{2g} = \frac{p_2}{\gamma} + h_2 + \frac{V_2^2}{2g} = H$$

H = constant along a stream tube



→ 1-D form of 1-D Bernoulli eq.

p, h, V = cross-sectional average values at each section → average values

- Assumptions made in deriving Eq. (6.62)

→ incompressibility + steadiness + irrotational motion+ constant viscosity (Newtonian fluid)

6.6 Irrotational Motion

In Eq. (6.57), viscosity term dropped out because $\nabla \cdot \vec{q} = 0$ (continuity Eq.).

→ Thus, Eq. (6.62) can be applied to either a viscous or inviscid fluid.

- Viscous flow

Velocity gradients result in viscous shear.

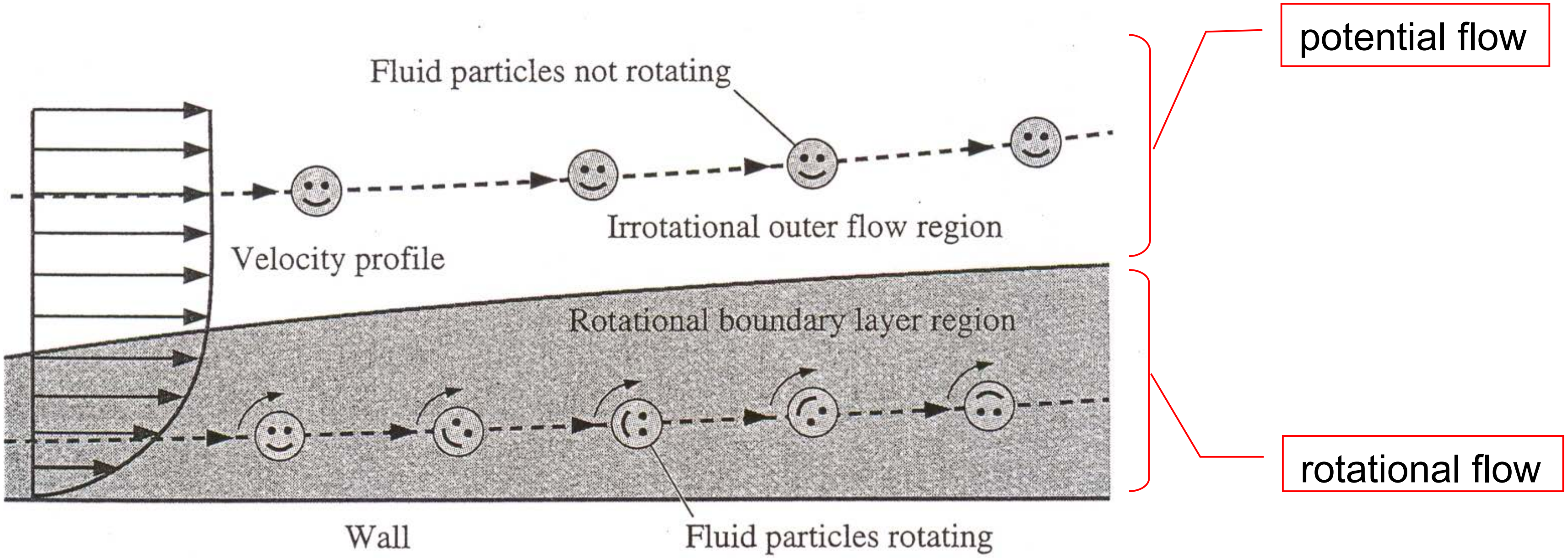
→ Viscosity causes a spread of vorticity (forced vortex).

→ Flow becomes rotational.

→ H in Eq. (6.62) varies throughout the fluid field.

→ Irrotational motion takes place only in a few special cases (irrotational vortex).

6.6 Irrotational Motion



6.6 Irrotational Motion

- Boundary layer flow (Ch. 8)

i) Flow within thin boundary layer - viscous flow- rotational flow

→ use boundary layer theory

ii) Flow outside the boundary layer - irrotational (potential) flow

→ use potential flow theory

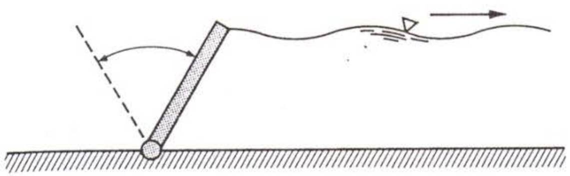
6.6 Irrotational Motion

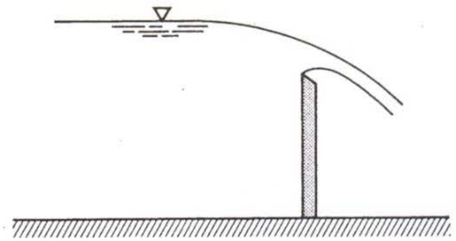
• Irrotational motion can never become rotational as long as only gravitational and pressure force acts on the fluid particles (without shear forces).

→ In real fluids, nearly irrotational flows may be generated if the motion is primarily a result of pressure and gravity forces.

[Ex] free surface wave motion generated by pressure forces (Fig. 6.8)

flow over a weir under gravity forces (Fig. 6.9)





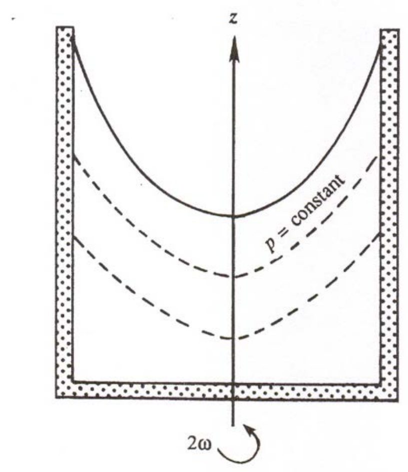
6.6 Irrotational Motion

- Vortex motion

i) Forced vortex - rotational flow

~ generated by the transmission of tangential shear stresses

→ rotating cylinder



ii) Free vortex - irrotational flow

~ generated by the gravity and pressure

→ drain in the tank bottom, tornado, hurricane