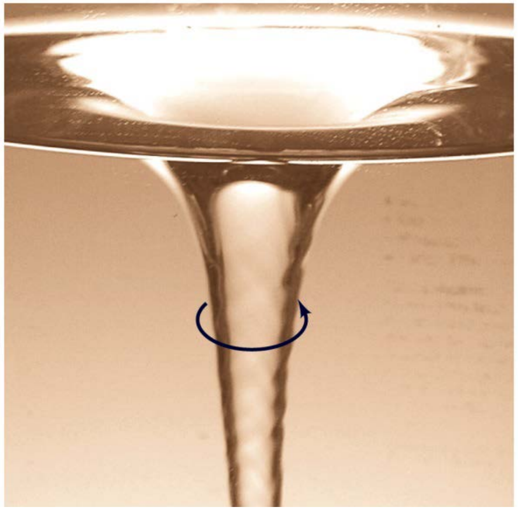
Chapter 6 Equations of Continuity and Motion

Session 6-3 Motions of viscous and inviscid fluids



Chapter 6 Equations of Continuity and Motion

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6.7.1 The Bernoulli equation for flow along a streamline

For inviscid flow

→ Assume no frictional (viscous) effects but compressible fluid flows

→ Bernoulli eq. can be obtained by integrating Navier-Stokes equation along a streamline.

Eq. (6.24a): N-S eq. for ideal compressible fluid ($\mu = 0$)

$$\rho\vec{g} - \nabla p + \cancel{\mu\nabla^2\vec{q}} + \cancel{\frac{\mu}{3}\nabla(\nabla\cdot\vec{q})} = \rho\frac{\partial\vec{q}}{\partial t} + \rho\left(\vec{q}\cdot\nabla\right)\vec{q}$$

$$\vec{g} - \frac{\nabla p}{\rho} = \frac{\partial\vec{q}}{\partial t} + \left(\vec{q}\cdot\nabla\right)\vec{q} \qquad (6.63)$$

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→ Euler's equation of motion for inviscid (ideal) fluid flow

$$\vec{g} = -g\nabla h$$

Substituting (6.26a) into (6.63) leads to

$$-g\nabla h - \frac{\nabla p}{\rho} = \frac{\partial \vec{q}}{\partial t} + (\vec{q}\cdot\nabla)\vec{q} \qquad (6.64)$$

$$\vec{i}dx + \vec{j}dy + \vec{k}dz$$

Multiply $d\vec{r}$ (element of streamline length) and integrate along the streamline

$$-g\int \nabla h\cdot d\vec{r} - \int \frac{1}{\rho}\nabla p\cdot d\vec{r} = \int \left(\frac{\partial \vec{q}}{\partial t}\right)\cdot d\vec{r} + \int \left[(\vec{q}\cdot\nabla)\vec{q}\right]\cdot d\vec{r} + C(t) \qquad (6.65)$$

6.7 Frictionless Flow

$$-gh-\int\frac{dp}{\rho}=\int\left(\frac{\partial\vec{q}}{\partial t}\right)\cdot d\vec{r}+\int\underbrace{\left[\left(\vec{q}\cdot\nabla\right)\vec{q}\right]\cdot d\vec{r}}_{\text{I}}+C\left(t\right) \qquad (6.66)$$

$$I=\left[\left(\vec{q}\cdot\nabla\right)\vec{q}\right]\cdot d\vec{r}=d\vec{r}\cdot\left[\left(\vec{q}\cdot\nabla\right)\vec{q}\right]=\vec{q}\cdot\left[\underbrace{\left(d\vec{r}\cdot\nabla\right)}_{\text{II}}\vec{q}\right]$$

By the way,

$$II=d\vec{r}\cdot\nabla=\frac{\partial(\ )}{\partial x}dx+\frac{\partial(\ )}{\partial y}dy+\frac{\partial(\ )}{\partial z}dz$$

$$\therefore\ \left(d\vec{r}\cdot\nabla\right)\vec{q}=\frac{\partial\vec{q}}{\partial x}dx+\frac{\partial\vec{q}}{\partial y}dy+\frac{\partial\vec{q}}{\partial z}dz=d\vec{q}$$

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$$I = \vec{q} \cdot d\vec{q} = d\left(\frac{q^2}{2}\right)$$

$$\therefore \int \left[\left(\vec{q} \cdot \nabla\right)\vec{q}\right] \cdot d\vec{r} = \int d\left(\frac{q^2}{2}\right) = \frac{q^2}{2}$$

Thus, Eq. (6.66) becomes

$$\int \frac{dp}{\rho} + gh + \frac{q^2}{2} + \int \left(\frac{\partial q}{\partial t}\right) \cdot d\vec{r} = -C\left(t\right) \qquad (6.67)$$

For steady motion, $\dfrac{\partial \vec{q}}{\partial t} = 0; C\left(t\right) \rightarrow C$

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$$\int \frac{dp}{\rho} + gh + \frac{q^2}{2} = const. \quad \text{along a streamline} \qquad (6.68)$$

For incompressible fluids, $\rho =$ const.

$$\frac{p}{\rho} + gh + \frac{q^2}{2} = const.$$

Divide by g

$$\frac{p}{\gamma} + h + \frac{q^2}{2g} = C \quad \text{along a streamline} \qquad (6.69)$$

6.7 Frictionless Flow

→ Bernoulli equation for steady, frictionless, incompressible fluid flow

→ Eq. (6.69) is identical to Eq. (6.22). Constant *C* is varying from one streamline to another in a rotational flow, Eq. (6.69); it is invariant throughout the fluid for irrotational flow, Eq. (6.22).

6.7.2 Summary of Bernoulli equation forms

• Bernoulli equations for steady, incompressible flow

1) For irrotational flow

$$H = \frac{p}{\gamma} + h + \frac{q^2}{2g} = \text{ constant throughout the flow field} \qquad (6.70)$$

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2) For frictionless flow (rotational)

$$H = \frac{p}{\gamma} + h + \frac{q^2}{2g} = \text{ constant along a streamline} \quad (6.71)$$

3) For 1-D frictionless flow (rotational)

$$H = \frac{p}{\gamma} + h + Ke\frac{V^2}{2g} = \text{ constant along finite pipe} \quad (6.72)$$

4) For steady flow with friction ~ include head loss h_L

$$\frac{p_1}{\gamma} + h_1 + \frac{q_1^2}{2g} = \frac{p_2}{\gamma} + h_2 + \frac{q_2^2}{2g} + h_L \quad (6.73)$$

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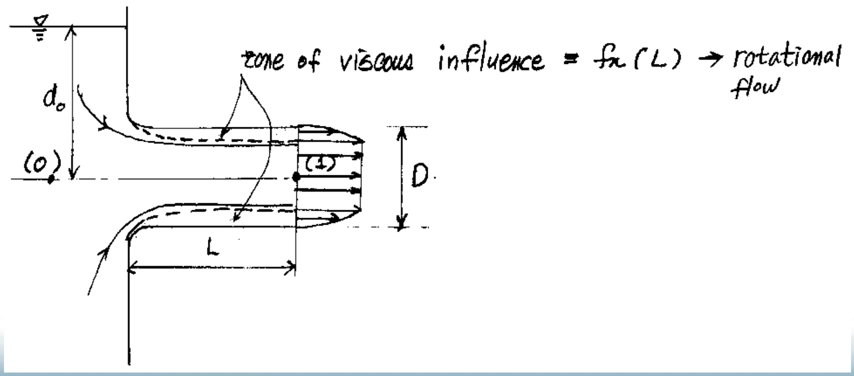
6.7.3 Applications of Bernoulli's equation to flows of real fluids

(1) Efflux from a short tube

- Zone of viscous action (boundary layer): frictional effects cannot be neglected.
- Flow in the reservoir and central core of the tube: primary forces are pressure and gravity forces. → irrotational flow
- Apply Bernoulli eq. along the centerline streamline between (0) and (1)



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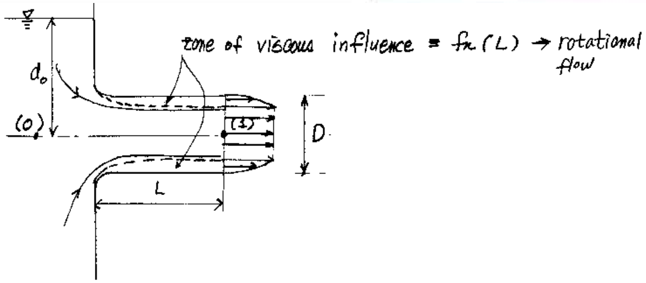

$$\frac{p_0}{\gamma} + z_0 + \frac{q_0^2}{2g} = \frac{p_1}{\gamma} + z_1 + \frac{q_1^2}{2g}$$

p_0 = hydrostatic pressure = γd_0 , $p_1 = p_{atm}$ → $p_{1_{gage}} = 0$

$$z_0 = z_1$$

$q_0 = 0$ (neglect velocity at the large reservoir)

$$\therefore \frac{q_1^2}{2g} = d_0 \qquad q_1 = \sqrt{2gd_0} \quad \rightarrow \text{Torricelli's result} \qquad (6.74)$$

If we <u>neglect thickness of the zone of viscous influence</u>

$$Q = \frac{\pi D^2}{4} q_1$$

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(2) Stratified flow

During summer months, large reservoirs and lakes become thermally stratified.

→ At thermocline, temperature changes rapidly with depth.

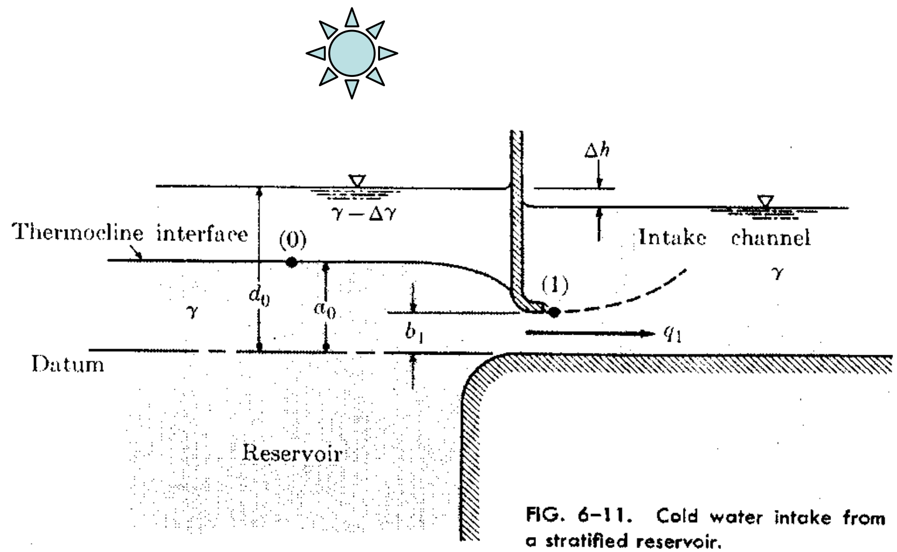

FIG. 6-11. Cold water intake from a stratified reservoir.

• Selective withdrawal: Colder water is withdrawn into the intake channel with a velocity q_1 (uniform over the height b_1) in order to provide cool condenser water for thermal (nuclear) power plant.

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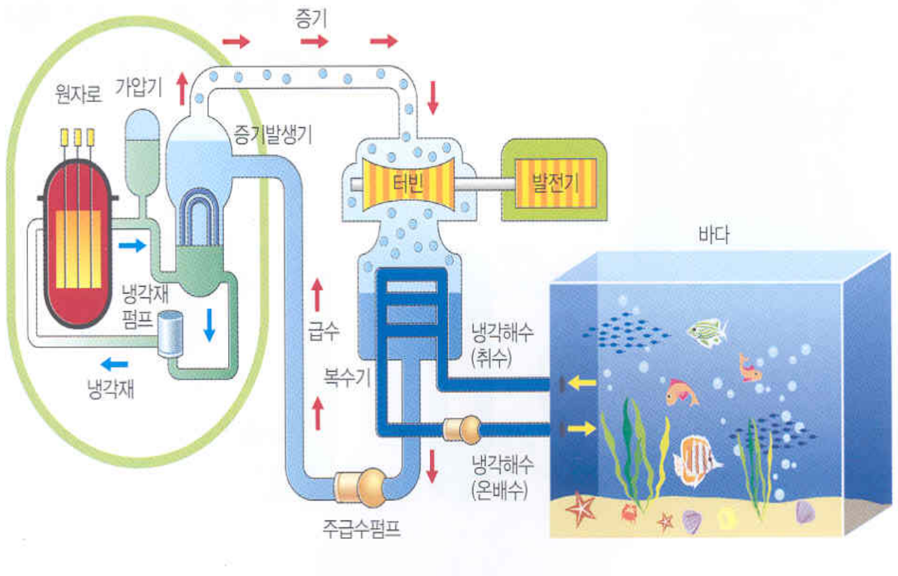

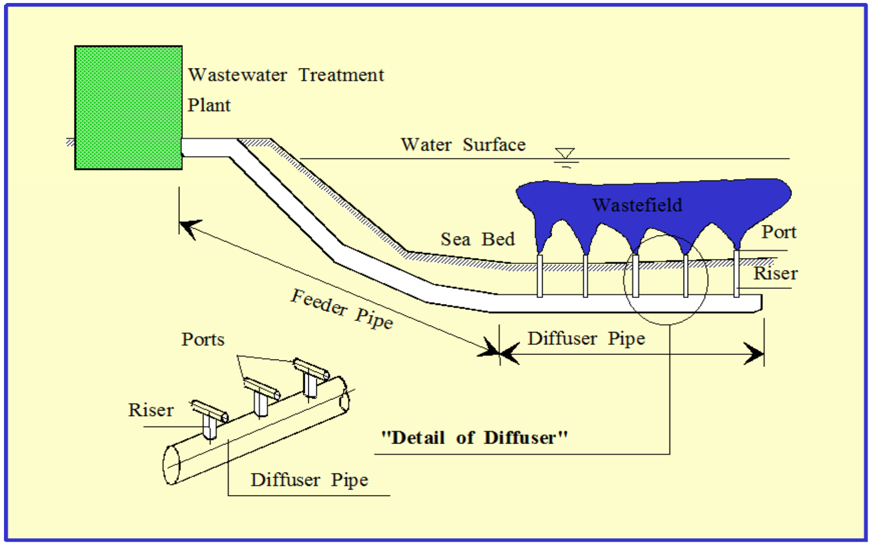

6.7 Frictionless Flow

Apply Bernoulli eq. between points (0) and (1)

$$\frac{p_0}{\gamma} + a_0 + \frac{q_0^{\ 2}}{2g} = \frac{p_1}{\gamma} + b_1 + \frac{q_1^2}{2g} \quad (6.75)$$

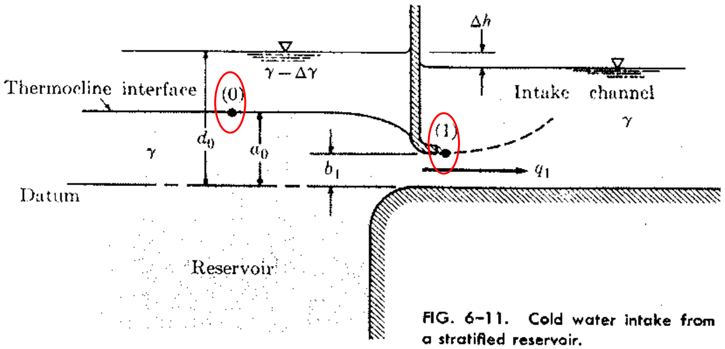

FIG. 6-11. Cold water intake from a stratified reservoir.

$$q_0 \cong 0$$

p_0 = hydrostatic pressure = $(\gamma - \Delta\gamma)(d_0 - a_0)$

$$p_1 = \gamma\left(d_0 - \Delta h - b_1\right)$$

$$\therefore \quad \frac{q_1^{\ 2}}{2g} = \Delta h - \frac{\Delta\gamma}{\gamma}\left(d_0 - a_0\right) \quad (6.76)$$

$$q_1 = \left[2g\left\{\Delta h - \frac{\Delta\gamma}{\gamma}\left(d_0 - a_0\right)\right\}\right]^{\frac{1}{2}} \quad (6.77)$$

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For isothermal (unstratified) case, $a_0 = d_0$

$$q_1 = \sqrt{2g\Delta h} \quad \rightarrow \text{Torricelli's result} \tag{6.78}$$

6.7 Frictionless Flow

(3) Velocity measurements with the Pitot tube (Henri Pitot, 1732)

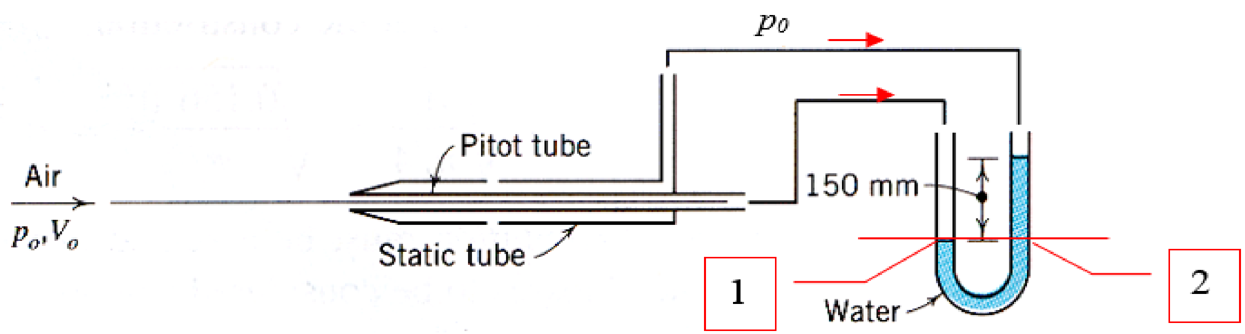
→ Measure velocity from stagnation or impact pressure

$$\frac{p_0}{\gamma} + h_0 + \frac{q_0^{\ 2}}{2g} = \frac{p_s}{\gamma} + h_s + \frac{q_s^{\ 2}}{2g}$$

$$h_0 = h_s, \quad q_s = 0 \qquad (6.79)$$

$$\therefore \quad \frac{q_0^{\ 2}}{2g} = \frac{p_s - p_0}{\gamma} = \Delta h \qquad (6.80)$$

P_s



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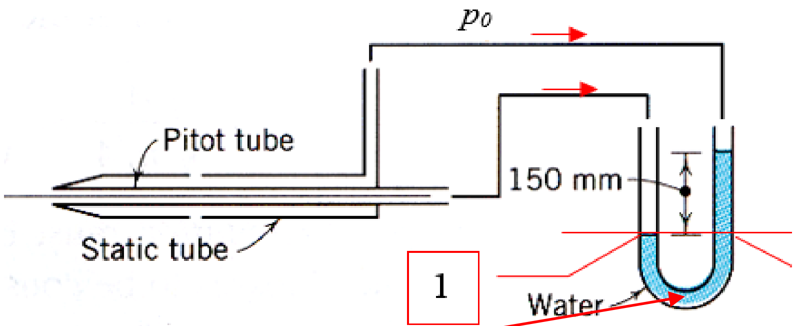
- Pitot-static tube

$$q_0 = \sqrt{\frac{2(p_s - p_0)}{\rho}} \qquad \text{(A)}$$



By the way,

$$p_1 = p_s + \gamma \Delta h = p_2 = p_0 + \gamma_m \Delta h$$

$$p_s - p_0 = \Delta h(\gamma_m - \gamma) \qquad \text{(B)}$$

Combine (A) and (B)

$$q_0 = \sqrt{\frac{2\Delta h(\gamma_m - \gamma)}{\rho}}$$

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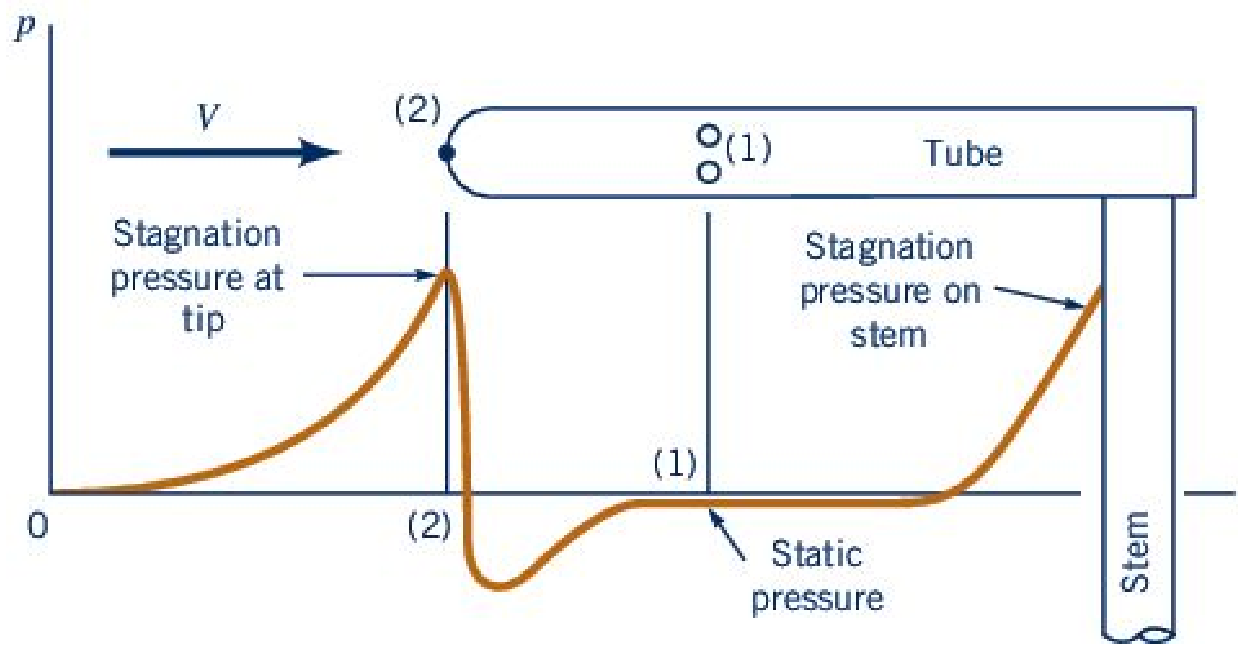

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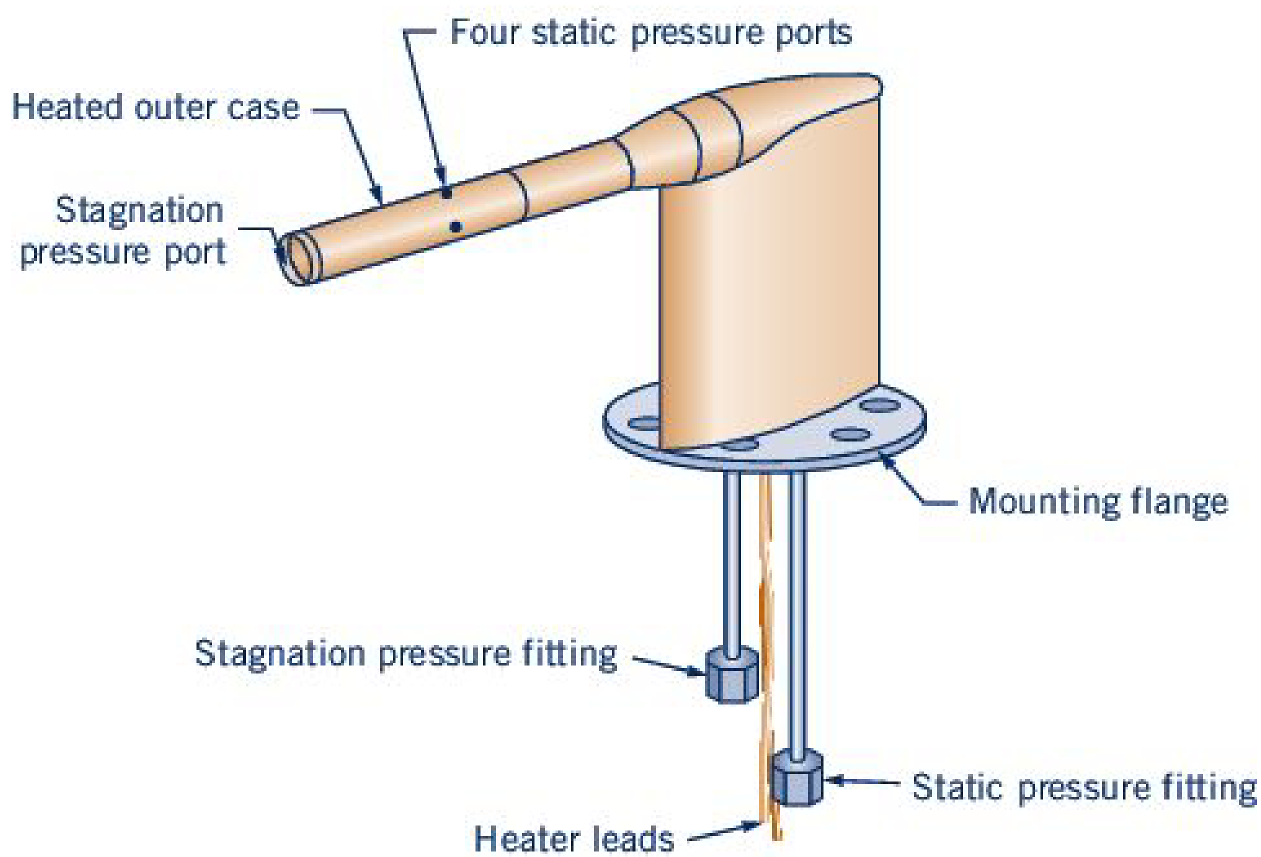

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