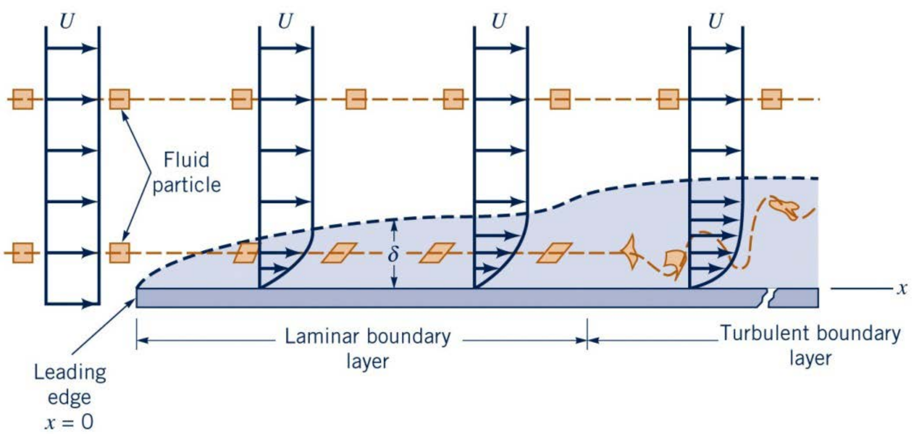
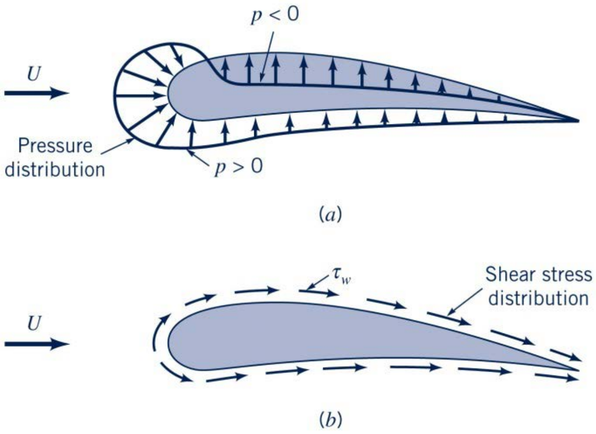
Chapter 7

Specialized Equations in Fluid Dynamics





Chapter 7 Specialized Equations in Fluid Dynamics

Contents

Objectives

- Discuss special cases of flow motion
- Derive equations for creeping motion
- Derive equation for 2-D boundary layers and integral equation
- Study flow resistance and drag force

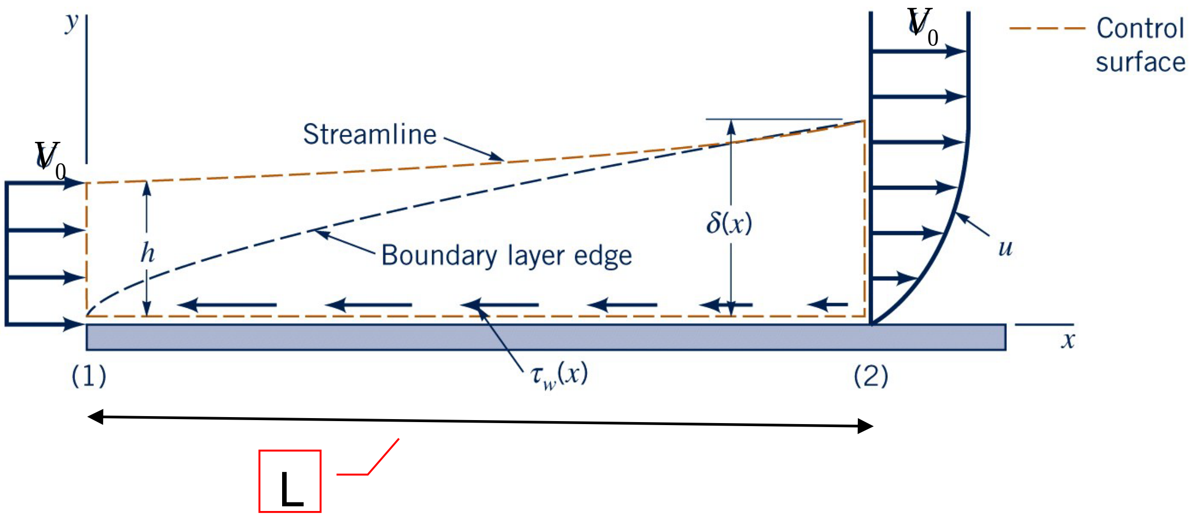
7.2 Equations for Creeping Motion and 2-D Boundary Layers

7.2.2 Equations for 2-D boundary layers

(1) Two-dimensional boundary layer equations: Prandtl

→ simplification of the N-S Eq. using order-of-magnitude arguments

→ 2D dimensionless N-S eq. for incompressible fluid (omit gravity)



L, V_0 - constant reference values

7.2 Equations for Creeping Motion and 2-D Boundary Layers

Within thin and small curvature boundary layer

$$u \gg v, \qquad x \gg y$$

$$\frac{\partial u}{\partial y} \gg \frac{\partial u}{\partial x}$$

$$\frac{\partial p}{\partial y} \; is\ small \sim may\ be\ neglected$$

dimensionless boundary-layer thickness δ°

$$\delta^{\circ} = \frac{\delta}{L} \rightarrow \boxed{\delta^{\circ} \ll 1}$$

$\therefore$ scale for decreasing order

$$\frac{1}{\delta^{\circ 2}} > \frac{1}{\delta^{\circ}} > 1 > \delta^{\circ} > \delta^{\circ 2}$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

Order of magnitude

$$x^\circ \sim O\,(1) \qquad x^\circ = \frac{x}{L}$$

$$y^\circ \sim O\,(\delta^\circ) \qquad y^\circ = \frac{y}{L}$$

$$u^\circ \sim O\,(1) \qquad u^\circ = \frac{u}{V_0}$$

$$v^\circ \sim O\,(\delta^\circ) \qquad v^\circ = \frac{v}{V_0}$$

$$\frac{\partial u^\circ}{\partial x^\circ} \sim O\,(1)$$

$$\frac{\partial v^\circ}{\partial y^\circ} \sim O\,(1) \leftarrow continuity \left(\frac{\partial v^\circ}{\partial y^\circ} = -\frac{\partial u^\circ}{\partial x^\circ} \right) \qquad p^\circ = \frac{p}{\rho V_0^2}$$

$$\frac{\partial u^\circ}{\partial y^\circ} \sim O\,(\frac{1}{\delta^\circ})$$

$$\frac{\partial v^\circ}{\partial x^\circ} \sim O\,(\delta^\circ)$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

$$\frac{\partial^2 u^\circ}{\partial (x^\circ)^2} = \frac{\partial}{\partial x^\circ}\left(\frac{\partial u^\circ}{\partial x^\circ}\right) \sim O(1)$$

$$\frac{\partial^2 v^\circ}{\partial (y^\circ)^2} = \frac{\partial}{\partial y^\circ}\left(\frac{\partial v^\circ}{\partial y^\circ}\right) \sim O(\frac{1}{\delta^\circ})$$

$$\frac{\partial u^\circ}{\partial t^\circ} = \frac{\partial u^\circ}{\partial x^\circ}\frac{\partial x^\circ}{\partial t^\circ} = u^\circ \frac{\partial u^\circ}{\partial x^\circ} \sim O(1)$$

$$\frac{\partial v^\circ}{\partial t^\circ} = \frac{\partial v^\circ}{\partial x^\circ}\frac{\partial x^\circ}{\partial t^\circ} = u^\circ \frac{\partial v^\circ}{\partial x^\circ} \sim O(\delta^\circ)$$

$$\mathrm{Re} = \frac{\rho v y}{\mu} \sim O(\delta^{\circ 2})$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

$$x: \frac{\partial u^\circ}{\partial t^\circ} + u^\circ \frac{\partial u^\circ}{\partial x^\circ} + v^\circ \frac{\partial u^\circ}{\partial y^\circ} = -\frac{\partial p^\circ}{\partial x^\circ} + \frac{1}{\mathrm{Re}}\left(\cancel{\frac{\partial^2 u^\circ}{\partial x^{\circ 2}}} + \frac{\partial^2 u^\circ}{\partial y^{\circ 2}} \right) \tag{7.3}$$

1   1x1 $\delta^\circ \times 1/\delta^\circ$ $\delta^{\circ 2}(1+1/\delta^{\circ 2})$ → 1

$$y: \cancel{\frac{\partial v^\circ}{\partial t^\circ}} + u^\circ \cancel{\frac{\partial v^\circ}{\partial x^\circ}} + v^\circ \cancel{\frac{\partial v^\circ}{\partial y^\circ}} = -\cancel{\frac{\partial p^\circ}{\partial y^\circ}} + \frac{1}{\mathrm{Re}}\left(\cancel{\frac{\partial^2 v^\circ}{\partial x^{\circ 2}}} + \cancel{\frac{\partial^2 v^\circ}{\partial y^{\circ 2}}} \right)$$

δ° $1\times\delta^\circ$ $\delta^\circ\times 1$ $\delta^{\circ 2}(\delta^\circ + 1/\delta^\circ)$ → δ°

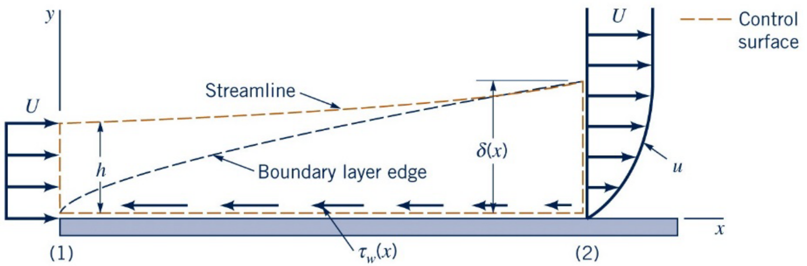
$$Continuity: \frac{\partial u^\circ}{\partial x^\circ} + \frac{\partial v^\circ}{\partial y^\circ} = 0$$

1   1

7.2 Equations for Creeping Motion and 2-D Boundary Layers

Therefore, eliminate all terms of order less than unity in Eq. (7.3) and revert to dimensional terms

$$\boxed{\begin{aligned} &\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} \\ &\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \end{aligned}}$$



(7.7)

→ Prandtl's 2-D boundary-layer equation

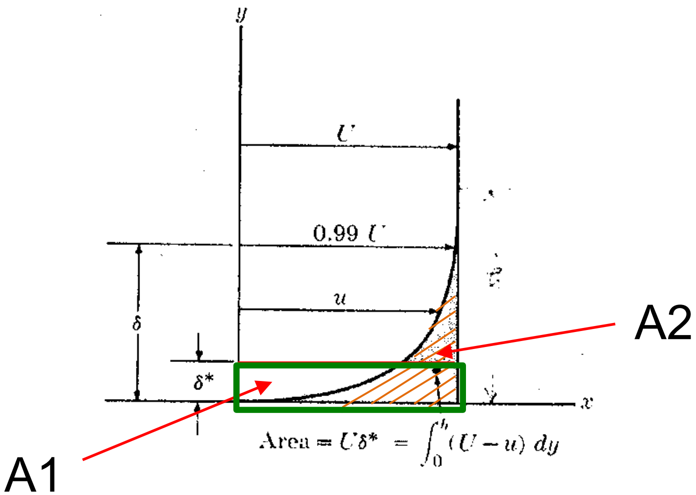
$$BC: 1)\quad y=0\,;\, u=0, v=0$$

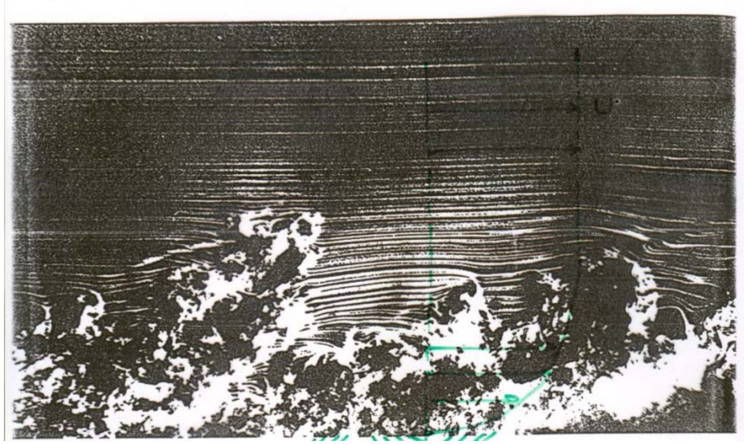
$$2)\quad y=\infty\,;\, u=U(x) \qquad (7.8)$$

Unknowns: *u, v, p*; Eqs. = 2 → needs assumptions for *p*

7.2 Equations for Creeping Motion and 2-D Boundary Layers

7.2.3 Boundary - layer thickness definitions





Intermittent nature

(1) Boundary-layer thickness,

~ The point separating the boundary layer from the <u>zone of negligible viscous influence</u> is not a sharp one. → very intermittent

δ = distance to the point where the velocity is within 1% of the free-stream velocity, *U*

$$@\, y = \delta \rightarrow u_\delta = 0.99U$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

(2) Mass displacement thickness, $\delta^* \, (\delta_1)$

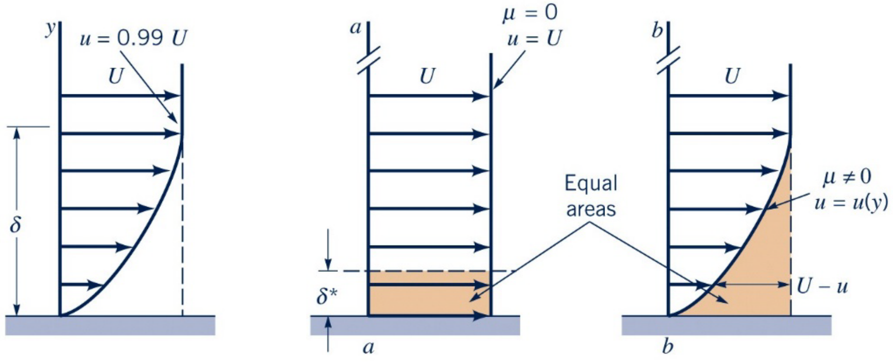
~ is the thickness of an imaginary layer of fluid of velocity *U*.

~ is the thickness of mass flux rate equal to the amount of defect

$$A_1 = A_2$$

$$\rho U \delta^* = \underbrace{\rho \int_0^h (U-u)\,dy}_{mass\ defect} \qquad h \geq \delta$$

$$\boxed{\therefore \ \delta^* = \int_0^h \left(1 - \frac{u}{U}\right) dy} \qquad (7.9)$$



7.2 Equations for Creeping Motion and 2-D Boundary Layers

[Re] mass flux = mass/time

$$= \rho Q = \rho UA = \rho U \delta^* \times 1$$

(3) Momentum thickness, $\theta(\delta_2)$

→ Velocity retardation within δ causes a reduction in the rate of momentum flux.

→ θ is the thickness of an imaginary layer of fluid of velocity U for which the momentum flux rate equals the reduction caused by the velocity profile.

$$\rho \theta U^2 = \rho \int_0^h (U-u)u dy = \rho \int_0^h (Uu - u^2) dy$$

$$\boxed{\therefore \theta = \int_0^h \frac{u}{U}\left(1 - \frac{u}{U}\right) dy} \qquad (7.10)$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

[Re] momentum in θ = $mass \times velocity = \rho\theta U \times U = \rho\theta U^2$

momentum in shaded area = $\int [\rho(U-u) \times u] dy$

$$\boxed{\delta > \delta^* > \theta}$$

(4) Energy thickness, δ_3

$$\frac{1}{2}\rho U^3 \delta_3 = \frac{1}{2}\int_0^h \rho u (U^2 - u^2) dy$$

$$\boxed{\therefore \delta_3 = \int_0^h \frac{u}{U}\left(1 - \frac{u^2}{U^2}\right) dy}$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

[Re]

1) Batchelor (1985):

displacement thickness = distance through which streamlines just outside the boundary layer are displaced laterally by the <u>retardation of fluid</u> in the boundary layer.

2) Schlichting (1979):

displacement thickness = distance by which the external streamlines are shifted owing to the formation of the boundary layer.

7.2 Equations for Creeping Motion and 2-D Boundary Layers

7.2.4 Integral momentum equation for 2-D boundary layers

Integrate Prandtl's 2-D boundary-layer equations

$$\cancel{\frac{\partial u}{\partial t}} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \cancel{-\frac{1}{\rho}\frac{\partial p}{\partial x}} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Assumptions:

constant density $d\rho = 0$

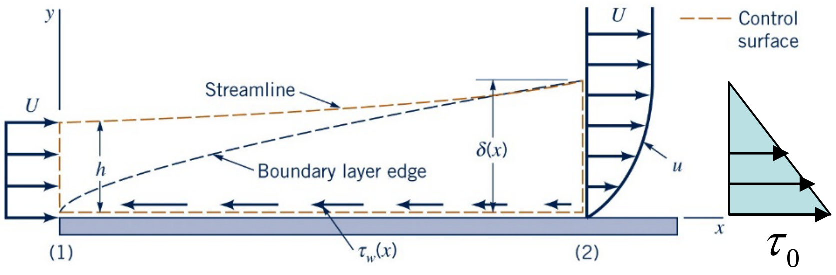
steady motion $\frac{\partial ()}{\partial t} = 0$

pressure gradient = 0 $\frac{\partial p}{\partial x} = 0$

BC's:

$$@\, y = h\,;\quad \tau = 0,\quad u = U$$

$$@\, y = 0\,;\quad \tau = \tau_0,\quad u = 0$$



7.2 Equations for Creeping Motion and 2-D Boundary Layers

Prandtl's 2-D boundary-layer equations become as follows:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} \qquad \text{(A)}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad \text{(B)}$$

Integrate Eq. (A) w.r.t. y

$$\int_{y=0}^{y=h\geq\delta}\left(\underbrace{u\frac{\partial u}{\partial x}}_{①}+\underbrace{v\frac{\partial u}{\partial y}}_{②}\right)dy=\underbrace{\frac{\mu}{\rho}\int_{y=0}^{y=h}\frac{\partial^2 u}{\partial y^2}\,dy}_{③} \qquad \text{(C)}$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

$$③ = \mu \int_0^h \frac{\partial^2 u}{\partial y^2} dy = \int_0^h \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) dy = \int_0^h \frac{\partial \tau}{\partial y} dy = [\tau]_0^h$$

$$= \tau\big|_{y=h} - \tau\big|_{y=0} = 0 - \tau_0 = -\tau_0$$

$$② = \int_0^h v \frac{\partial u}{\partial y} dy = \underbrace{\int_0^h \frac{\partial uv}{\partial y} dy}_{④} - \underbrace{\int_0^h u \frac{\partial v}{\partial y} dy}_{⑤} \qquad \text{(D)}$$

[Re] Integration by parts: $\int vu' dy = uv - \int uv' dy$

$$④ = \int_0^h \frac{\partial uv}{\partial y} dy = [uv]_0^h = Uv_h - 0 = Uv$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

Continuity Eq.: $\dfrac{\partial v}{\partial y} = -\dfrac{\partial u}{\partial x}$

$$\rightarrow \quad v = -\int_0^h \frac{\partial u}{\partial x} dy \qquad \text{(i)}$$

Substitute (i) into ⑤

$$⑤ = \int_0^h u\left(-\frac{\partial u}{\partial x}\right) dy = -\int_0^h u \frac{\partial u}{\partial x} dy \qquad \text{(ii)}$$

Substitute (i) into ④

$$④ = Uv = -U\int_0^h \frac{\partial u}{\partial x} dy$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

Eq. (D) becomes

$$\int_0^h v\frac{\partial u}{\partial y}dy = -U\int_0^h \frac{\partial u}{\partial x}dy + \int_0^h u\frac{\partial u}{\partial x}dy \qquad \text{(E)}$$

Then, (C) becomes

$$\int_0^h u\frac{\partial u}{\partial x}dy - U\int_0^h \frac{\partial u}{\partial x}dy + \int_0^h u\frac{\partial u}{\partial x}dy = -\frac{\tau_0}{\rho} \qquad \text{(F)}$$

For steady motion with and U = const., (F) becomes

$$\frac{\tau_0}{\rho} = U\int_0^h \frac{\partial u}{\partial x}dy - 2\int_0^h u\frac{\partial u}{\partial x}dy = \int_0^h \frac{\partial Uu}{\partial x}dy - \int_0^h \frac{\partial u^2}{\partial x}dy$$

$$= \int_0^h \frac{\partial}{\partial x}[u(U-u)]dy = \frac{\partial}{\partial x}\underbrace{\int_0^h u(U-u)\,dy}_{\theta U^2} = \frac{\partial}{\partial x}(\theta U^2)$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

where θ = momentum thickness

$$\boxed{\frac{\tau_0}{\rho} = \frac{\partial}{\partial x}(U^2\theta) = U^2\frac{\partial\theta}{\partial x}} \qquad (7.18)$$

$$\frac{\partial\theta}{\partial x} = \frac{\tau_0}{\rho U^2} = \left(\frac{u^*}{U}\right)^2$$

Introduce local surface (frictional) <u>resistance coefficient</u> C_f

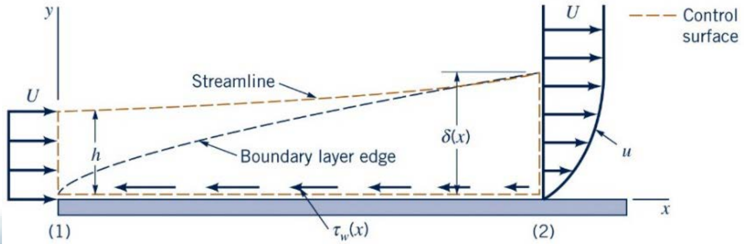
$$C_f = \frac{D_f}{\frac{\rho}{2}u^2 A_f} = \frac{\tau_0}{\frac{\rho}{2}U^2}$$

$$\boxed{D_f = \frac{\rho}{2}C_f A_f u^2}$$

$$\boxed{\tau_0 = \frac{\rho}{2}C_f U^2} \qquad (7.19)$$

Combine (7.18) with (7.19)

$$\boxed{C_f = 2\frac{\partial\theta}{\partial x}} \qquad (7.20)$$



7.2 Equations for Creeping Motion and 2-D Boundary Layers

[Re] Integral momentum equation for unsteady motion

→ unsteady motion: $\dfrac{\partial()}{\partial t} \neq 0$

→ pressure gradient, $\dfrac{\partial p}{\partial x} \neq 0$

First, simplify Eq. (7.7) for external flow where viscous influence is negligible.

$$\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + v\cancel{\frac{\partial U}{\partial y}} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\mu}{\rho}\cancel{\frac{\partial^2 U}{\partial y^2}}$$

$$\rho\frac{\partial U}{\partial t} + \rho U\frac{\partial U}{\partial x} = -\frac{\partial p}{\partial x} \qquad \text{(A)}$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

Substitute (A) into (7.7)

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\underbrace{\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial x}}_{\boxed{-\frac{1}{\rho}\frac{\partial p}{\partial x}}}+\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}$$

Integrate

$$\underbrace{\int_0^h \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}dy}_{①}=\int_0^h\left\{\underbrace{\frac{\partial u}{\partial t}-\frac{\partial U}{\partial t}}_{②}+\underbrace{u\frac{\partial u}{\partial x}-U\frac{\partial U}{\partial x}}_{③}+\underbrace{v\frac{\partial u}{\partial y}}_{④}\right\}dy \qquad \text{(B)}$$

$$①:\ \int_0^h \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}dy=-\frac{\tau_0}{\rho}$$

$$②:\ \int_0^h\left(\frac{\partial u}{\partial t}-\frac{\partial U}{\partial t}\right)dy=\int_0^h\frac{\partial}{\partial t}(u-U)dy=\frac{\partial}{\partial t}\underbrace{\int_0^h(u-U)dy}_{\boxed{-U\delta^*}}=-\frac{\partial}{\partial t}U\delta^*$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

$$③ = \underbrace{\int_0^h \left(u\frac{\partial u}{\partial x} - u\frac{\partial U}{\partial x} \right) dy}_{③\text{-}1} + \underbrace{\int \left(u\frac{\partial U}{\partial x} - U\frac{\partial U}{\partial x} \right) dy}_{③\text{-}2}$$

$$③\text{-}1 = \int_0^h \left\{ u\frac{\partial}{\partial x}(u-U) \right\} dy$$

$$③\text{-}2 = \int_0^h \left\{ (u-U)\frac{\partial U}{\partial x} \right\} dy = \frac{\partial U}{\partial x}\underline{\int_0^h (u-U)dy} = \frac{\partial U}{\partial x}(-U\delta^*)$$

$$④ = \int_0^h v\frac{\partial u}{\partial y}dy \underset{Eq.(E)}{=} -U\int_0^h \frac{\partial u}{\partial x}dy + \int u\frac{\partial u}{\partial x}dy = \int_0^h (u-U)\frac{\partial u}{\partial x}dy$$

$$\boxed{Eq.(E)}$$

7.2 Equations for Creeping Motion and 2-D Boundary Layers

Combine ③-1 and ④

$$\int_0^h u\frac{\partial}{\partial x}(u-U)dy+\int_0^h (u-U)\frac{\partial u}{\partial x}dy=\int_0^h\left[u\frac{\partial}{\partial x}(u-U)+(u-U)\frac{\partial u}{\partial x}\right]dy$$

$$=\int_0^h\frac{\partial}{\partial x}\{u(u-U)\}dy=\frac{\partial}{\partial x}\int_0^h \underline{u(u-U)dy}=\frac{\partial}{\partial x}(-\theta U^2)$$

Substituting all these into (B) yields

$$-\frac{\tau_0}{\rho}=-\frac{\partial}{\partial t}(U\delta^*)-U\frac{\partial U}{\partial x}\delta^*-\frac{\partial}{\partial x}(\theta U^2)$$

$$\boxed{\frac{\tau_0}{\rho}=\frac{\partial}{\partial x}(U^2\theta)+U\frac{\partial U}{\partial x}\delta^*+\frac{\partial}{\partial t}(U\delta^*)} \qquad (7.21)$$

→ Karman's integral momentum eq.

7.3 The notion of resistance, drag, and lift

→ Study Ch.15 (D&H)

Resistance to motion = drag of a fluid on an immersed body in the direction of flow

♦ Dynamic (surface) force exerted on the rigid boundary by moving fluid are

1) Tangential force caused by shear stresses due to viscosity and velocity gradients at the boundary surfaces

2) Normal force caused by pressure intensities which vary along the surface due to dynamic effects

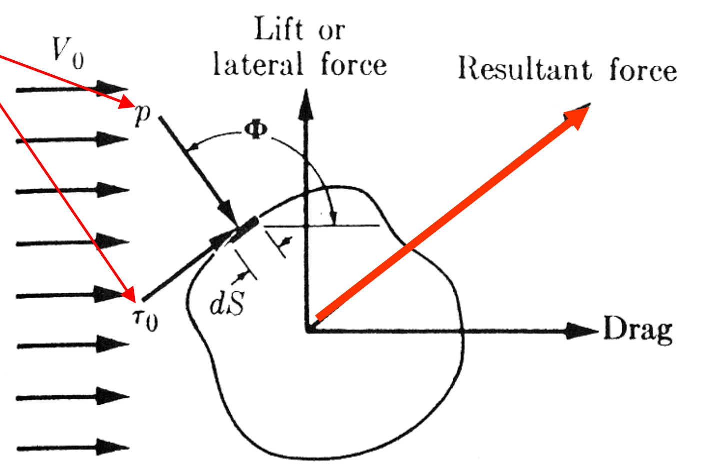


FIG. 8–7. Definition diagram for flow-induced forces.

7.3 The notion of resistance, drag, and lift

♦ Resultant force = vector sum of the normal and tangential surface forces integrated over the complete surface

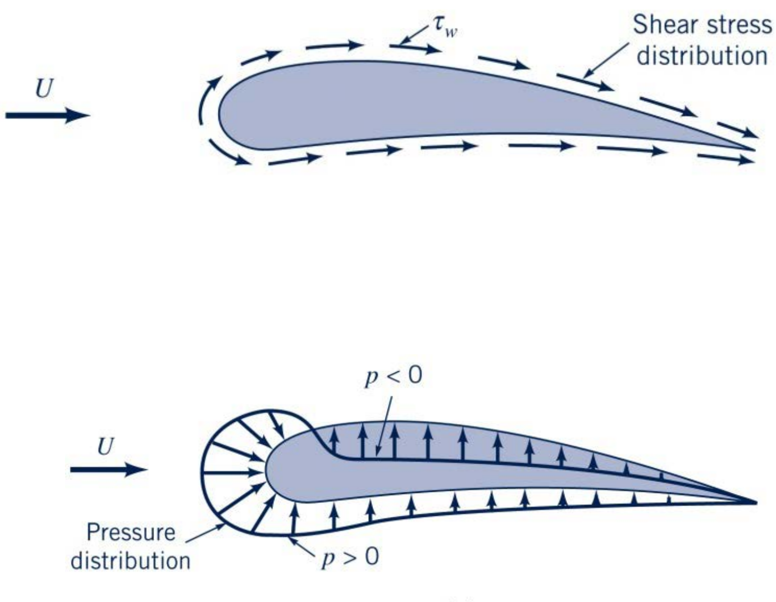
~ resultant force is divided into two components:

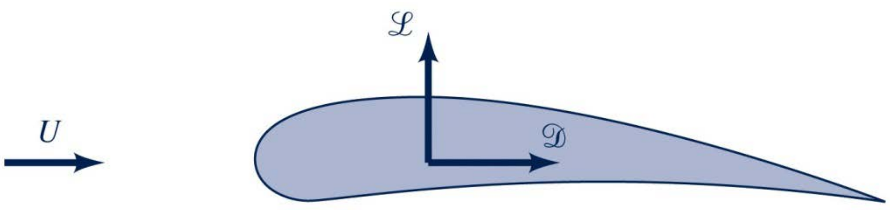
1) **Drag force** = component of the resultant force in the direction of relative velocity V_0

2) **Lift force** = component of the resultant force normal to the relative velocity V_0

~ Both drag and lift include frictional and pressure components.

7.3 The notion of resistance, drag, and lift





(c)

7.3 The notion of resistance, drag, and lift

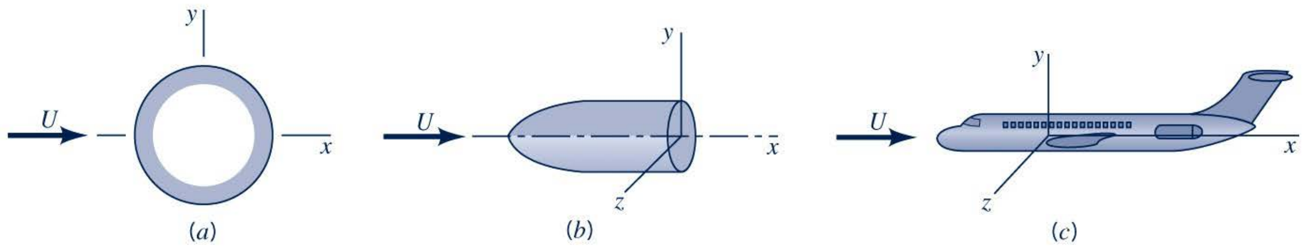
① Frictional drag = surface resistance = skin drag

② Pressure drag = form drag

~ depends largely on shape or form of the body

For airfoil, hydrofoil, and slim ships: surface drag > form drag

For bluff objects like spheres, bridge piers: surface drag < form drag



7.3 The notion of resistance, drag, and lift

7.3.1 Drag force

♦ Total drag, *D*

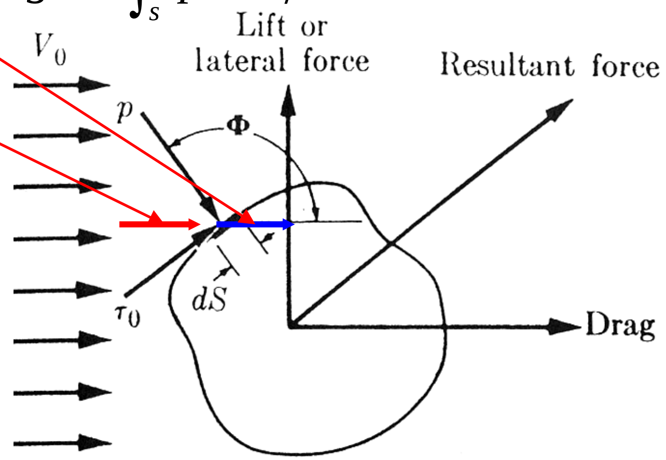
$$D = D_f + D_p$$

where $D_f = \text{frictional drag} = \int_s \tau_0 \sin\phi \, ds$

$$D_p = \text{pressure drag} = -\int_s p \cos\phi \, ds$$

$$\sin\phi = \sin(90^\circ + \alpha) = \cos\alpha$$

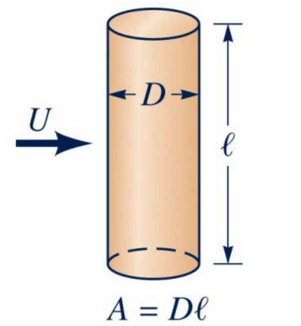
$$\cos\phi = \cos(90^\circ + \alpha) = -\sin\alpha$$



7.3 The notion of resistance, drag, and lift

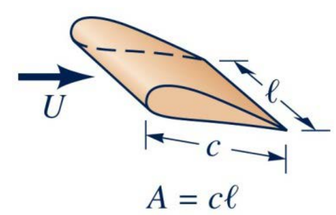
♦ Drag coefficients, C_{D_f}, C_{D_p}

$$\boxed{\begin{aligned} D_f &= C_f \rho \frac{V_0^2}{2} A_f \\ D_p &= C_{D_p} \rho \frac{V_0^2}{2} A_P \end{aligned}}$$



$A_f = \pi D l$

$A_p = D l$



$A_f = 2 c l$

$A_p = t l$

Where A_f = actual area over which shear stresses act to produce D_f

A_p = frontal (projected) area normal to the V_0 velocity

7.3 The notion of resistance, drag, and lift

♦ Total drag coefficient C_D

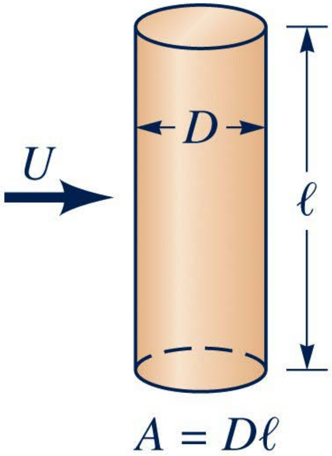
$$D = C_D \rho \frac{V_0^2}{2} A$$

where A = frontal area normal to V_O

$$C_D = C_{D_f} + C_{D_p}$$

$$Cf = C_{D_f}\left(\frac{A_P}{A_f}\right)$$

$$C_D = C_D(geometry, \mathrm{Re})$$

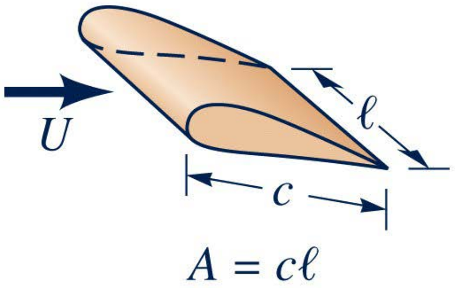


[Re] Dimensional Analysis → Ch. 15

$$D = f_1(\rho, \mu, V, L)$$

$$\frac{D}{\rho L^2 V^2} = f_2\left(\frac{\rho V L}{\mu}\right) = f_2(\mathrm{Re}) = C_D$$

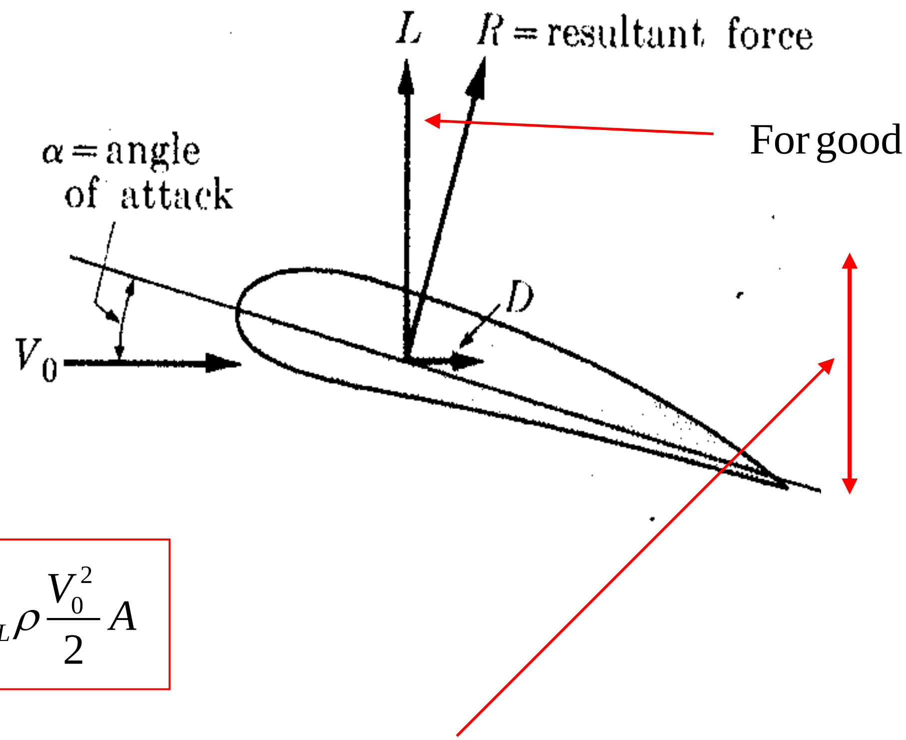
$$\therefore D = C_D \frac{\rho}{2} A V^2$$



7.3 The notion of resistance, drag, and lift

7.3.1 Lift force

For lift forces, it is not customary to separate the frictional and pressure components.



♦ Total lift, L

$$\boxed{L = C_L \rho \frac{V_0^2}{2} A}$$

where C_L = lift coefficient; A = <u>largest projected area</u> of the body

Homework Assignment

Homework Assignment # 7

Due: 1 week from today

1. Derive the equation of falling velocity of the sediment particle in the still water. Use Stokes law.