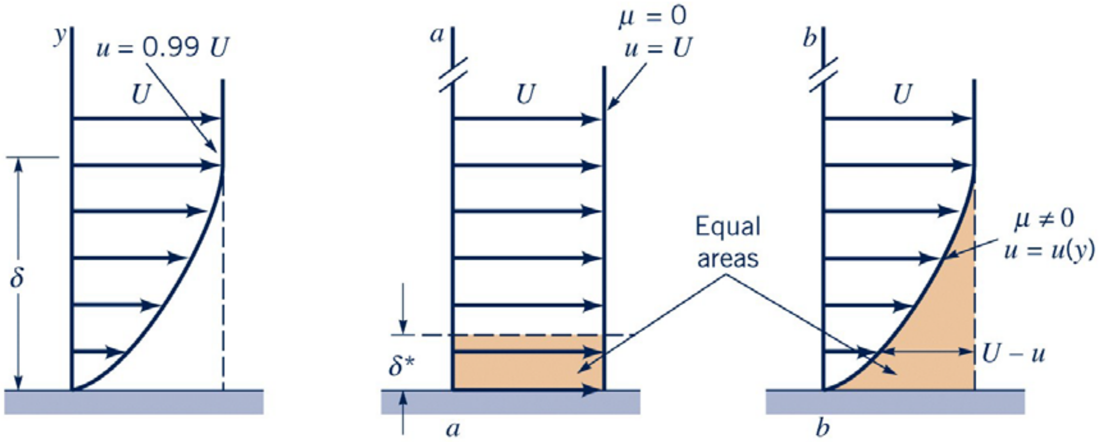
Chapter 9

Turbulent Boundary-Layer Flows



Chapter 9 Turbulent Boundary-Layer Flows

Contents

Objectives

- Study wall turbulence
- Derive equations of velocity distribution and friction coefficient for both smooth and rough walls

9.3 Mean-flow characteristics for turbulent boundary layer

- Relations describing the mean-flow characteristics in boundary layer

- It is useful to predict velocity magnitude and relation between velocity and wall shear or pressure gradient forces.

- It is desirable that these relations should not require knowledge of the turbulence details.

- **For laminar boundary layer**, it is possible to obtain a solution by integrating the equations of motion (Prandtl's 2D boundary layers equation, Eq. (10.7))

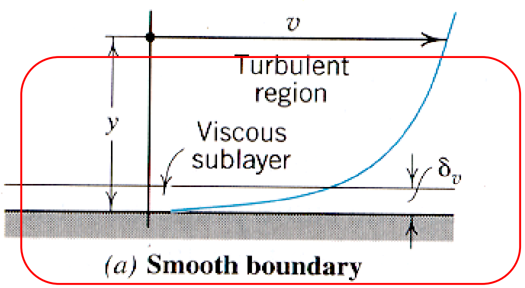
- However, **the turbulent boundary layer** is composed of zones of different types of flow, and effective viscosity varies from wall out through the layer.

- Thus, theoretical solution is not practical for the general non-uniform boundary layer → use semiempirical procedure

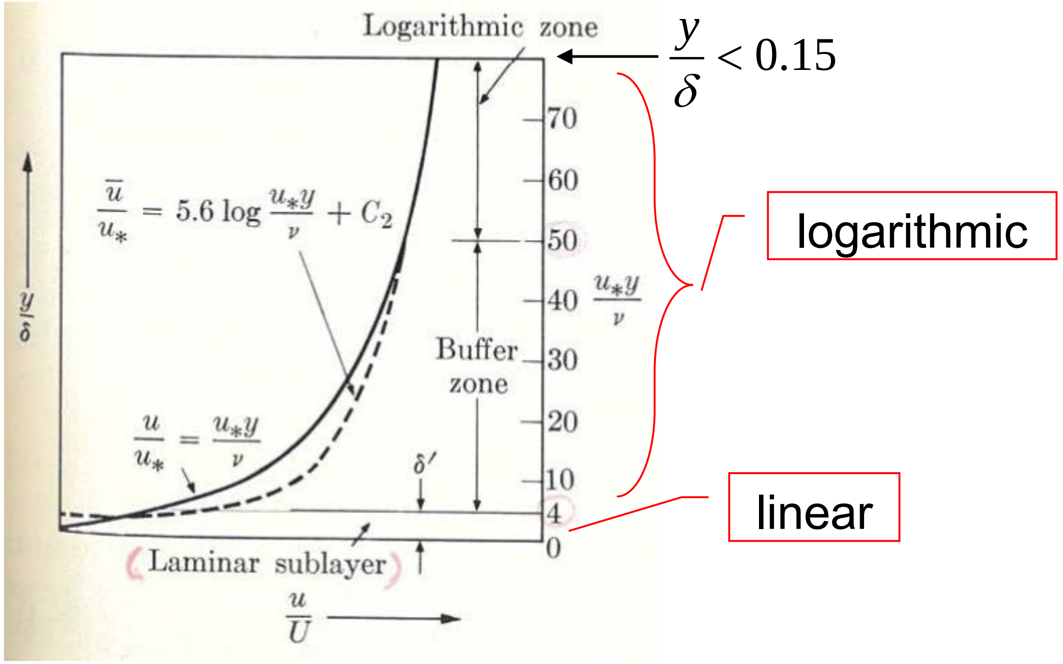
9.3 Mean-flow characteristics for turbulent boundary layer

9.3.1 Universal velocity and friction laws: smooth walls

I. Inner law (wall law)



(a) Smooth boundary



Velocity profile on smooth walls

9.3 Mean-flow characteristics for turbulent boundary layer

1) laminar sublayer: $0 < \frac{u_* y}{\nu} \le 4$

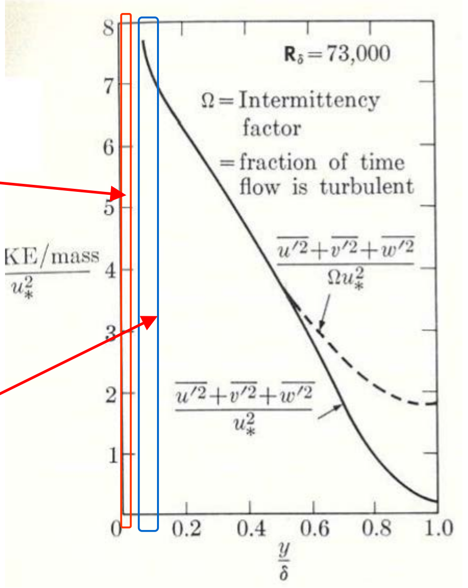
$\frac{d\bar{u}}{dy}$ ~ linear

$$\rightarrow \quad \frac{u}{u_*} = \frac{u_* y}{\nu} \qquad (9.6)$$

~ Mean shear stress is controlled by the dynamic molecular viscosity.

→ Reynolds stress is negligible. → Mean flow is laminar.

~ energy of velocity fluctuation ≈ 0



2) buffer zone: $4 < \frac{u_* y}{\nu} < 30 \sim 70$

~ Viscous and Reynolds stress are of the same order.

→ Both laminar flow and turbulence flow exist.

~ Sharp peak in the turbulent energy occurs (Fig. 9.4).

9.3 Mean-flow characteristics for turbulent boundary layer

3) turbulent zone - inner region: $\frac{u_* y}{\nu} > 30 \sim 70$, and $\frac{y}{\delta} < 0.15$

~ fully turbulent flow

~ inner law zone/wall law

~ Intensity of turbulence decreases.

~ velocity equation: logarithmic function

$$\boxed{\frac{\bar{u}}{u_*} = 5.6 \log \frac{u_* y}{\nu} + C_2}$$

- Outer law (velocity defect law)

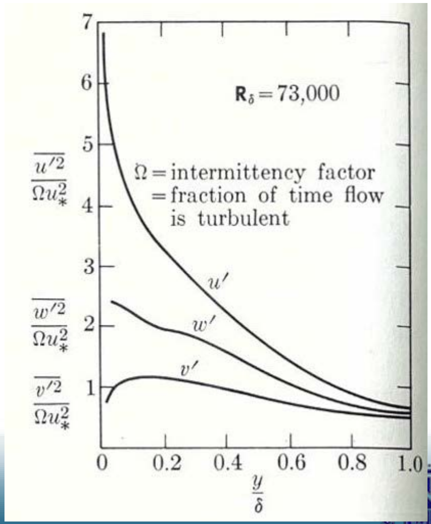
$$\frac{y}{\delta} > 0.15$$

$$\frac{U - \bar{u}}{u_*} = -8.6 \log\left(\frac{y}{\delta}\right)$$

$$\frac{\bar{u}}{u_*} = \frac{U}{u_*} + 8.6 \log\left(\frac{y}{\delta}\right)$$

4) turbulent zone-outer region: $0.15\delta < y < 0.4\delta$

~ outer law, velocity-defect law



5) intermittent zone: $0.4\delta < y < 1.2\delta$

~ Flow is intermittently turbulent and non-turbulent.

9.3 Mean-flow characteristics for turbulent boundary layer

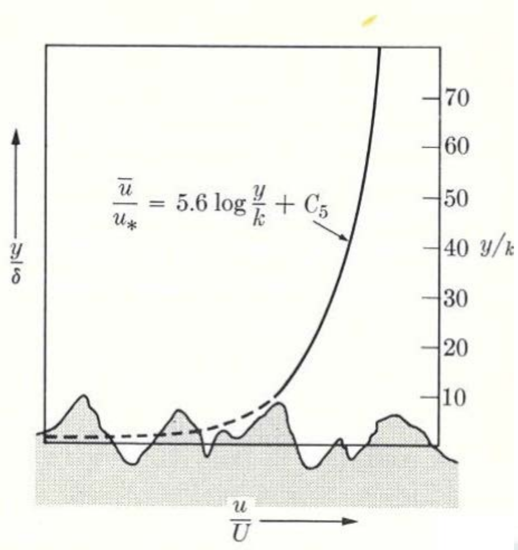
6) non-turbulent zone

~ external flow zone

~ potential flow

[Cf] Rough wall:

→ Laminar sublayer is destroyed by the roughness elements.



9.3 Mean-flow characteristics for turbulent boundary layer

- **Law of wall (Inner law)** $y < 0.15\delta$

- close to smooth boundaries (molecular viscosity dominant)
- Law of wall assumes that the relation between wall shear stress and velocity at distance y from the wall depends only on fluid density and viscosity
- Dimensional analysis yields

$$f(\overline{u}, u_*, y, \rho, \mu) = 0$$

$$\frac{\overline{u}}{u_*} = f\left(\frac{u_* y}{\nu}\right) \qquad (9.3)$$

9.3 Mean-flow characteristics for turbulent boundary layer

i) Laminar sublayer

- mean velocity, $\bar{u} \equiv u$

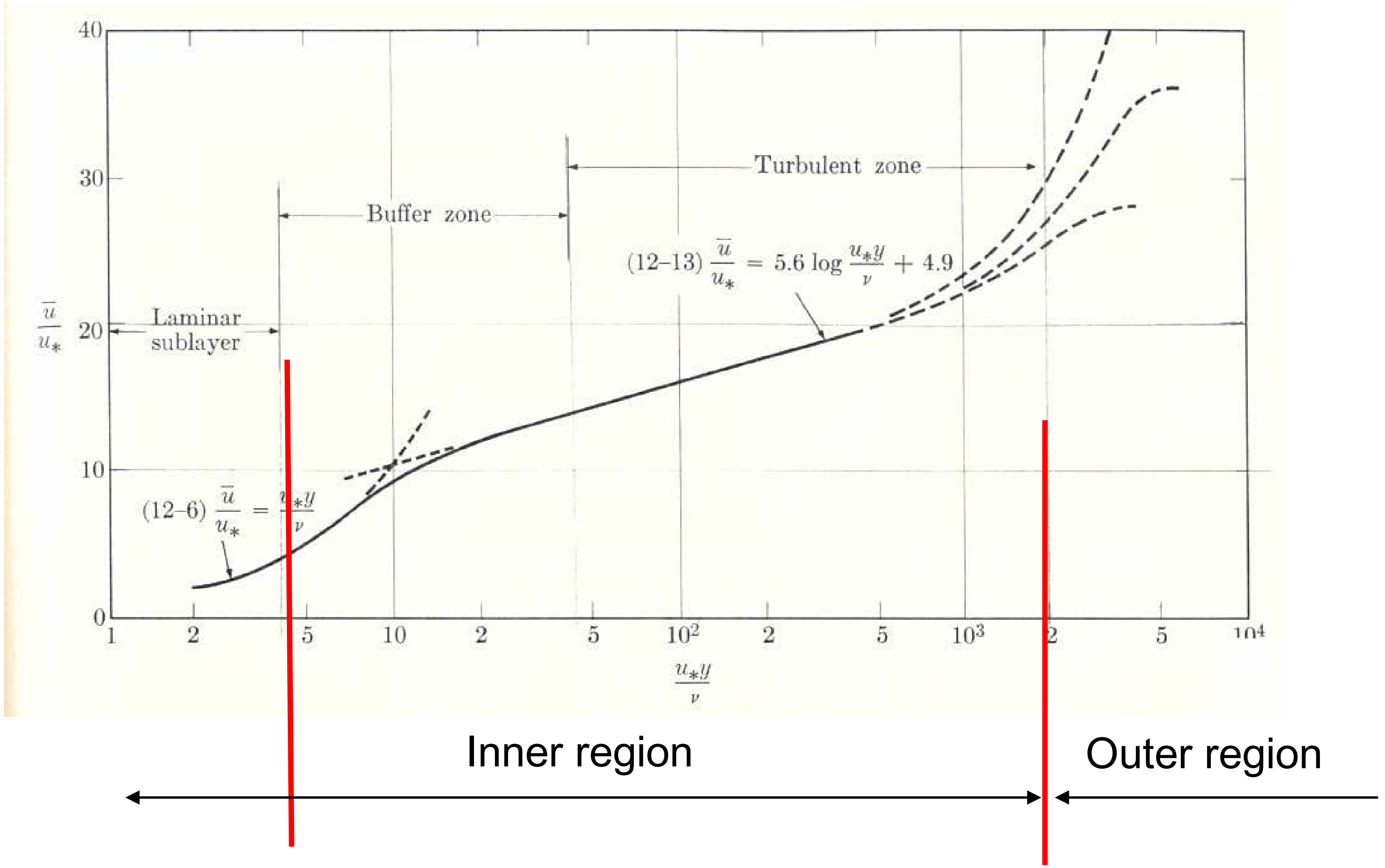
- velocity gradient, $\dfrac{\partial u}{\partial y} \sim \text{constant} \equiv \dfrac{u}{y}$

- shear stress, $\tau \approx \tau_0 = \mu \left.\dfrac{\partial u}{\partial y}\right|_{y=0} \equiv \mu \dfrac{u}{y}$ (9.5)

- shear velocity, $u_* = \sqrt{\dfrac{\tau}{\rho}} \rightarrow u_*^2 = \dfrac{\tau}{\rho} = \dfrac{\mu}{\rho}\dfrac{u}{y}$

$$\frac{u}{u_*} = \frac{u_* y}{\nu} \qquad (9.6)$$

9.3 Mean-flow characteristics for turbulent boundary layer



9.3 Mean-flow characteristics for turbulent boundary layer

- Thickness of laminar sublayer

- define thickness of laminar sublayer (δ') as the value of y which makes

$$\frac{u_* y}{\nu} = 4$$

$$\delta' = \frac{4\nu}{u_*} = \frac{4\nu}{\sqrt{\dfrac{\tau_0}{\rho}}} = \frac{4\nu}{\left(\dfrac{c_f \rho U^2/2}{\rho}\right)^{\frac{1}{2}}} = \frac{4\nu}{U\sqrt{c_f/2}} \qquad (9.7)$$

Where $\tau_0 = c_f \rho \dfrac{u^2}{2}$; $\quad c_f = local\ \ shear\ stress\ \ coeff.$

- c_f decreases slowly with increasing Reynolds number, $\mathrm{Re}_x = \dfrac{Ux}{\nu}$

Thus, δ' increases with distance along the surface.

Re] For the laminar sublayer in pipe flows

$$\delta' = 11.6 \frac{\nu}{u_*}$$

9.3 Mean-flow characteristics for turbulent boundary layer

ii) Turbulent region

- Start with equation of 2D turbulent boundary layer, Eq. (8.8a)

$$\rho\left(\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}\right)=-\frac{\partial \bar{p}}{\partial x}-\rho\frac{\partial \overline{u'^2}}{\partial x}+\mu\frac{\partial^2 \bar{u}}{\partial y^2}-\rho\frac{\partial \overline{u'v'}}{\partial y}$$

- In the turbulent zone of near wall region, the mean shear stress remains nearly equal to the wall shear; $\tau \approx \tau_0$

$$\rho\left(\frac{\partial \bar{u}}{\partial t}+\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}\right)+\frac{\partial \bar{p}}{\partial x}+\rho\frac{\partial \overline{u'^2}}{\partial x}=\frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{u}}{\partial y}-\rho\overline{u'v'}\right)=\frac{\partial \tau}{\partial y}\approx\frac{\partial \tau_0}{\partial y}=0$$

- Hence a solution is obtained if we have a relation for τ_0

- We can use Prandtl's mixing length theory for near wall, $l=\kappa y$

$$\tau_0 \approx \tau = \rho l^2\left|\frac{d\bar{u}}{dy}\right|\frac{d\bar{u}}{dy}=\rho\kappa^2 y^2\left(\frac{d\bar{u}}{dy}\right)^2 \qquad (1)$$

9.3 Mean-flow characteristics for turbulent boundary layer

Integrate (1)

$$\bar{u} = \frac{u_*}{\kappa}\ln y + C_1 \qquad u_* = \sqrt{\frac{\tau_0}{\rho}}$$

$$\boxed{\frac{\bar{u}}{u_*} = \frac{1}{\kappa}\ln y + C_1} \qquad (9.10)$$

Substitute BC [$\bar{u} = 0$ at $y = y'$] into (9.10)

$$0 = \frac{1}{\kappa}\ln y' + C_1 \qquad (5)$$

$$\therefore C_1 = -\frac{1}{\kappa}\ln y'$$

Assume $y' \propto \dfrac{\nu(m^2/s)}{u_*(m/s)} \rightarrow y' = C\dfrac{\nu}{u_*}$

Then (5) becomes

$$C_1 = -\frac{1}{\kappa}\ln y' = -\frac{1}{\kappa}\ln\left(C\frac{\nu}{u_*}\right) = C_2 - \frac{1}{\kappa}\ln\frac{\nu}{u_*} \qquad (9.11)$$

9.3 Mean-flow characteristics for turbulent boundary layer

Substitute (9.11) into (9.10)

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa}\ \ln y + C_2 - \frac{1}{\kappa}\ \ln\frac{\nu}{u_*} = \frac{1}{\kappa}\ \ln\left(\frac{u_* y}{\nu}\right) + C_2$$

Changing the logarithm to base 10 yields

$$\boxed{\frac{\bar{u}}{u_*} = \frac{2.3}{\kappa}\log_{10}\left(\frac{u_* y}{\nu}\right) + C_2} \qquad (9.12)$$

Empirical values of κ and C_2 for inner region of the boundary layer

$$\kappa = 0.41\,; \quad C_2 = 4.9$$

$$\boxed{\frac{\bar{u}}{u_*} = 5.6\ \log\left(\frac{u_* y}{\nu}\right) + 4.9,\ \ 30 \sim 70 < \frac{u_* y}{\nu},\ \ and\ \frac{y}{\delta} < 0.15} \qquad (9.13)$$

→ Prandtl's velocity distribution law; inner law; wall law

9.3 Mean-flow characteristics for turbulent boundary layer

[Re] Prandtl's turbulent boundary layer equation

x-eq.

$$\rho\left(\frac{\partial \overline{u}}{\partial t}+\overline{u}\frac{\partial \overline{u}}{\partial x}+\overline{v}\frac{\partial \overline{u}}{\partial y}\right)=-\frac{\partial \overline{p}}{\partial x}-\rho\frac{\partial \overline{u'^2}}{\partial x}+\mu\frac{\partial^2 \overline{u}}{\partial y^2}-\rho\frac{\partial \overline{u'v'}}{\partial y} \qquad (8.8a)$$

y-eq.

$$0=-\frac{\partial}{\partial y}(\overline{p}+\rho\overline{v'^2}) \qquad (8.8b)$$

Continuity eq.:

$$\frac{\partial \overline{u}}{\partial x}+\frac{\partial \overline{v}}{\partial y}=0 \qquad (8.8c)$$

Solution to these equations: $\overline{u}(x,y,t),\ \overline{v}(x,y,t),\ \overline{p}(x,y,t)$

$$\frac{\overline{u}}{u_*}=5.6\ \log\left(\frac{u_* y}{\nu}\right)+4.9 \quad \rightarrow \quad \overline{u}=\overline{u}(y)$$

9.3 Mean-flow characteristics for turbulent boundary layer

II. Outer law; Velocity-defect law

- In the outer reaches of the turbulent boundary layer for both smooth and rough walls, Reynolds stresses dominate the viscous stresses to produce the velocity profile.
- It was observed that the velocity defect (reduction) at *y*-values was dependent on the magnitude of the wall shear stress

$$\frac{U-\bar{u}}{u_*} = g\left(\frac{y}{\delta}\right) \qquad (9.14)$$

- A logarithmic relation for the function *g* can be obtained by assuming that Eq. (9.12) will give $\bar{u} = U \; at \;\; y = \delta$

$$\frac{U}{u_*} = \frac{2.3}{\kappa}\log\left(\frac{u_*\delta}{\nu}\right) + C_2{}' \qquad (2)$$

9.3 Mean-flow characteristics for turbulent boundary layer

Subtract (9.12) from (2)

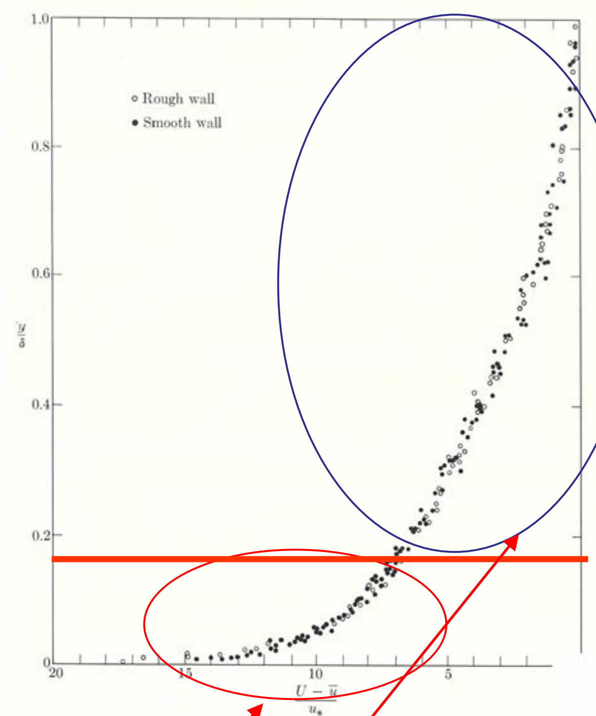
$$\frac{U}{u_*} = \frac{2.3}{\kappa}\log\left(\frac{u_*\delta}{\nu}\right) + C_2'$$

$$-\quad \frac{\bar{u}}{u_*} = \frac{2.3}{\kappa}\log\left(\frac{u_* y}{\nu}\right) + C_2$$

$$\frac{U-\bar{u}}{u_*} = \frac{2.3}{\kappa}\left\{\log\left(\frac{u_*\delta}{\nu}\right) - \log\left(\frac{u_* y}{\nu}\right)\right\} + C_2' - C_2$$

$$= \frac{2.3}{\kappa}\log\left(\frac{u_*\delta}{\nu}\frac{\nu}{u_* y}\right) + C_3 = -\frac{2.3}{\kappa}\log\left(\frac{y}{\delta}\right) + C_3$$

$$\boxed{\frac{U-\bar{u}}{u_*} = -\frac{2.3}{\kappa}\log\left(\frac{y}{\delta}\right) + C_3} \qquad (9.15)$$



- A single log-equation does not fit the data over the entire boundary layer.
- Instead <u>one equation will fit an inner region overlapping with Eq. (9.12),</u> while <u>second equation will approximate the outer region</u>.

9.3 Mean-flow characteristics for turbulent boundary layer

i) Inner region; $\frac{y}{\delta} \leq 0.15$

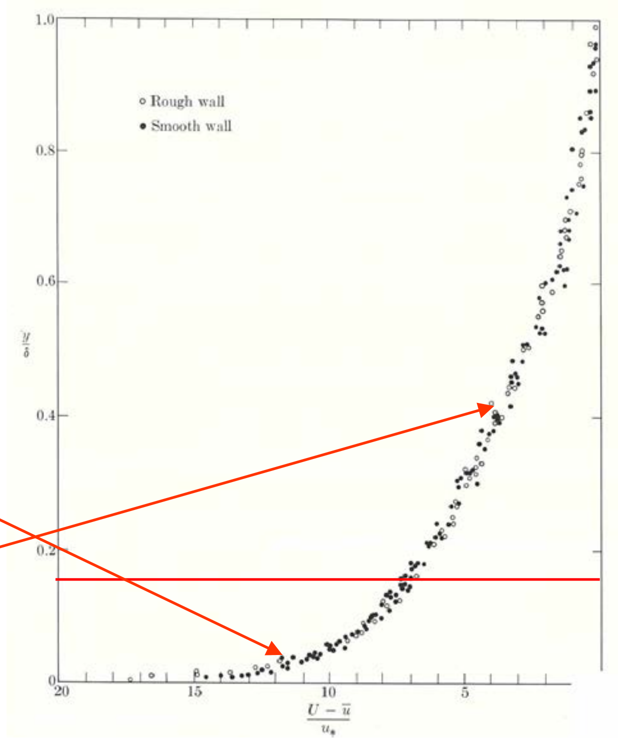
→ $\kappa = 0.41,\ C_3 = 2.5$

$$\boxed{\frac{U-\bar{u}}{u_*} = -5.6\log\left(\frac{y}{\delta}\right) + 2.5} \qquad (9.16)$$

ii) Outer region; $\frac{y}{\delta} > 0.15$

→ $\kappa = 0.267,\ C_3 = 0$

$$\boxed{\frac{U-\bar{u}}{u_*} = -8.6\log\left(\frac{y}{\delta}\right)} \qquad (9.17)$$

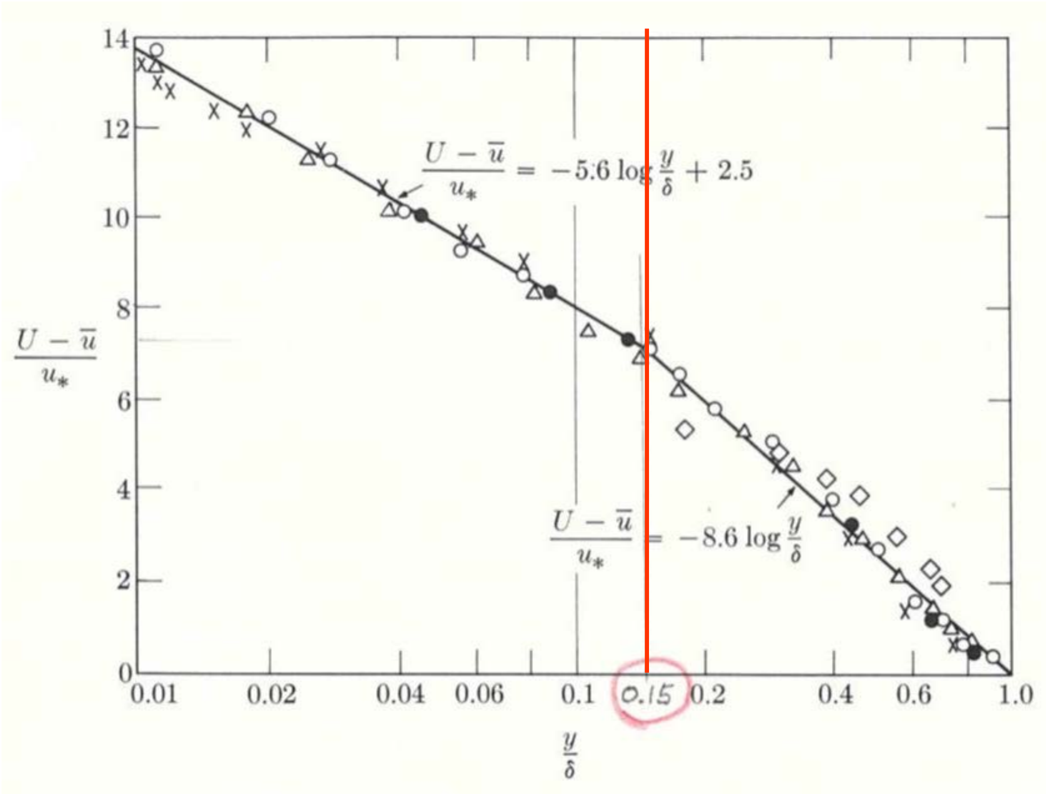


→ Eqs. (9.16) & (9.17) apply to both smooth and rough surfaces.

→ Eq. (9.16) = Eq. (9.13) $\quad \frac{\bar{u}}{u_*} = 5.6\ \log\left(\frac{u_* y}{\nu}\right) + 4.9,\ 30 \sim 70 < \frac{u_* y}{\nu},\ \ and\ \frac{y}{\delta} < 0.15$

9.3 Mean-flow characteristics for turbulent boundary layer

- The velocity-defect law is applicable for both smooth and rough walls



Velocity profile equations

	Wall law	**Velocity-defect law**	
	Smooth wall	Smooth wall	Rough wall
Outer region $\frac{y}{\delta} > 0.15$	–	$\frac{U-\bar{u}}{u_*} = -8.6\log\left(\frac{y}{\delta}\right)$	
Inner region $30 \sim 70 < \frac{u_* y}{\nu}$ $\frac{y}{\delta} < 0.15$	$\frac{\bar{u}}{u_*} = 5.6\ \log\left(\frac{u_* y}{\nu}\right) + 4.9$	$\frac{U-\bar{u}}{u_*} = -5.6\log\left(\frac{y}{\delta}\right) + 2.5$	
Laminar sublayer	$\frac{u}{u_*} = \frac{u_* y}{\nu}$	–	

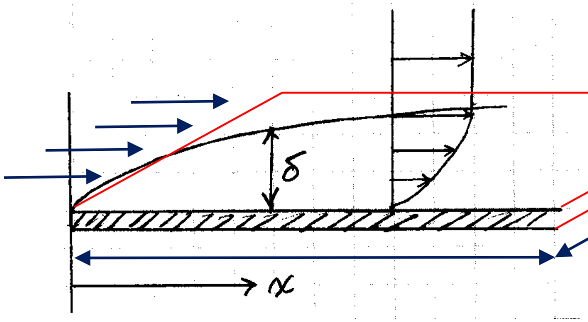
9.3 Mean-flow characteristics for turbulent boundary layer

III. Surface-resistance formulas

(1) Local shear-stress coefficient on smooth walls

velocity profile ↔ shear-stress equations

$$u_* = \sqrt{\tau / \rho} = \sqrt{\frac{c_f}{2}}U \rightarrow \frac{U}{u_*} = \sqrt{\frac{2}{c_f}}$$



where c_f = local shear stress coefficient

[Re] $\tau_0 = \frac{1}{2}\rho c_f U^2 \rightarrow \frac{\tau_0}{\rho} = \frac{c_f}{2}U^2$

$$D_f = \tau_0 A = \frac{1}{2}A\rho c_f U^2$$

$$D = D_f + D_p$$

$$u_* = \sqrt{\frac{\tau_0}{\rho}} = \sqrt{\frac{c_f}{2}}U \quad (9.18)$$

9.3 Mean-flow characteristics for turbulent boundary layer

(i) Assume that logarithmic law will give the relation

Substituting $\bar{u} = U \; at \;\; y = \delta$ into Eq. (9.12) yields

$$\frac{U}{u_*} = \frac{2.3}{\kappa} \log\left(\frac{u_* \delta}{\nu}\right) + C_4 \qquad \text{(A)}$$

Substitute (9.18) into (A)

$$\boxed{\sqrt{\frac{2}{c_f}} = \frac{2.3}{\kappa} \log\left(\frac{U\delta}{\nu}\sqrt{\frac{c_f}{2}}\right) + C_4 = \frac{2.3}{\kappa} \log\left(\mathrm{Re}_\delta \sqrt{\frac{c_f}{2}}\right) + C_4} \qquad (9.19)$$

- c_f is a function of Reynolds number for smooth walls.
- c_f is not given explicitly.

9.3 Mean-flow characteristics for turbulent boundary layer

(ii) For explicit expression, use displacement thickness and momentum thickness θ instead of δ

Clauser: $$\frac{1}{\sqrt{c_f}} = 3.96 \log \mathrm{Re}_{\delta^*} + 3.04 \qquad (9.20)$$

Squire and Young: $$\frac{1}{\sqrt{c_f}} = 4.17 \log \mathrm{Re}_{\theta} + 2.54 \qquad (9.21)$$

where $\mathrm{Re}_{\delta^*} = \dfrac{U\delta^*}{\nu}$; $\mathrm{Re}_{\theta} = \dfrac{U\theta}{\nu}$

$$\mathrm{Re}_{\delta^*}, \ \mathrm{Re}_{\theta} = f(\mathrm{Re}_x) \qquad (9.22)$$

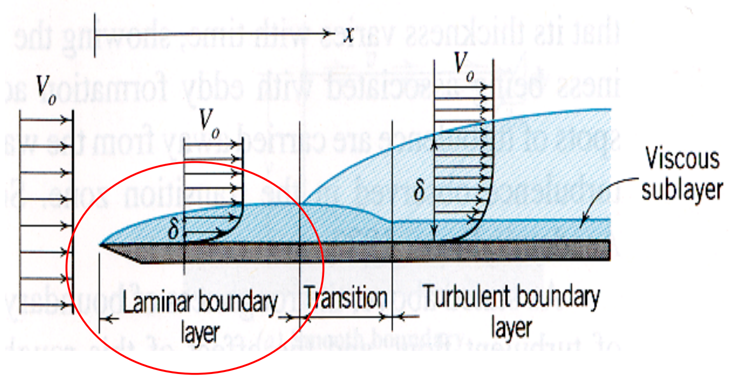
9.3 Mean-flow characteristics for turbulent boundary layer

iii) Karman's relation

~ assume turbulence boundary layer all the way from the leading edge (i.e., no preceding stretch of laminar boundary layer)

$$\frac{1}{\sqrt{c_f}} = 4.15\log(\mathrm{Re}_x\, c_f) + 1.7 \qquad (9.23)$$

- Karman's equation is useful for ships and aircraft wings where the laminar boundary layer is insignificant.



9.3 Mean-flow characteristics for turbulent boundary layer

iv) Explicit equation by Schultz-Grunow (1940)

$$\boxed{c_f = \frac{0.370}{(\log \mathrm{Re}_x)^{2.58}}} \qquad (9.24)$$

Comparison of (9.23) and (9.24)

$$c_f \approx 0.001 \sim 0.01$$

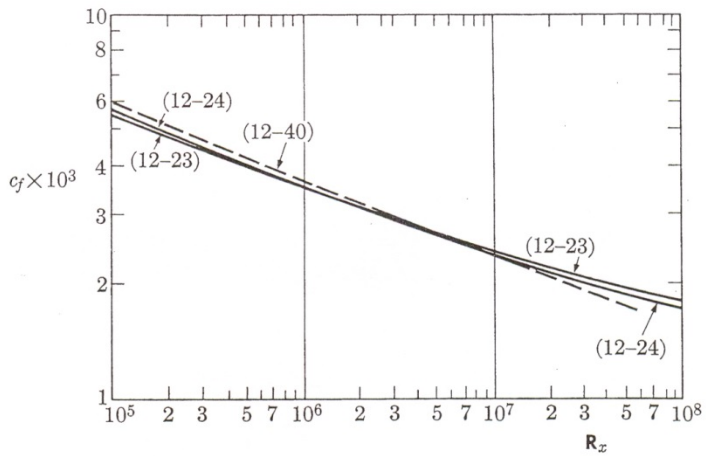


FIG. 12–9. Local coefficient of resistance.

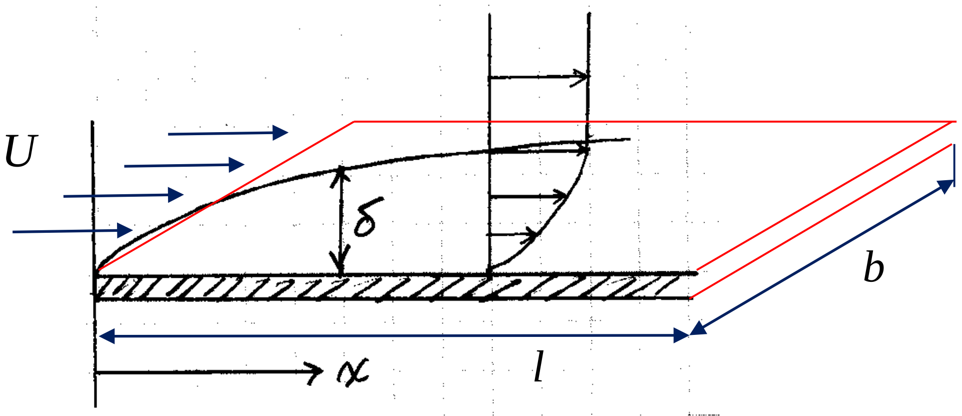
9.3 Mean-flow characteristics for turbulent boundary layer

(2) Average shear-stress coefficient on smooth walls

Consider average shear-stress coefficient over a distance l along a flat plate of a width b

$$Total\ \ frictional\, drag\,(D_f) = \tau \times bl = \frac{1}{2} C_f \rho U^2 bl$$

$$C_f \equiv \frac{D}{bl\rho U^2/2}$$



9.3 Mean-flow characteristics for turbulent boundary layer

i) Schoenherr (1932)

$$\frac{1}{\sqrt{C_f}} = 4.13\log(\mathrm{Re}_l C_f) \tag{9.26}$$

- assume turbulence boundary layer all the way from the leading edge
- Similar to von Karman's equation

where $\mathrm{Re}_l = \dfrac{Ul}{\nu}$

ii) Schultz-Grunow

$$\boxed{C_f = \frac{0.427}{(\log \mathrm{Re}_l - 0.407)^{2.64}}} \quad , \; 10^6 < \mathrm{Re}_l < 10^9 \tag{9.27}$$

9.3 Mean-flow characteristics for turbulent boundary layer

Comparison of (9.26) and (9.27)

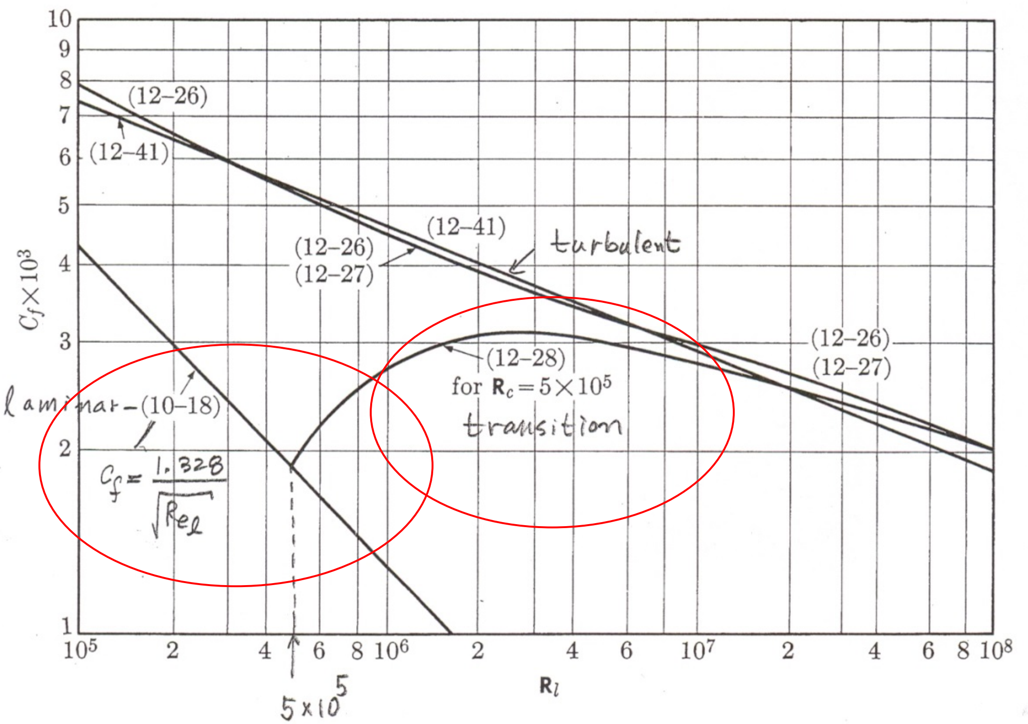
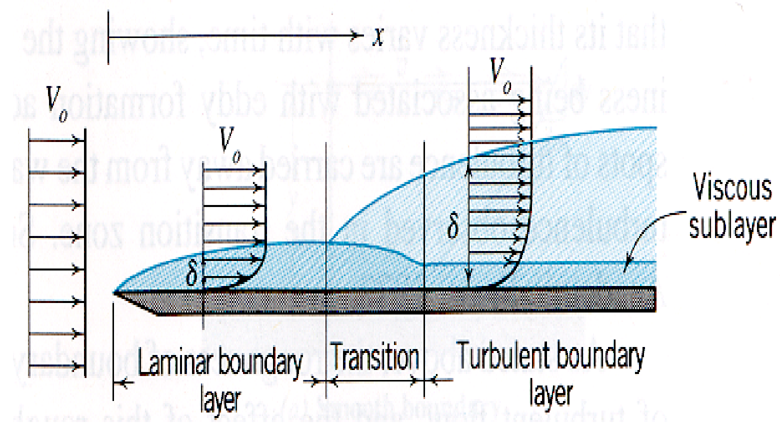


FIG. 12–10. Average coefficient of resistance for flat plates.



9.3 Mean-flow characteristics for turbulent boundary layer

3) Transition formula

■ Boundary layer developing on a smooth flat plate

~ At upstream end (leading edge), laminar boundary layer develops because viscous effects dominate due to inhibiting effect of the smooth wall on the development of turbulence.

~ Thus, when there is a significant stretch of laminar boundary layer preceding the turbulent layer, total friction is the laminar portion up to x_{crit} plus the turbulent portion from x_{crit} to l.

~ Therefore, average shear-stress coefficient is lower than the prediction by Eqs. (9.26) or (9.27).

→ Use transition formula

$$\boxed{C_f = \frac{0.427}{(\log \mathrm{Re}_l - 0.407)^{2.64}} - \frac{A}{\mathrm{Re}_l}} \qquad (9.28)$$

9.3 Mean-flow characteristics for turbulent boundary layer

where A / Re_l = correction term = $f(\mathrm{Re}_{crit})$, $\mathrm{Re}_{crit} = \dfrac{Ux_{crit}}{\nu}$

→ A = 1,060~3,340 (Table 9.2, p. 240)

- Since A is a function of Re_{crit}, the curve in Fig. 10 is given for $\mathrm{Re}_{crit} = 5 \times 10^5$
- For flow along smooth walls

$$300,000 < \mathrm{Re}_{crit} < 600,000$$

- For laminar flow:

$$C_f = \frac{1.328}{\mathrm{Re}_l^{1/2}}$$

9.3 Mean-flow characteristics for turbulent boundary layer

Surface Resistance Formulas for Boundary Layers with $d\bar{p}/dx = 0$

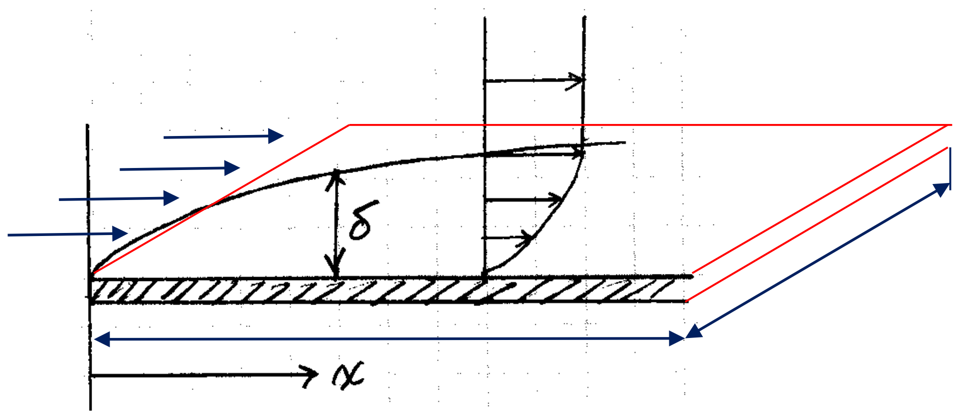
	Smooth walls	Rough walls
Local Shear		
Universal equations		
Clauser (12–20)	$1/\sqrt{c_f} = 3.96 \log \mathbf{R}_{\delta^*} + 3.04$	(12–46) $\frac{1}{\sqrt{c_f}} = 3.96 \log \frac{\delta}{k} + C_8$; $C_8 = f$ (size, shape, and distribution of roughness)
Squire and Young (12–21)	$1/\sqrt{c_f} = 4.17 \log \mathbf{R}_\theta + 2.54$	
von Kármán (12–23)	$1/\sqrt{c_f} = 4.15 \log (\mathbf{R}_x c_f) + 1.7$	
Schultz-Grunow (12–24)	$c_f = \frac{0.370}{(\log \mathbf{R}_x)^{2.58}}$	
Power law (12–40)	$c_f = \frac{0.0466}{\mathbf{R}_\delta^{1/4}} = \frac{0.059}{\mathbf{R}_x^{1/5}}$	
Average Shear		
Universal equations		
Schoenherr (12–26)	$1/\sqrt{C_f} = 4.13 \log (\mathbf{R}_l C_f)$	
Schultz-Grunow (12–27)	$C_f = \frac{0.427}{(\log \mathbf{R}_l - 0.407)^{2.64}}$	
Power law (12–41)	$C_f = \frac{0.074}{\mathbf{R}_l^{1/5}}$	
Transition formula		
Schultz-Grunow-Prandtl (12–28)	$C_f = \frac{0.427}{(\log \mathbf{R}_l - 0.407)^{2.64}} - \frac{A}{\mathbf{R}_l}$; $A = f(\mathbf{R}_{\text{crit}})$ as given in Table 12–2	

9.3 Mean-flow characteristics for turbulent boundary layer

[Ex. 9.1] Turbulent boundary-layer velocity and thickness

An aircraft flies at 25,000 ft with a speed of 410 mph (600 ft/s).

Compute the following items for the boundary layer at a distance 10 ft from the leading edge of the wing of the craft.



9.3 Mean-flow characteristics for turbulent boundary layer

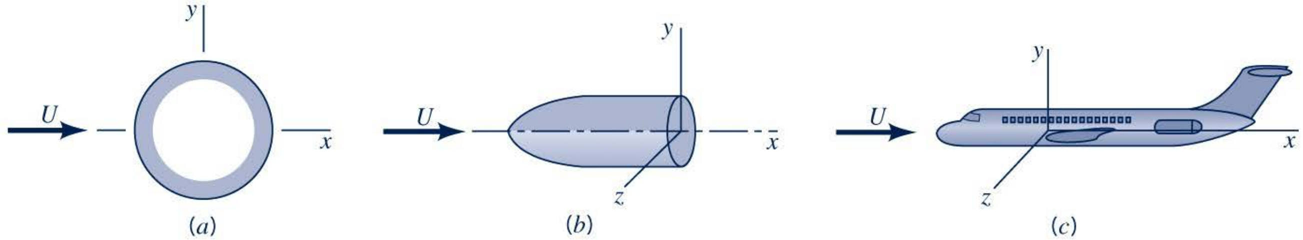
① Frictional drag = surface resistance = skin drag

② Pressure drag = form drag

~ depends largely on shape or form of the body

For airfoil, hydrofoil, and slim ships: surface drag > form drag

For bluff objects like spheres, bridge piers: surface drag < form drag

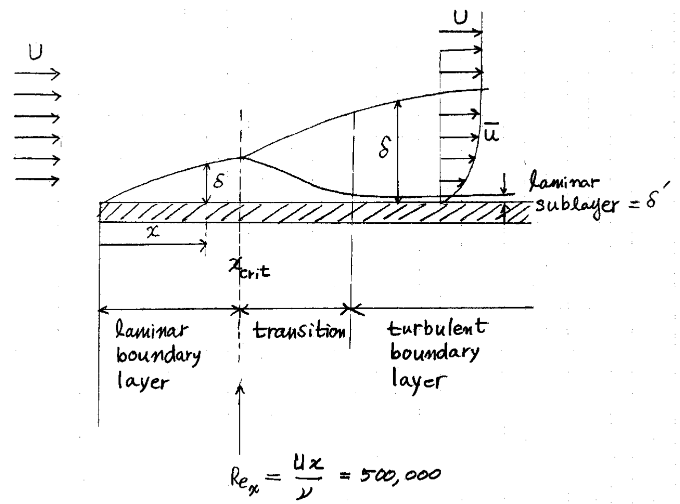


9.3 Mean-flow characteristics for turbulent boundary layer

(a) Thickness δ' (laminar sublayer; $\delta' = \dfrac{4\nu}{u_*}$) at $x = 10ft$

Air at El. 25,000 ft: $\nu = 3\times10^{-4}\ \ ft^2/s$

$$\rho = 1.07\times10^{-3}\ slug/ft^3$$



Find $\quad \mathrm{Re}_{crit} = 5\times10^5 = \dfrac{600(x)}{3\times10^{-4}}$

$$\therefore x_{crit} = 0.25\,ft \sim \text{negligible comparaed to } l = 10\,ft$$

Therefore, assume that <u>turbulent boundary layer</u> develops <u>all the way from the leading edge</u>.

$$\mathrm{Re}_x = \frac{Ux}{\nu} = \frac{600(10)}{(3\times10^{-4})} = 2\times10^7$$

9.3 Mean-flow characteristics for turbulent boundary layer

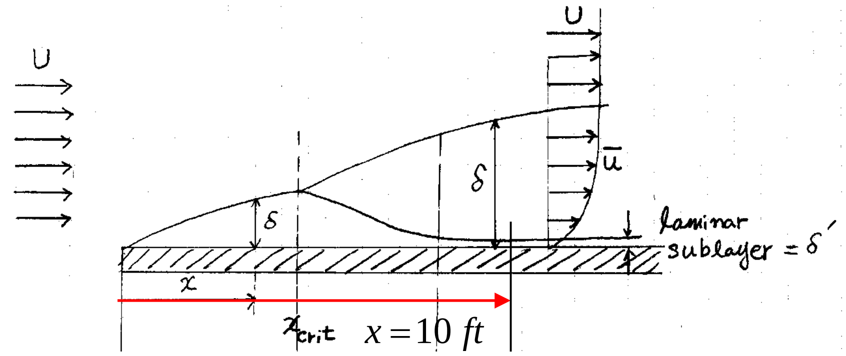
Use Schultz-Grunow eq., (9.24) to compute c_f

$$c_f = \frac{0.370}{(\log \mathrm{Re}_x)^{2.58}} = \frac{0.370}{\left\{\log\left(2\times10^7\right)\right\}^{2.58}} = \frac{0.370}{(7.30)^{2.58}} = \boxed{0.0022}$$

$$\tau_0 = \frac{\rho}{2} c_f U^2 = \frac{1}{2}(1.07\times10^{-3})(0.0022)(600)^2 = \boxed{0.422\, lb / ft^2}$$

$$u_* = \sqrt{\frac{\tau_0}{\rho}} = \sqrt{\frac{0.422}{1.07\times10^{-3}}} = \boxed{19.8\ ft / s}$$

$$\delta' = \frac{4\nu}{u_*} = \frac{4(3\times10^{-4})}{19.8} = 0.61\times10^{-4}\ ft = \boxed{7.3\times10^{-4}\ in}$$



9.3 Mean-flow characteristics for turbulent boundary layer

(b) Velocity $\bar{u}$ at $y = \delta'$

Eq. (9.6): $\dfrac{u}{u_*} = \dfrac{u_* y}{\nu}$

$$\therefore u = \frac{u_*^2 \delta'}{\nu} = \frac{(19.8)^2 (0.61\times10^{-4})}{(3\times10^{-4})} = 79.7\ ft/s \rightarrow \underline{13\%\ \text{ of }\ U}$$

$\uparrow$

$y = \delta'$

$[Cf]\ U = 600\ ft/s$

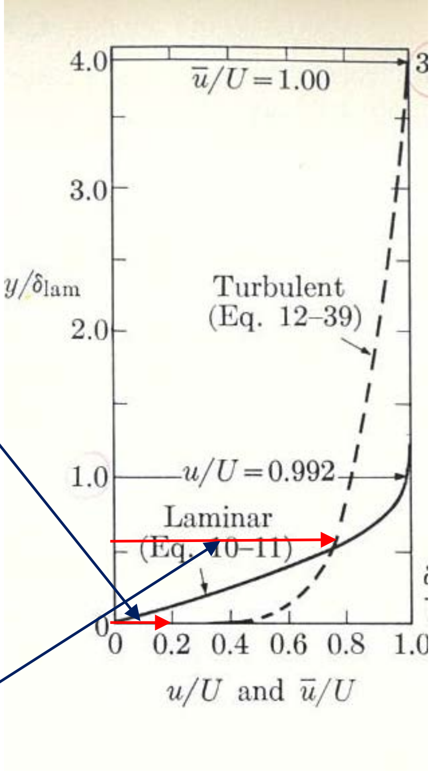
(c) Velocity $\bar{u}$ at $y/\delta = 0.15$

Use Eq. (9.16) – outer law

$$\frac{U - \bar{u}}{u_*} = -5.6 \log\left(\frac{y}{\delta}\right) + 2.5$$

$$\frac{600 - \bar{u}}{19.8} = -5.6\ \underline{\log(0.15)} + 2.5$$

$$\bar{u} = 600 - 91.35 - 49.5 = 459.2\ ft/s \rightarrow \underline{76\%\ \text{ of }\ U}$$



9.3 Mean-flow characteristics for turbulent boundary layer

[Cf] $\bar{u} = U + 5.6\, u_* \log\left(\dfrac{y}{\delta}\right) - 2.5 u_*$

(d) Distance y at $y/\delta = 0.15$ and thickness δ

Use Eq. (9.13) – inner law

$$\frac{\bar{u}}{u_*} = 5.6 \log\left(\frac{u_* y}{\nu}\right) + 4.9$$

$$At\,\frac{y}{\delta} = 0.15: \frac{459}{19.8} = 5.6 \log\left(\frac{19.8y}{3\times10^{-4}}\right) + 4.9$$

$$\log\left(\frac{19.8y}{3\times10^{-4}}\right) = 3.26; \quad \frac{19.8y}{3\times10^{-4}} = 1839$$

$$y = 0.028' = 0.33in \approx 0.8cm$$

(B)

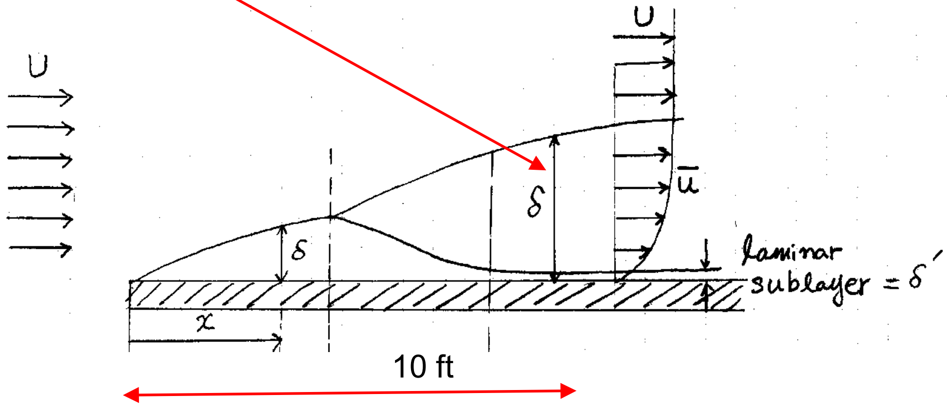
9.3 Mean-flow characteristics for turbulent boundary layer

Substitute (B) into

$$\delta = \frac{y}{0.15} = \boxed{0.186' = 2.24in} \approx 5.7cm$$

$$\text{At } x = 10': \ \frac{\delta}{\delta'} = \frac{0.186}{0.61 \times 10^{-4}} = 3049 \approx 3 \times 10^{3}$$

$$\boxed{\frac{\delta'}{\delta} = 0.0003}$$



9.3 Mean-flow characteristics for turbulent boundary layer

[Ex. 9.2] Surface resistance on a smooth boundary given as Ex. 9.1

(a) Displacement thickness δ^*

$$\delta^* = \int_0^h \left(1 - \frac{u}{U}\right) dy \qquad (7.9)$$

$$\frac{\delta^*}{\delta} = \int_0^{h/\delta} \left(1 - \frac{\bar{u}}{U}\right) d\left(\frac{y}{\delta}\right), \qquad h/\delta \geq 1 \qquad \text{(A)}$$

Neglect laminar sublayer and approximate buffer zone with Eq. (9.16)

$$(i) \quad y/\delta < 0.15, \quad \frac{U - \bar{u}}{u_*} = -5.6 \log\left(\frac{y}{\delta}\right) + 2.5 \leftarrow (9.16)$$

Divide (9.16) by U

$$\boxed{\ln x = 2.3 \log x}$$

$$\therefore 1 - \frac{\bar{u}}{U} = -5.6 \frac{u_*}{U} \log \frac{y}{\delta} + 2.5 \frac{u_*}{U} = \left(-2.43 \ln \frac{y}{\delta} + 2.5\right) \frac{u_*}{U} \qquad \text{(B)}$$

9.3 Mean-flow characteristics for turbulent boundary layer

$$(ii) \quad y/\delta > 0.15, \qquad \frac{U-\bar{u}}{u_*} = -8.6\log\left(\frac{y}{\delta}\right) \leftarrow (9.17)$$

$$\therefore 1-\frac{\bar{u}}{U} = -8.6\frac{u_*}{U}\log\left(\frac{y}{\delta}\right) = -3.74\frac{u_*}{U}\ln\left(\frac{y}{\delta}\right) \qquad \text{(C)}$$

Substituting (B) and (C) into (A) yields

$$\therefore \frac{\delta^*}{\delta} = \int_{\delta'/\delta}^{0.15}\left(-2.43\ln\frac{y}{\delta}+2.5\right)\frac{u_*}{U}d\left(\frac{y}{\delta}\right) + \int_{0.15}^{1.0}\left(-3.74\ln\frac{y}{\delta}\right)\frac{u_*}{U}d\left(\frac{y}{\delta}\right)$$

$$= \frac{u_*}{U}\left[\left[-2.43\left\{\frac{y}{\delta}\ln\frac{y}{\delta}-\frac{y}{\delta}\right\}+2.5\frac{y}{\delta}\right]_{0.0003}^{0.15} + \left[-3.74\left\{\frac{y}{\delta}\ln\frac{y}{\delta}-\frac{y}{\delta}\right\}\right]_{0.15}^{1}\right]$$

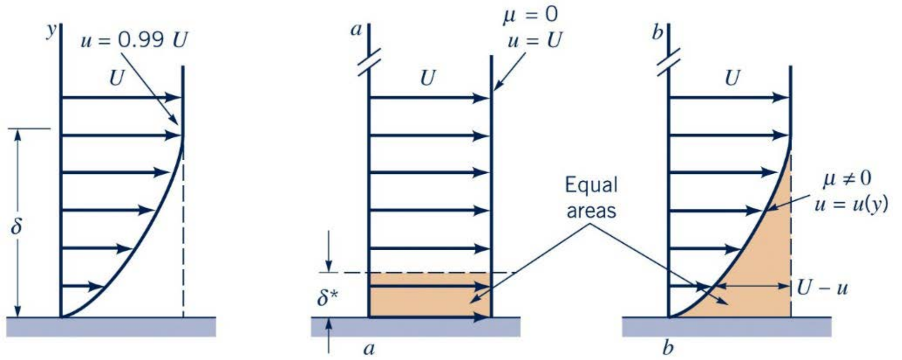
$$\cong 3.74\frac{u_*}{U} = 3.74\frac{(19.8)}{600} = 0.1184$$

$$\boxed{\int \ln x \, dx = x\ln x - x}$$

9.3 Mean-flow characteristics for turbulent boundary layer

$$\delta^* = 0.1184\delta = 0.1184\,(0.186) = 0.022\,ft$$

$$\frac{\delta^*}{\delta} = 0.1184 \quad \rightarrow \underline{11.8\%}$$



(b) Local surface-resistance coeff. c_f

Use Eq. (9.20) by Clauser

$$\frac{1}{\sqrt{c_f}} = 3.96 \log \mathrm{Re}_{\delta^*} + 3.04 \quad \leftarrow \quad \mathrm{Re}_{\delta^*} = \frac{600 \times 0.022}{3 \times 10^{-4}} = 44{,}000$$

$$= 3.96 \log(44{,}000) + 3.04$$

$$\therefore\ c_f = 2.18 \times 10^{-3} = \underline{0.00218}$$

[Cf] c_f = 0.0022 by Schultz-Grunow Eq.