Lecture 24

Turbulent Boundary-Layer Flows (2)
Lecture 24 Turbulent Boundary-Layer Flows (2)

Contents

24.1 Velocity Profiles
24.2 Surface Resistance Formulas

Objectives

- Derive equations of velocity distribution and friction coefficient for both smooth and rough walls
24.1 Velocity Profiles

- Relations describing the mean-flow characteristics in boundary layer
  - It is useful to predict velocity magnitude and relation between velocity and wall shear or pressure gradient forces.
  - It is desirable that these relations should not require knowledge of the turbulence details.

- For laminar boundary layer, it is possible to obtain a solution by integrating the equations of motion (Prandtl’s 2D boundary layers equation, Eq. (17.7))

- However, the turbulent boundary layer is composed of zones of different types of flow, and effective viscosity varies from wall out through the layer.

- Thus, theoretical solution is not practical for the general non-uniform boundary layer → use semiempirical procedure
24.1 Velocity Profiles

24.1.1 Velocity profile regions

- No single equation will describe the velocity profile over its entire thickness $\delta$.

- Close to the smooth boundary, a law of wall applies.

- For outer reaches of boundary layers for both smooth and rough walls, a velocity-defect law applies.
24.1 Velocity Profiles

- Inner layer might be divided into three regions
  ① Viscous sublayer
  ② Buffer zone
  ③ Turbulent zone

Regions of inner law (wall law)
24.1 Velocity Profiles

1) Laminar sublayer: \[ 0 < \frac{u_* y}{v} \leq 4 \]
   - Velocity profile \( \frac{d\bar{u}}{dy} \) is very nearly linear.
   - Mean shear stress is controlled by the dynamic \textit{molecular viscosity}.
   - Reynolds stress is negligible, and mean flow is laminar.
   - Energy of velocity fluctuation is nearly zero.

2) Buffer zone: \[ 4 < \frac{u_* y}{v} < 30 \sim 70 \]
   - Viscous and Reynolds stress are of the same order.
   - Both laminar flow and turbulent flow exist.
   - Sharp peak in the turbulent energy occurs.
3) Turbulent zone - **inner region**: $\frac{u_* y}{v} > 30 \sim 70, \text{and } \frac{y}{\delta} < 0.15$

- Flow is fully turbulent.
- Intensity of turbulence decreases from its peak value.
- Velocity profile is logarithmic.
24.1 Velocity Profiles

- **Regions of velocity-defect law**

- Outer layer might be divided into two regions.
  1. **Turbulent zone-outer region**: $0.15\delta < y < 0.4\delta$
  2. **Intermittent zone**: $0.4\delta < y < 1.2\delta$

- In the intermittent zone, flow is intermittently turbulent and non-turbulent.
24.1 Velocity Profiles

- All zones merge smoothly into a continuous mean velocity profile.
- On rough wall, laminar sublayer is destroyed by the roughness elements.
24.1 Velocity Profiles

24.1.2 Law of wall (Inner law)

- Close to smooth boundaries, molecular viscosity is dominant.

- Law of wall assumes that the relation between wall shear stress and velocity at distance $y$ from the wall depends only on fluid density and viscosity.

- Dimensional analysis yields

$$f(\bar{u}, u_*, y, \rho, \mu) = 0$$

$$\frac{\bar{u}}{u_*} = f\left(\frac{u_* y}{\nu}\right)$$

(24.1)
24.1 Velocity Profiles

- A unique law of wall, Eq. (24.1), shows $\frac{\bar{u}}{u_*}$ as a single line up to $\frac{u_*y}{\nu} = 2,000$.
- At higher values of $\frac{u_*y}{\nu}$, a single velocity law no longer holds because viscosity is no longer a major factor.
24.1 Velocity Profiles

1) Laminar sublayer

- The mean velocity is

$$\bar{u} \equiv u$$

- The shear stress may be assumed constant and equal to $\tau_0$

$$\tau \approx \tau_0 = \mu \frac{\partial u}{\partial y} \bigg|_{y=0} \equiv \frac{\mu}{\nu} u$$ (24.2)

- Then Eq. 24.2 becomes

$$\frac{u}{u_*} = \frac{u_* y}{v}$$ (24.3)


24.1 Velocity Profiles

- Thickness of laminar sublayer

- Define thickness of laminar sublayer ($\delta'$) as the value of $y$ which makes

$$\frac{u_* y}{v} = \frac{u_* \delta'}{v} = 4$$

$$\delta' = \frac{4v}{u_*} = \frac{4v}{\sqrt{\frac{\tau_0}{\rho}}} = \frac{4v}{\left(\frac{c_f \rho U^2}{2 \rho}\right)^{1/2}} = \frac{4v}{U \sqrt{c_f/2}}$$

(24.4)

where $\tau_0 = c_f \rho \frac{u^2}{2}$; $c_f$ = local shear stress coeff.

- $c_f$ decreases slowly with increasing Reynolds number, $Re_x = \frac{Ux}{v}$

Thus, $\delta'$ increases with distance along the surface.

For the laminar sublayer in pipe flows

$$\delta' = 11.6 \frac{v}{u_*}$$
24.1 Velocity Profiles

2) Turbulent region

- Start with equation of 2D turbulent boundary layer, Eq. (21.15a)

\[
\rho \left( \frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} \right) = - \frac{\partial p}{\partial x} - \rho \frac{\partial \bar{u}^2}{\partial x} + \mu \frac{\partial^2 \bar{u}}{\partial y^2} - \rho \frac{\partial \bar{u}'v'}{\partial y}
\]

- In the turbulent zone of near wall region, the mean shear stress remains nearly equal to the wall shear; \( \tau \approx \tau_0 \)

\[
\rho \left( \frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} \right) + \frac{\partial p}{\partial x} + \rho \frac{\partial \bar{u}^2}{\partial x} = \frac{\partial}{\partial y} \left( \mu \frac{\partial \bar{u}}{\partial y} - \rho \bar{u}'v' \right) = \frac{\partial \tau}{\partial y} \approx \frac{\partial \tau_0}{\partial y} = 0
\]

- Hence a solution is obtained if we have a relation for \( \tau_0 \)

- We can use Prandtl’s mixing length theory for near wall, \( l = \kappa y \)

\[
\tau_0 \approx \tau = \rho l^2 \left| \frac{d\bar{u}}{dy} \right| \left( \frac{d\bar{u}}{dy} \right) = \rho \kappa^2 y^2 \left( \frac{d\bar{u}}{dy} \right)^2
\]  

(1)
24.1 Velocity Profiles

Integrate (1)

\[ \bar{u} = \frac{u_*}{\kappa} \ln y + C_1 \]

\[ u_* = \sqrt{\frac{\tau_0}{\rho}} \]

\[ \frac{\bar{u}}{u_*} = \frac{1}{\kappa} \ln y + C_1 \]

(24.5)

Substitute BC \( \bar{u} = 0 \) at \( y = y' \) into (24.5)

\[ 0 = \frac{1}{\kappa} \ln y' + C_1 \]

(24.6)

\[ : C_1 = -\frac{1}{\kappa} \ln y' \]

Assume \( y' \propto \frac{v(m^2/s)}{u_*(m/s)} \rightarrow y' = C \frac{v}{u_*} \)

Then (24.6) becomes

\[ C_1 = -\frac{1}{\kappa} \ln y' = -\frac{1}{\kappa} \ln \left( C \frac{v}{u_*} \right) = C_2 - \frac{1}{\kappa} \ln \frac{v}{u_*} \]

(24.7)
24.1 Velocity Profiles

Substitute (24.7) into (24.5)

$$\frac{\bar{u}}{u_*} = \frac{1}{\kappa} \ln y + C_2 - \frac{1}{\kappa} \ln \frac{v}{u_*} = \frac{1}{\kappa} \ln \left( \frac{u_* y}{v} \right) + C_2$$

Changing the logarithm to base 10 yields

$$\frac{\bar{u}}{u_*} = \frac{2.3}{\kappa} \log_{10} \left( \frac{u_* y}{v} \right) + C_2$$

(24.8)

Empirical values of $\kappa$ and $C_2$ for inner region of the boundary layer, $\frac{y}{\delta} < 0.15$

$$\kappa = 0.41; \quad C_2 = 4.9$$

$$\frac{\bar{u}}{u_*} = 5.6 \log \left( \frac{u_* y}{v} \right) + 4.9, \quad 30 \sim 70 < \frac{u_* y}{v}, \text{ and } \frac{y}{\delta} < 0.15$$

(24.9)

→ Prandtl's velocity distribution law; inner law; wall law
24.1 Velocity Profiles

[Re] Prandtl’s turbulent boundary layer equation

\( x \)-eq. \[ \rho \left( \frac{\partial \bar{u}}{\partial t} + \bar{u} \frac{\partial \bar{u}}{\partial x} + \bar{v} \frac{\partial \bar{u}}{\partial y} \right) = -\frac{\partial \bar{p}}{\partial x} - \rho \frac{\partial \bar{u}_{\text{v}}^2}{\partial x} + \mu \frac{\partial^2 \bar{u}}{\partial y^2} - \rho \frac{\partial \bar{u} v'}{\partial y} \] (24.10a)

\( y \)-eq. \[ 0 = -\frac{\partial}{\partial y} (\bar{p} + \rho \bar{v}_{\text{v}}^2) \] (24.10b)

Continuity eq.: \[ \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \] (24.10c)

Solution to these equations: \( \bar{u}(x, y, t), \bar{v}(x, y, t), \bar{p}(x, y, t) \)

\[ \frac{\bar{u}}{u_*} = 5.6 \log \left( \frac{u_* y}{v} \right) + 4.9 \quad \rightarrow \quad \bar{u} = \bar{u}(y) \]
24.1 Velocity Profiles

24.1.3 Velocity-defect law

- In the outer reaches of the turbulent boundary layer for both smooth and rough walls, Reynolds stresses dominate the viscous stresses to produce the velocity profile.

- It was observed that the velocity defect (reduction) at $y$-values was dependent on the magnitude of the wall shear stress

$$\frac{U - \bar{u}}{u_*} = g \left( \frac{y}{\delta} \right) \quad (24.11)$$

- A logarithmic relation for the function $g$ can be obtained by assuming that Eq. (24.8) will give $\bar{u} = U$ at $y = \delta$.

$$\frac{U}{u_*} = \frac{2.3}{\kappa} \log \left( \frac{u_*\delta}{\nu} \right) + C_2 \quad (24.12)$$
24.1 Velocity Profiles

Subtract (24.8) from (24.12)

\[
\frac{U}{u_*} = \frac{2.3}{\kappa} \log \left( \frac{u_* \delta}{\nu} \right) + C_2',
\]

\[
- \quad \frac{u}{u_*} = \frac{2.3}{\kappa} \log \left( \frac{u_* y}{\nu} \right) + C_2
\]

\[
\frac{U - \bar{u}}{u_*} = \frac{2.3}{\kappa} \left\{ \log \left( \frac{u_* \delta}{\nu} \right) - \log \left( \frac{u_* y}{\nu} \right) \right\} + C_2' - C_2
\]

\[
= \frac{2.3}{\kappa} \log \left( \frac{u_* \delta}{\nu \ u_* y} \right) + C_3 = - \frac{2.3}{\kappa} \log \left( \frac{y}{\delta} \right) + C_3
\]

\[
\frac{U - \bar{u}}{u_*} = - \frac{2.3}{\kappa} \log \left( \frac{y}{\delta} \right) + C_3
\] (24.13)

- A single log-equation does not fit the data over the entire boundary layer.
- Instead one equation will fit an inner region overlapping with Eq. (24.8), while second equation will approximate the outer region.
24.1 Velocity Profiles

i) Inner region; \( \frac{y}{\delta} \leq 0.15 \)

\[ \kappa = 0.41, \quad C_3 = 2.5 \]

\[
\frac{U - \bar{u}}{u_*} = -5.6 \log \left( \frac{y}{\delta} \right) + 2.5
\]  \hspace{1cm} (24.14)

ii) Outer region; \( \frac{y}{\delta} > 0.15 \)

\[ \kappa = 0.267, \quad C_3 = 0 \]

\[
\frac{U - \bar{u}}{u_*} = -8.6 \log \left( \frac{y}{\delta} \right)
\]  \hspace{1cm} (24.15)

\[ \rightarrow \text{Eqs. (24.14) & (24.15) apply to both smooth and rough surfaces.} \]

\[ \rightarrow \text{Eq. (24.14) = Eq. (24.9)} \]

\[
\frac{\bar{u}}{u_*} = 5.6 \log \left( \frac{u_* y}{v} \right) + 4.9
\]

[Re] Eq. (24.12) - Eq. (24.14) = Eq. (24.9)
### Velocity Profile equations

<table>
<thead>
<tr>
<th></th>
<th>Wall law</th>
<th>Velocity-defect law</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Smooth wall</td>
<td>Smooth wall</td>
</tr>
</tbody>
</table>

#### Outer region

- $\frac{y}{\delta} > 0.15$

- $u = u_* y$

#### Inner region

- $30 \sim 70 < \frac{u_* y}{v}$

- $\frac{\bar{u}}{u_*} = 5.6 \log \left( \frac{u_* y}{v} \right) + 4.9$

- $\frac{U - \bar{u}}{u_*} = -8.6 \log \left( \frac{y}{\delta} \right)$

- $\frac{y}{\delta} < 0.15$

- $u = \frac{u_* y}{v}$

- $\frac{U - \bar{u}}{u_*} = -5.6 \log \left( \frac{y}{\delta} \right) + 2.5$

#### Laminar sublayer

- $\frac{u}{u_*} = \frac{u_* y}{v}$

- $\frac{U - \bar{u}}{u_*} = -5.6 \log \left( \frac{y}{\delta} \right)$

#### Overlapping region

- $30 \sim 70 \leq \frac{u_* y}{v}$
24.2 Surface Resistance Formulas

24.2.1 Local shear stress coefficient

- Local shear-stress coefficient on smooth walls

velocity profile ↔ shear-stress equations

\[ u_* = \sqrt{\frac{\tau}{\rho}} = \sqrt{\frac{c_f}{2}} U \rightarrow \frac{U}{u_*} = \sqrt{\frac{2}{c_f}} \]

where \( c_f = \) local shear stress coefficient

[Re] Skin drag force

\[ D_f = \tau_0 A = \frac{1}{2} A \rho c_f U^2 \]

\[ \tau_0 = \frac{1}{2} \rho c_f U^2 \rightarrow \frac{\tau_0}{\rho} = \frac{c_f}{2} U^2 \]

\[ u_* = \sqrt{\frac{\tau_0}{\rho}} = \sqrt{\frac{c_f}{2}} U \quad (24.16) \]
(i) Assume that logarithmic law will give the relation

Substituting $\bar{u} = U$ at $y = \delta$ into Eq. (24.8) yields

$$U \overline{u_\ast} = \frac{2.3}{\kappa} \log \left( \frac{u_\ast \delta}{v} \right) + C_4$$

(A)

Substitute (24.16) into (A)

$$\sqrt{\frac{2}{c_f}} = \frac{2.3}{\kappa} \log \left( \frac{U \delta}{v} \sqrt{\frac{c_f}{2}} \right) + C_4 = \frac{2.3}{\kappa} \log \left( \text{Re}_\delta \sqrt{\frac{c_f}{2}} \right) + C_4$$

(24.17)

- $c_f$ is a function of Reynolds number $\text{Re}_\delta$ for smooth walls.
- $c_f$ is not given explicitly.
(ii) For explicit expression, use displacement thickness and momentum thickness $\theta$ instead of $\delta$

Clauser: $\frac{1}{\sqrt{c_f}} = 3.96 \log \text{Re}_{\delta^*} + 3.04$  \hspace{1cm} (24.18)

Squire and Young: $\frac{1}{\sqrt{c_f}} = 4.17 \log \text{Re}_\theta + 2.54$  \hspace{1cm} (24.19)

where $\text{Re}_{\delta^*} = \frac{U \delta^*}{v}$ ; $\text{Re}_\theta = \frac{U \theta}{v}$

$\text{Re}_{\delta^*}$, $\text{Re}_\theta = f(\text{Re}_x)$  \hspace{1cm} (24.20)
iii) Karman's relation

- Assume turbulence boundary layer all the way from the leading edge (i.e., no preceding stretch of laminar boundary layer)

\[ \frac{1}{\sqrt{c_f}} = 4.15 \log(Re_x c_f) + 1.7 \]  

(24.21)

- Karman's equation is useful for ships and aircraft wings where the laminar boundary layer is insignificant.
iv) Explicit equation by Schultz-Grunow (1940)

\[ c_f = \frac{0.370}{(\log \text{Re}_x)^{2.58}} \]  

(24.22)

Comparison of (24.21) and (24.22)

\[ c_f \approx 0.001 \sim 0.01 \]
24.2 Surface Resistance Formulas

24.2.2 Average shear stress coefficient

- Average shear-stress coefficient on smooth walls

Consider average shear-stress coefficient over a distance \( l \) along a flat plate of a width \( b \)

\[
\text{Total frictional drag } (D_f) = \tau \times bl = \frac{1}{2} C_f \rho U^2 b l
\]

\[
C_f = \frac{D}{bl \rho U^2 / 2}
\]
24.2 Surface Resistance Formulas

i) Schoenherr (1932)

\[
\frac{1}{\sqrt{C_f}} = 4.13 \log(Re_l C_f)
\]  

(24.23)

- Assume turbulence boundary layer all the way from the leading edge
- Similar to von Karman’s equation

where \( Re_l = \frac{Ul}{\nu} \)

ii) Schultz-Grunow

\[
C_f = \frac{0.427}{(\log Re_l - 0.407)^{2.64}}, \quad 10^6 < Re_l < 10^9
\]  

(24.24)
24.2 Surface Resistance Formulas

Comparison of (24.23) and (24.24)
3) Transition formula

- Boundary layer developing on a smooth flat plate
- At upstream end (leading edge), laminar boundary layer develops because viscous effects dominate due to inhibiting effect of the smooth wall on the development of turbulence.
- Thus, when there is a significant stretch of laminar boundary layer preceding the turbulent layer, total friction is the laminar portion up to $x_{crit}$ plus the turbulent portion from $x_{crit}$ to $l$.
- Therefore, average shear-stress coefficient is lower than the prediction by Eqs. (24.23) or (24.24).

→ Use transition formula

$$C_f = \frac{0.427}{(\log Re_l - 0.407)^{2.64}} - \frac{A}{Re_l}$$  \hspace{1cm} (24.25)
24.2 Surface Resistance Formulas

where \( A / Re_l = \text{correction term} = f(Re_{crit}), Re_{crit} = \frac{Ux_{crit}}{\nu} \)

- Since \( A \) is a function of \( Re_{crit} \), the curve in Fig. 10 is given for \( Re_{crit} = 5 \times 10^5 \)
- For flow along smooth walls

\[ 300,000 < Re_{crit} < 600,000 \]

<table>
<thead>
<tr>
<th>( Re_{crit} )</th>
<th>3 ( \times 10^5 )</th>
<th>4 ( \times 10^5 )</th>
<th>5 ( \times 10^5 )</th>
<th>6 ( \times 10^5 )</th>
<th>10^6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A )</td>
<td>1,060</td>
<td>1,400</td>
<td>1,740</td>
<td>2,080</td>
<td>3,340</td>
</tr>
</tbody>
</table>

- For laminar flow:

\[
C_f = \frac{1.328}{Re_l^{1/2}}
\]

(17.26)
### 24.2 Surface Resistance Formulas

<table>
<thead>
<tr>
<th>Local Shear</th>
<th>Smooth walls</th>
<th>Rough walls</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Universal equations</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Clauser (12–20)</td>
<td>$1/ \sqrt{c_f} = 3.96 \log R_s^* + 3.04$</td>
<td>$(12–46) \quad \frac{1}{\sqrt{c_f}} = 3.96 \log \frac{\delta}{k} + C_8$</td>
</tr>
<tr>
<td>Squire and Young (12–21)</td>
<td>$1/ \sqrt{c_f} = 4.17 \log R_s + 2.54$</td>
<td>$C_8 = f \text{ (size, shape, and distribution of roughness)}$</td>
</tr>
<tr>
<td>von Kármán (12–23)</td>
<td>$1/ \sqrt{c_f} = 4.15 \log (R_s c_f) + 1.7$</td>
<td></td>
</tr>
<tr>
<td>Schultz-Grunow (12–24)</td>
<td>$c_f = \frac{0.370}{(\log R_s)^{2.58}}$</td>
<td></td>
</tr>
<tr>
<td><strong>Power law</strong> (12–40)</td>
<td>$c_f = \frac{0.0466}{R_s^{1/4}} = \frac{0.059}{R_s^{1/5}}$</td>
<td></td>
</tr>
</tbody>
</table>

| Average Shear | |
| **Universal equations** | |
| Schoenherr (12–26) | $1/ \sqrt{C_f} = 4.13 \log (R_t C_f)$ |
| Schultz-Grunow (12–27) | $C_f = \frac{0.427}{(\log R_t - 0.407)^{2.64}}$ |
| **Power law** (12–41) | $C_f = \frac{0.074}{R_t^{1/5}}$ |
| **Transition formula** | |
| Schultz-Grunow-Prandtl (12–28) | $C_f = \frac{0.427}{(\log R_t - 0.407)^{2.64}} - \frac{A}{R_t}$ |

$A = f(R_{crit})$ as given in Table 12–2
[Ex. 24.1] Turbulent boundary-layer velocity and thickness

An aircraft flies at 25,000 ft with a speed of 410 mph (600 ft/s). Compute the following items for the boundary layer at a distance 10 ft from the leading edge of the wing of the craft.
24.2 Surface Resistance Formulas

[Re] Drag force

① Frictional drag = surface resistance = skin drag

② Pressure drag = form drag

~ depends largely on shape or form of the body

For airfoil, hydrofoil, and slim ships: surface drag > form drag

For bluff objects like spheres, bridge piers: surface drag < form drag
(a) Thickness $\delta'$ (laminar sublayer; $\delta' = \frac{4v}{u_*}$) at $x = 10$ ft

Air at El. 25,000 ft: $\nu = 3 \times 10^{-4}$ ft$^2$/s

$\rho = 1.07 \times 10^{-3}$ slug/ft$^3$

Find $Re_{crit} = 5 \times 10^5 = \frac{600(x)}{3 \times 10^{-4}}$

$\therefore x_{crit} = 0.25$ ft ~ negligible compared to $l = 10$ ft

Therefore, assume that turbulent boundary layer develops all the way from the leading edge.

$Re_x = \frac{Ux}{\nu} = \frac{600(10)}{(3 \times 10^{-4})} = 2 \times 10^7$
24.2 Surface Resistance Formulas

Use Schultz-Grunow eq., (24.22) to compute $c_f$

$$c_f = \frac{0.370}{(\log \text{Re}_x)^{2.58}} = \frac{0.370}{\left\{ \log \left(2 \times 10^7\right) \right\}^{2.58}} = \frac{0.370}{(7.30)^{2.58}} = 0.0022$$

$$\tau_0 = \frac{\rho}{2} c_f U^2 = \frac{1}{2} (1.07 \times 10^{-3})(0.0022)(600)^2 = 0.422 \text{ lb/ft}^2$$

$$u_* = \sqrt{\frac{\tau_0}{\rho}} = \sqrt{\frac{0.422}{1.07 \times 10^{-3}}} = 19.8 \text{ ft/s}$$

$$\delta' = \frac{4\nu}{u_*} = \frac{4(3 \times 10^{-4})}{19.8} = 0.61 \times 10^{-4} \text{ ft} = 7.3 \times 10^{-4} \text{ in}$$
24.2 Surface Resistance Formulas

(b) Velocity $\bar{u}$ at $y = \delta'$

Eq. (24.1): $\frac{u}{u_*} = \frac{u_* y}{v}$

$u = \frac{u_*^2 \delta'}{v} = \frac{(19.8)^2 (0.61 \times 10^{-4})}{(3 \times 10^{-4})} = 79.7 \text{ ft/s} \rightarrow 13\% \text{ of } U$

$U = 600 \text{ ft/s}$

(c) Velocity $\bar{u}$ at $y / \delta = 0.15$

Use Eq. (24.14) - outer law

$$\frac{U - \bar{u}}{u_*} = -5.6 \log \left( \frac{y}{\delta} \right) + 2.5$$

$$\frac{600 - \bar{u}}{19.8} = -5.6 \log (0.15) + 2.5$$

$$\bar{u} = 600 - 91.35 - 49.5 = 459.2 \text{ ft/s} \rightarrow 76\% \text{ of } U$$
24.2 Surface Resistance Formulas

[Cf] \[ \bar{u} = U + 5.6 \ u_\ast \ \log \left( \frac{y}{\delta} \right) - 2.5u_\ast \]

(d) Distance \( y \) at \( y / \delta = 0.15 \) and thickness \( \delta \)
Use Eq. (24.9) - inner law

\[ \frac{\bar{u}}{u_\ast} = 5.6 \ \log \left( \frac{u_\ast y}{v} \right) + 4.9 \]

At \( \frac{y}{\delta} = 0.15 \):
\[ \frac{459}{19.8} = 5.6 \ \log \left( \frac{19.8y}{3 \times 10^{-4}} \right) + 4.9 \]

\[ \log \left( \frac{19.8y}{3 \times 10^{-4}} \right) = 3.26; \quad \frac{19.8y}{3 \times 10^{-4}} = 1839 \]

\[ y = 0.028' = 0.33in \approx 0.8cm \]

(B)
24.2 Surface Resistance Formulas

Substitute (B) into

\[ \delta = \frac{y}{0.15} = 0.186' = 2.24\text{in} \approx 5.7\text{cm} \]

At \( x = 10' \):

\[ \frac{\delta'}{\delta} = \frac{0.186}{0.61 \times 10^{-4}} = 3049 \approx 3 \times 10^3 \]

\[ \frac{\delta'}{\delta} = 0.0003 \]
24.2 Surface Resistance Formulas

[Ex. 24.2] Surface resistance on a smooth boundary given as Ex. 24.1

(a) Find displacement thickness $\delta^*$

$$
\delta^* = \int_0^h \left(1 - \frac{u}{U}\right) dy
$$

(17.4)

$$
\frac{\delta^*}{\delta} = \int_0^{h/\delta} \left(1 - \frac{\bar{u}}{U}\right) d\left(\frac{y}{\delta}\right), \quad h/\delta \geq 1
$$

(A)

Neglect laminar sublayer and approximate buffer zone with Eq. (24.14)

(i) $y/\delta < 0.15, \quad \frac{U - \bar{u}}{u_*} = -5.6 \log\left(\frac{y}{\delta}\right) + 2.5 \quad \leftarrow (24.14)$

Divide (24.14) by $U$

$$
1 - \frac{\bar{u}}{U} = -5.6 \frac{u_*}{U} \log\frac{y}{\delta} + 2.5 \frac{u_*}{U} = \left(-2.43 \ln\frac{y}{\delta} + 2.5\right) \frac{u_*}{U}
$$

(B)
(ii)  \( y / \delta > 0.15, \quad \frac{U - \bar{u}}{u_*} = -8.6 \log \left( \frac{y}{\delta} \right) \) \( \leftarrow \) (24.15)

\[ \therefore 1 - \frac{\bar{u}}{U} = -8.6 \frac{u_*}{U} \log \left( \frac{y}{\delta} \right) = -3.74 \frac{u_*}{U} \ln \left( \frac{y}{\delta} \right) \] (C)

Substituting (B) and (C) into (A) yields

\[ \therefore \frac{\delta^*}{\delta} = \int_{\delta / \delta}^{0.15} \left( -2.43 \ln \frac{y}{\delta} + 2.5 \right) \frac{u_*}{U} d \left( \frac{y}{\delta} \right) + \int_{0.15}^{1.0} \left( -3.74 \ln \frac{y}{\delta} \right) \frac{u_*}{U} d \left( \frac{y}{\delta} \right) \]

\[ = \frac{u_*}{U} \left[ -2.43 \left\{ \frac{y}{\delta} \ln \frac{y}{\delta} - \frac{y}{\delta} \right\} + 2.5 \frac{y}{\delta} \right]^{0.15}_{0.003} + \left[ -3.74 \left\{ \frac{y}{\delta} \ln \frac{y}{\delta} - \frac{y}{\delta} \right\} \right]^{1}_{0.15} \]

\[ \cong 3.74 \frac{u_*}{U} = 3.74 \frac{19.8}{600} = 0.1184 \]

\[ \int \ln x \, dx = x \ln x - x \]
24.2 Surface Resistance Formulas

\[ \delta^* = 0.1184 \delta = 0.1184 (0.186) = 0.022 \text{ ft} \]

\[ \frac{\delta^*}{\delta} = 0.1184 \rightarrow 11.8\% \]

(b) Local surface-resistance coeff. \( c_f \)

Use Eq. (24.18) by Clauser

\[ \frac{1}{\sqrt{c_f}} = 3.96 \log \text{Re}_{\delta^*} + 3.04 \quad \leftarrow \quad \text{Re}_{\delta^*} = \frac{600 \times 0.022}{3 \times 10^{-4}} = 44,000 \]

\[ = 3.96 \log (44,000) + 3.04 \]

\[ \therefore \quad c_f = 2.18 \times 10^{-3} = 0.00218 \]

[\text{Cf}] \quad c_f = 0.0022 \text{ by Schultz-Grunow Eq.}