differential equations of heat transfer

$$\sum \mathbf{F} = \iint_{\text{c.s.}} \mathbf{v}\rho\left(\mathbf{v}\cdot\mathbf{n}\right)dA + \frac{\partial}{\partial t}\iiint_{\text{c.v.}} \rho\mathbf{v}dV$$

$$\boxed{\rho\frac{\mathrm{D}\mathbf{v}}{\mathrm{Dt}} = -\nabla \mathrm{P} + \mu\nabla^2\mathbf{v} + \rho\mathbf{g}}$$

$$\frac{\delta \mathrm{Q}}{\mathrm{dt}} - \frac{\delta \mathrm{W_s}}{\mathrm{dt}} - \frac{\delta \mathrm{W}_\mu}{\mathrm{dt}} = \iint_{\text{c.s.}}\left(\mathrm{e} + \frac{\mathrm{P}}{\rho}\right)\rho\left(\mathbf{v}\cdot\mathbf{n}\right)\mathrm{dA} + \frac{\partial}{\partial \mathrm{t}}\iiint_{\text{c.v.}} \mathrm{e}\rho\mathrm{dV}$$

$$\boxed{\rho c_p\frac{DT}{Dt} = -\nabla\cdot\mathbf{q} + \dot{q} + \boldsymbol{\tau} : \nabla\mathbf{v}}$$

$$\boxed{\rho c_p \frac{DT}{Dt} = -\nabla \cdot \mathbf{q} + \dot{q} + \boldsymbol{\tau} : \nabla \mathbf{v}}$$

$$\rho c_p \left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z}\right) = k\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right] + \dot{q} + \boldsymbol{\tau} : \nabla \mathbf{v}$$

$$\boldsymbol{\tau} : \nabla \mathbf{v} = \Phi = 2\mu\left[\left(\frac{\partial v_x}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial y}\right)^2 + \left(\frac{\partial v_z}{\partial z}\right)^2\right]$$

$$+\mu\left[\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right)^2 + \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right)^2 + \left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right)^2\right]$$

$$\frac{\partial T}{\partial t} = \alpha\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right]$$

$$\frac{\partial T}{\partial t} = \alpha\left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2}\right]$$

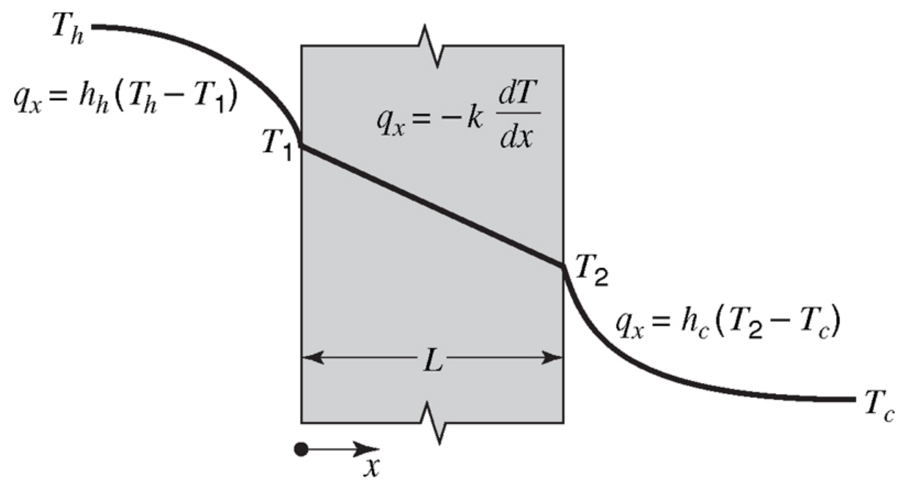
$$\frac{\partial T}{\partial t} = \alpha\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 T}{\partial \phi^2}\right]$$

boundary conditions

1) isothermal BC

2) insulated BC

3) combined BC



$$h_h\left(T_h - T\big|_{x=0}\right) = -k\frac{\partial T}{\partial x}\bigg|_{x=0}$$

$$h_c\left(T\big|_{x=L} - T_c\right) = -k\frac{\partial T}{\partial x}\bigg|_{x=L}$$

1-D steady state conduction

$$\nabla^2 T = 0 \qquad \frac{d}{dx}\left(x^i \frac{dT}{dx} \right) = 0$$

plane wall $\quad T = C_1 x + C_2 \qquad$ at $x = 0 \quad T = T_1 \qquad$ at $x = L \quad T = T_2$

$$T = T_1 - \frac{T_1 - T_2}{L} x \qquad q_x = -kA\frac{dT}{dx} = \frac{kA}{L}\left(T_1 - T_2\right)$$

hollow cylinder

$$\frac{d}{dr}\left(r \frac{dT}{dt} \right) = 0$$

at $r = r_i \quad T = T_i \qquad$ at $r = r_o \quad T = T_o$

$$T\left(r\right) = T_i - \frac{T_i - T_o}{\ln\left(r_o / r_i\right)} \ln\frac{r}{r_i} \qquad \frac{q_r}{L} = \frac{2\pi k}{\ln\left(r_o / r_i\right)}\left(T_i - T_o\right)$$

hollow sphere

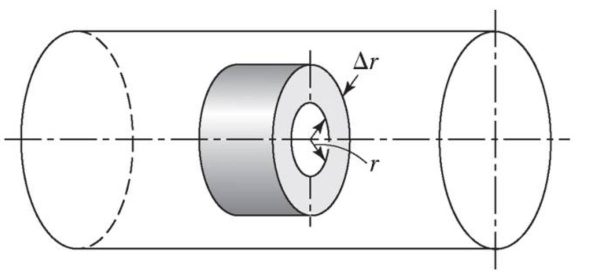
$$\frac{d}{dr}\left(r^2 \frac{dT}{dt} \right) = 0 \qquad T = T_i - \left(\frac{T_i - T_o}{1/r_i - 1/r_o} \right)\left(\frac{1}{r_i} - \frac{1}{r} \right) \qquad q = \frac{4\pi k\left(T_i - T_o\right)}{\dfrac{1}{r_i} - \dfrac{1}{r_0}}$$

1-D conduction in a circular cylinder with internal generation of energy

$$\left\{\begin{matrix}\text{rate of energy}\\ \text{conduction into}\\ \text{the element}\end{matrix}\right\}+\left\{\begin{matrix}\text{rate of energy}\\ \text{generation within}\\ \text{the element}\end{matrix}\right\}-\left\{\begin{matrix}\text{rate of energy}\\ \text{conduction out}\\ \text{of the element}\end{matrix}\right\}=\left\{\begin{matrix}\text{rate of accumulation}\\ \text{of energy}\\ \text{within the element}\end{matrix}\right\}$$

$$-k(2\pi rL)\left.\frac{\partial T}{\partial r}\right|_r+\dot{q}(2\pi rL\Delta r)-\left[-k(2\pi rL)\left.\frac{\partial T}{\partial r}\right|_{r+\Delta r}\right]=\rho c_p\frac{\partial T}{\partial t}(2\pi rL\Delta r)$$

$$\dot{q}+\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)=\rho c_p\frac{\partial T}{\partial t}\qquad \dot{q}+\frac{k}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)=0$$

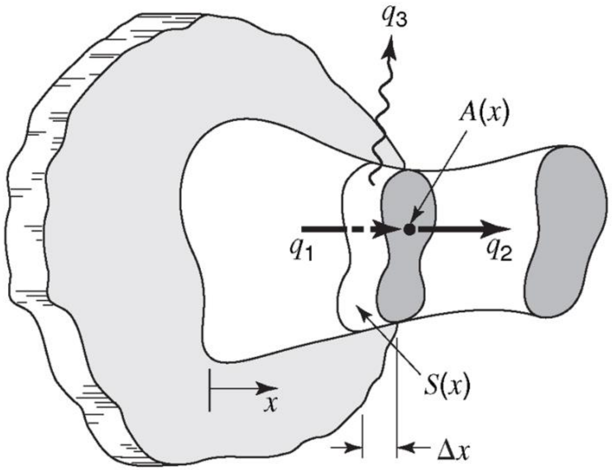


BC; 1) at center, 2) at wall

1. Cylindrical solid with homogeneous energy generation
2. Plane wall with variable energy generation

$$\boxed{\rho c_p\frac{DT}{Dt}=-\nabla\cdot\mathbf{q}+\dot{q}+\boldsymbol{\tau}:\nabla\mathbf{v}}$$

heat transfer from extended surfaces



$$q_1 = q_2 + q_3$$

$$kA\frac{dT}{dx}\bigg|_{x+\Delta x} - kA\frac{dT}{dx}\bigg|_{x} - hS\left(T - T_\infty\right) = 0$$

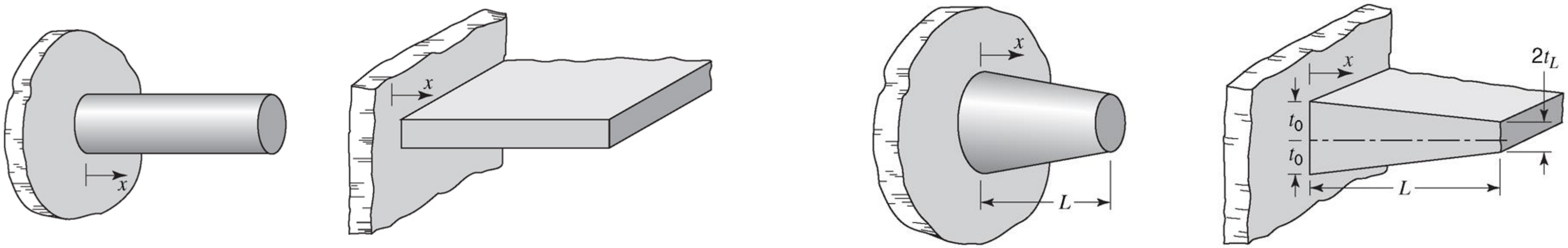
$$S = \Delta x \cdot P$$

$$\frac{kA\left(dT/dx\right)\big|_{x+\Delta x} - kA\left(dT/dx\right)\big|_{x}}{\Delta x} - hP\left(T - T_\infty\right) = 0$$

$$\frac{d}{dx}\left(kA\frac{dT}{dx}\right) - hP\left(T - T_\infty\right) = 0$$

assumption; T is a function of x only
; valid when the cross section is thin or when k is large

fin of uniform cross section



$$\frac{d^2T}{dx^2}-\frac{hP}{kA}\left(T-T_\infty\right)=0 \qquad \theta=T-T_\infty,\ m^2=hP/kA \qquad \frac{d^2\theta}{dx^2}-m^2\theta=0 \qquad \theta=c_1e^{mx}+c_2e^{-mx}$$

(a) $T=T_0$ at $x=0$

$T=T_L$ at $x=L$

$$\frac{\theta}{\theta_0}=\frac{T-T_\infty}{T_0-T_\infty}=\left(\frac{\theta_L}{\theta_0}-e^{-mL}\right)\left(\frac{e^{mx}-e^{-mx}}{e^{mL}-e^{-mL}}\right)+e^{-mx}$$

(b) $T=T_0$ at $x=0$

$\dfrac{dT}{dx}=0$ at $x=L$

$$\frac{\theta}{\theta_0}=\frac{T-T_\infty}{T_0-T_\infty}=\frac{\cosh\left[m\left(L-x\right)\right]}{\cosh mL}$$

(c) $T=T_0$ at $x=0$

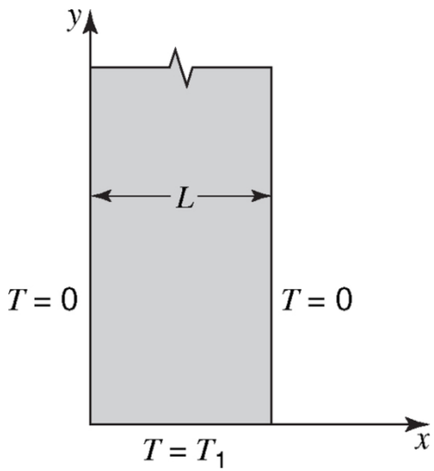
$-k\dfrac{dT}{dx}=h\left(T-T_\infty\right)$ at $x=L$

$$\frac{\theta}{\theta_0}=\frac{T-T_\infty}{T_0-T_\infty}=\frac{\cosh\left[m\left(L-x\right)\right]+\left(h/mk\right)\sinh\left[m\left(L-x\right)\right]}{\cosh mL+\left(h/mk\right)\sinh mL}$$

-> total heat transfer from an extended surface

-> fin efficiency

2-D systems



$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0$$

$$T=0 \quad \text{at} \quad x=0 \quad \text{for all values of } y$$

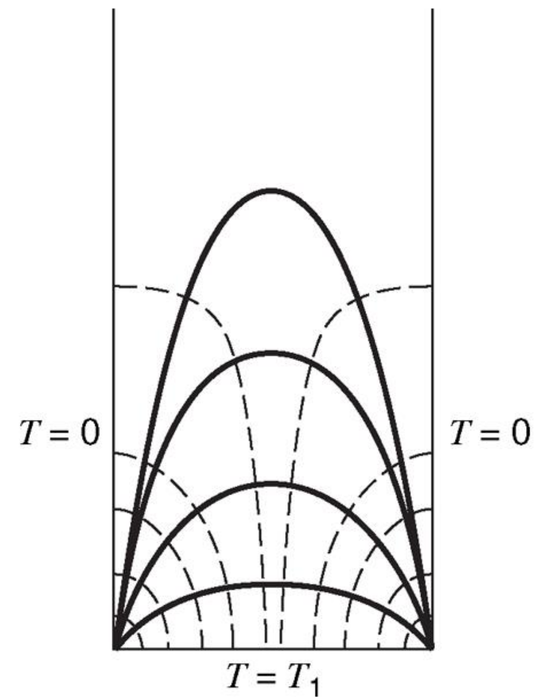
$$T=0 \quad \text{at} \quad x=L \quad \text{for all values of } y$$

$$T=T_1 \quad \text{at} \quad y=0 \quad \text{for } 0\le x\le L$$

$$T=0 \quad \text{at} \quad y=\infty \quad \text{for} \quad 0\le x\le L$$

$$T(x,y)=X(x)Y(y)$$

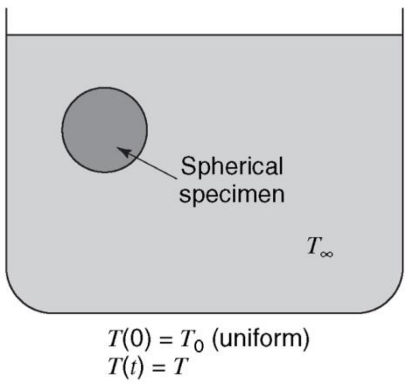
$$T=\frac{4T_1}{\pi}\sum_{n=0}^{\infty}\frac{e^{\left[-(2n+1)\pi y\right]/L}}{2n+1}\sin\frac{(2n+1)\pi x}{L}$$



unsteady-state conduction

lumped parameter analysis

T is a function of time & position
; if T varies with time only (large k)



$$\rho V c_p \frac{dT}{dt} = hA\left(T_\infty - T\right)$$

$$\frac{dT}{dt} = \frac{hA\left(T_\infty - T\right)}{\rho V c_p}$$

$$\frac{T - T_\infty}{T_0 - T_\infty} = e^{-hAt/\rho c_p V} = e^{-BiFo} \qquad \mathrm{Bi} = \frac{hV/A}{k} \qquad \mathrm{Fo} = \frac{\alpha t}{\left(V/A\right)^2}$$

error in lumped parameter analysis is less than 5% for Bi less than 0.1
; systems with negligible internal resistance

negligible surface resistance

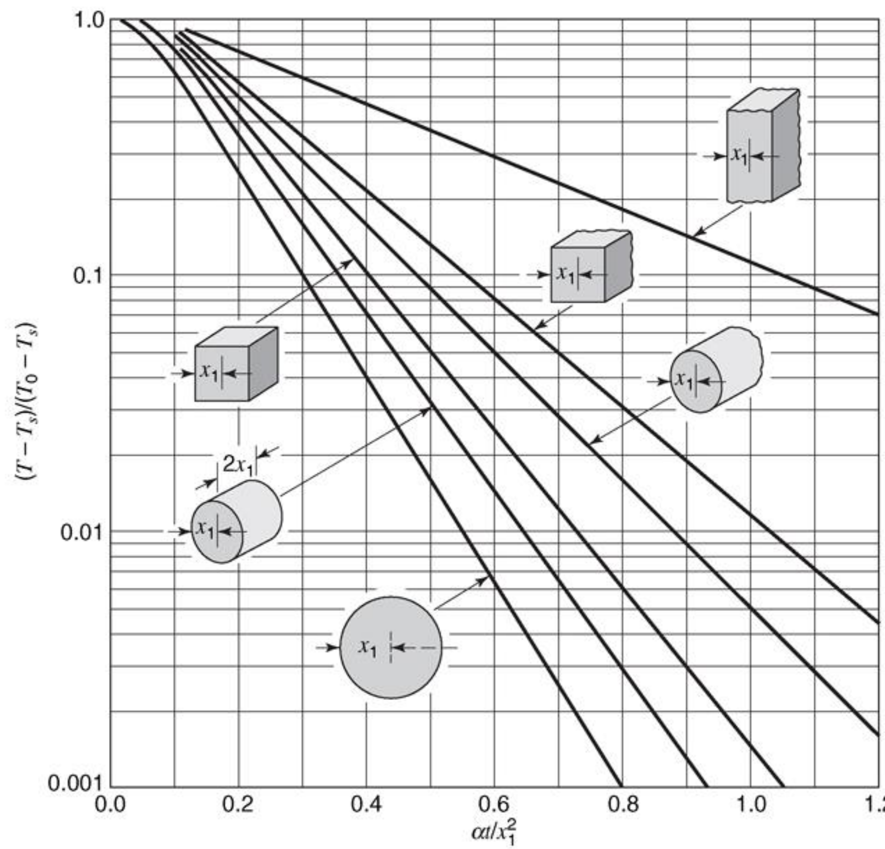
; surface temperature is constant for Bi>>1

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$T = T_0(x) \quad \text{at} \quad t = 0 \quad \text{for} \quad 0 \le x \le L$$
$$T = T_s \quad \text{at} \quad x = 0 \quad \text{for} \quad t > 0$$
$$T = T_s \quad \text{at} \quad x = L \quad \text{for} \quad t > 0$$

$$Y = \frac{T - T_s}{T_o - T_s} \qquad \frac{\partial Y}{\partial t} = \alpha \frac{\partial^2 Y}{\partial x^2}$$

$$\frac{T - T_s}{T_0 - T_s} = \frac{4}{\pi} \sum_{n,odd}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{L} x\right) e^{-(n\pi/2)^2 \mathrm{Fo}}$$



heating a body with finite surface and internal resistance

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$T = T_0 \quad \text{at} \quad t = 0$$

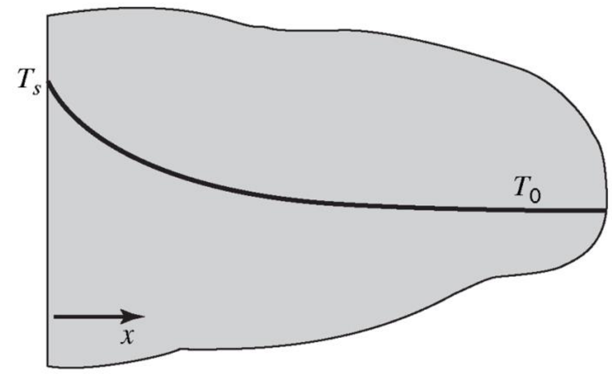
$$\frac{\partial T}{\partial x} = 0 \quad \text{at the centerline of the body}$$

$$-\frac{\partial T}{\partial x} = \frac{h}{k}\left(T - T_\infty\right) \quad \text{at the surface}$$

$$\frac{T - T_\infty}{T_0 - T_\infty} = 2\sum_{n=1}^{\infty} \frac{\sin\delta_n \cos\left(\delta_n x / x_1\right)}{\delta_n + \sin\delta_n \cos\delta_n} e^{-\delta_n^2 \mathrm{Fo}}$$

$$\delta_n \tan\delta_n = \frac{h x_1}{k}$$

heat transfer to a semi-infinite wall



$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

$$T = T_0 \quad \text{at} \quad t = 0 \quad \text{for all} \quad x$$

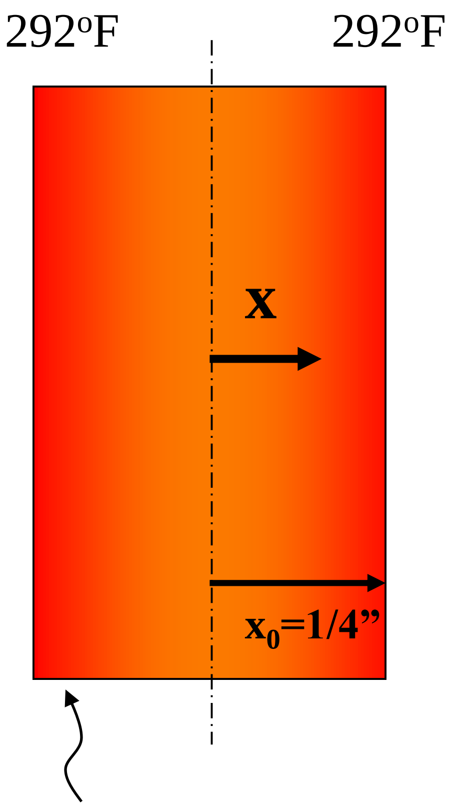
$$T = T_s \quad \text{at} \quad x = 0 \quad \text{for all } t$$

$$T \to T_0 \quad \text{as} \quad x \to \infty \quad \text{for all} \quad t$$

$$\frac{T_s - T}{T_s - T_0} = \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}} \right)$$

Separation of Variables

Unsteady state heat conduction in a rubber sheet



Curing at 292ºF
for 50min

$$\frac{\partial t}{\partial \theta} = \alpha \frac{\partial^2 t}{\partial x^2}$$

$$\text{I.C.} \quad : \quad \text{at } \theta = 0,\ t = t_0 = 70^\circ \text{F}$$

$$\text{B.C.1}: \quad \text{at } x = x_0 = 1/4'',\ t = t_s = 292^\circ \text{F}$$

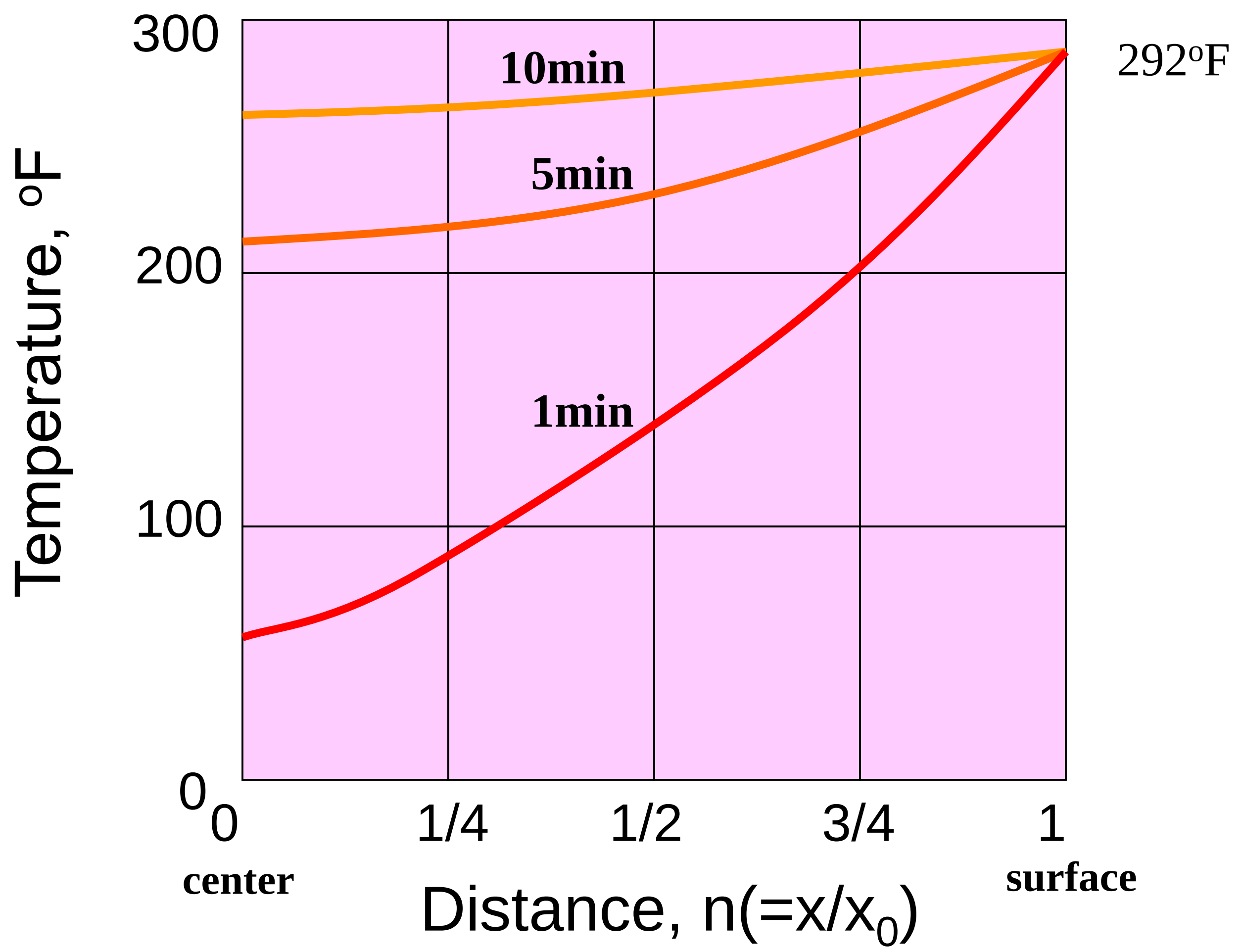
$$\text{B.C.2}: \quad \text{at } x = 0,\ dt/dx = 0$$

$$\theta = ?\ @\ x = 0 \text{ and } t = 290^\circ \text{F}$$

$$\alpha = \frac{k}{\rho C_p} = 0.0028\ ft^2/h$$

Separation of Variables

Temperature profiles in rubber sheet



18.7 min

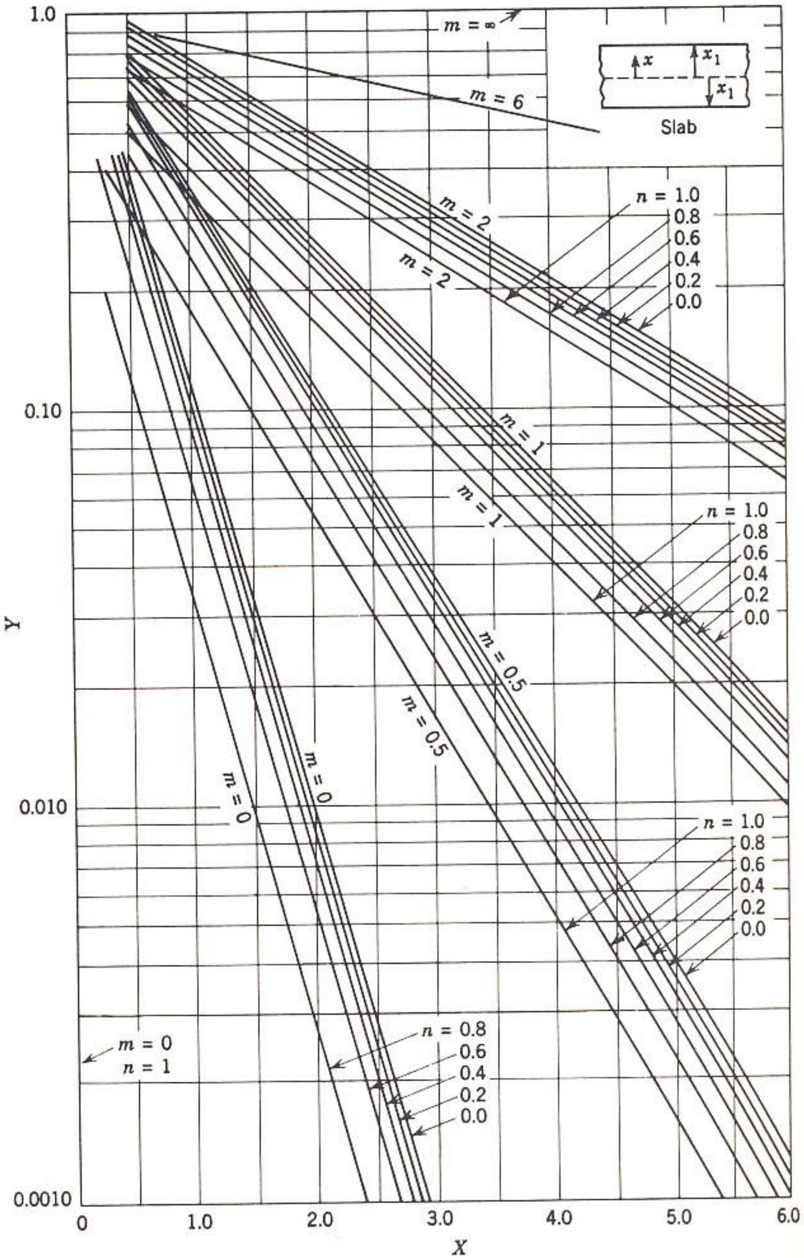


Figure F.1 Unsteady-state transport in a large flat slab.

$$x = \frac{k\theta}{\rho\, C_p r_m^2} = \frac{\alpha\theta}{r_m^2} : Fourier\ \ number$$

$$y = \frac{t_s - t}{t_s - t_o}$$

$$m = \frac{k}{hr_m} \rightarrow \frac{hr_m}{k} : Biot\ \ number(lumped\ \ parameter, B_i < 0.1)$$

$$n = \frac{r}{r_m}$$

Gurney-Lurie Chart

$$Y = \frac{t_s - t}{t_s - t_0} = \frac{292 - 290}{292 - 70} = 0.0090$$

$$X = \frac{\alpha\theta}{x_0^2} = \frac{k\theta}{\rho C_p x_0^2} = Fo = \frac{0.0028\theta}{(1/48)^2} = 6.44\theta$$

$$n = \frac{x}{x_0} = 0$$

$$m = \frac{k}{hx_0} = \frac{1}{Bi} = 0$$

*m is zero because the assumption of constant surface temperature implies that surface resistance to heat transfer is negligible (**h**=∞).*

Gurney-Lurie Chart

From Fig. 19-3,

$$\frac{t_s - t}{t_s - t_0} = 0.0090, \quad n = 0, \quad m = 0$$

we obtain

$$\frac{\alpha\theta}{x_0^2} = \frac{k\theta}{\rho C_p x_0^2} = 2.0$$

$$\frac{\alpha\theta}{x_0^2} = 6.44\theta$$

$$\theta = \frac{2.0}{6.44} = 0.31\,h = 19\,\mathrm{min}$$

cf. 18.7 min