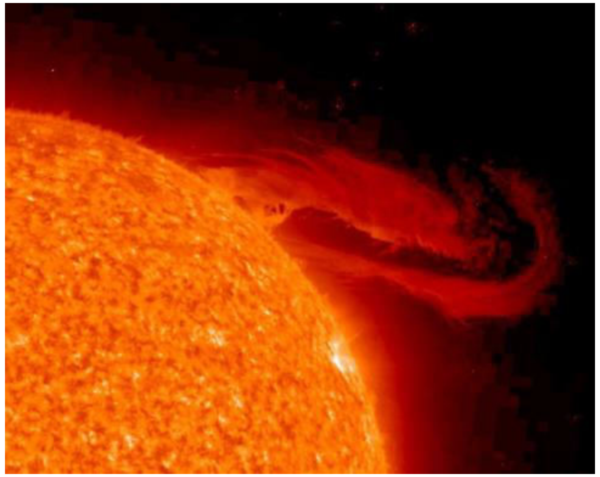
Introduction to Nuclear Fusion

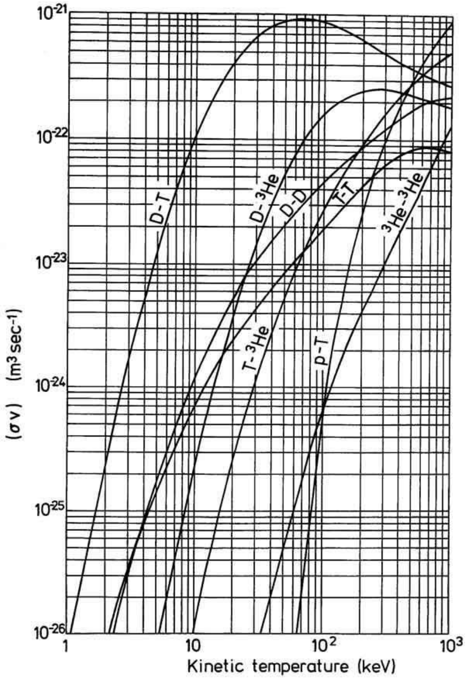
Prof. Dr. Yong-Su Na

How to describe a plasma?

Description of a Plasma

- **Thermonuclear fusion with high energy particles of 10-20 keV energy → plasma**



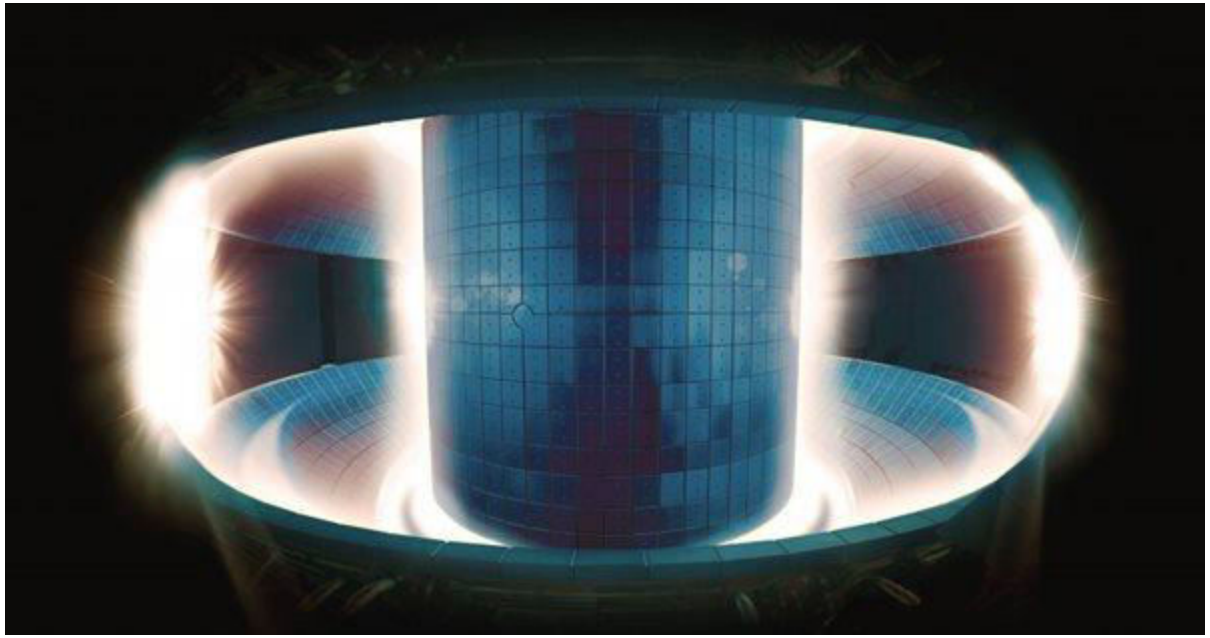


- **Three approaches to describe a plasma**

- Single particle approach
- Kinetic theory
- Fluid theory

http://http://dz-dev.net/blog/tag/eruption-solaire

Kinetic Approach of Plasmas



KSTAR

$n_e \sim 10^{20} m^{-3}$

$Volume \sim 18 m^3$

# of electrons: $\sim 10^{21}$

3 GHz of personal computer:
3×10^9 #/s ~ 10^{17} #/year
$\rightarrow 10^4$ years

Kinetic Approach of Plasmas

- **Boltzmann equation**

$$f_\alpha = f_\alpha\left(t, x, y, z, u_x, u_y, u_z\right)$$ Distribution function: number density of particles found 'near' a point in the 6-D space (**x**, **u**)

$$\frac{df_\alpha}{dt} =$$

$$= \frac{\partial f_\alpha}{\partial t} + \vec{u} \cdot \nabla f_\alpha + \vec{a} \cdot \nabla_{\vec{u}} f_\alpha$$

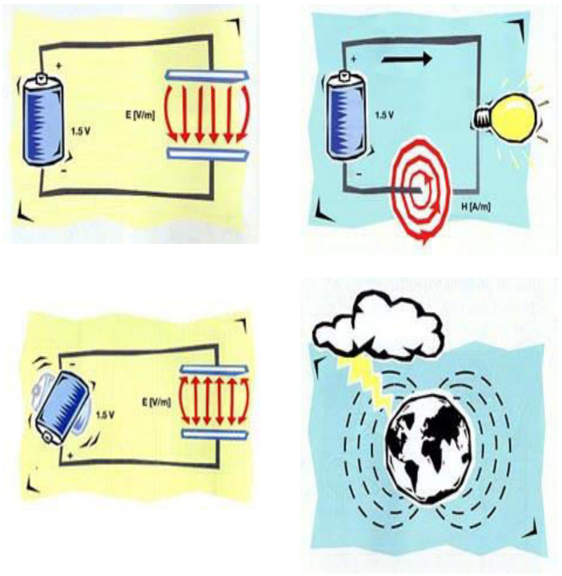
$$= \frac{\partial f_\alpha}{\partial t} + \vec{u} \cdot \nabla f_\alpha + \frac{q_\alpha}{m_\alpha}(\vec{E} + \vec{u} \times \vec{B}) \cdot \nabla_{\vec{u}} f_\alpha$$

$$= \left(\frac{\partial f_\alpha}{\partial t}\right)_c$$

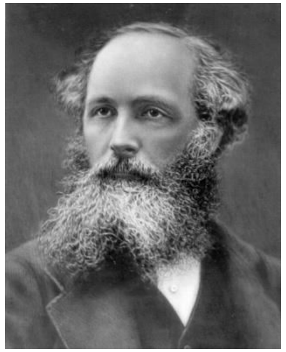
Kinetic Approach of Plasmas

- **Boltzmann equation**

$$\frac{df_\alpha}{dt} = \frac{\partial f_\alpha}{\partial t} + \vec{u}\cdot\nabla f_\alpha + \vec{a}\cdot\nabla_{\vec{u}} f_\alpha = \frac{\partial f_\alpha}{\partial t} + \vec{u}\cdot\nabla f_\alpha + \frac{q_\alpha}{m_\alpha}\left(\boxed{\vec{E}} + \vec{u}\times\boxed{\vec{B}}\right)\cdot\nabla_{\vec{u}} f_\alpha = \left(\frac{\partial f_\alpha}{\partial t}\right)_c$$



Ludwig Boltzmann (1844-1906)



James Clerk Maxwell (1831-1879)

$$\nabla\cdot\vec{E} = \frac{\sigma}{\varepsilon_0}$$

$$\nabla\cdot\vec{B} = 0$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\nabla\times\vec{B} = \mu_0\vec{J} + \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

$$\vec{J} \equiv \sum_{\alpha=e,i} q_\alpha \int \vec{u} f_\alpha d\vec{u}$$ current density

$$\sigma \equiv \sum_{\alpha=e,i} q_\alpha \int f_\alpha d\vec{u}$$ charge density

Wikipedia

Kinetic Approach of Plasmas

- **Boltzmann equation**

$$f_\alpha = f_\alpha\left(t, x, y, z, u_x, u_y, u_z\right)$$ Distribution function: number density of particles found 'near' a point in the 6-D space (**x**, **u**)

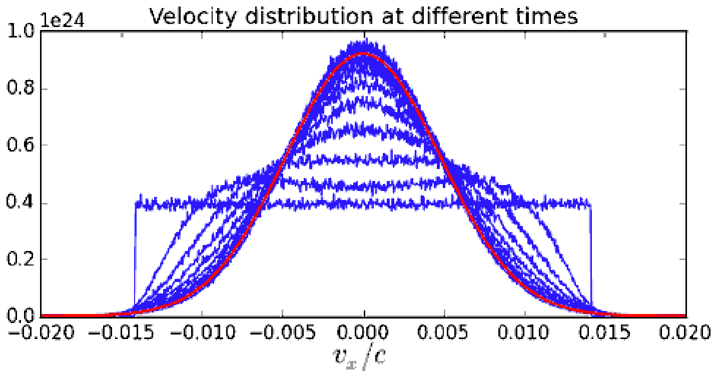
$$n_\alpha(\vec{r},t) \equiv \int f_\alpha d\vec{u}$$ number density of particles in physical space

$$\vec{v}_\alpha(\vec{r},t) \equiv \frac{1}{n_\alpha}\int \vec{u} f_\alpha d\vec{u}$$ Mean (fluid) velocity of particles in physical space

$$p(\vec{r},t) \equiv \frac{m_\alpha}{3}\int \vec{u}^2 f_\alpha d\vec{u}$$ Scalar pressure of particles in physical space

$$\langle Q \rangle \equiv \frac{1}{n_\alpha}\int Q f_\alpha d\vec{u}$$

Kinetic Approach of Plasmas



- **Boltzmann equation**

$$f_M(\vec{u}) = n\left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{m\vec{u}^2}{2T}\right)$$

Maxwellian velocity distribution in thermal equilibrium

$$n_\alpha(\vec{r},t) \equiv \int f_\alpha d\vec{u}$$

number density of particles in physical space

$$\vec{v}_\alpha(\vec{r},t) \equiv \frac{1}{n_\alpha}\int \vec{u} f_\alpha d\vec{u}$$

Mean (fluid) velocity of particles in physical space

$$p(\vec{r},t) \equiv \frac{m_\alpha}{3}\int \vec{u}^2 f_\alpha d\vec{u}$$

Scalar pressure of particles in physical space

$$\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle = \frac{1}{n}\int u_x^2 f_M(\vec{r},\vec{u},t) d\vec{u} = \frac{T}{m}$$

Mean square velocity

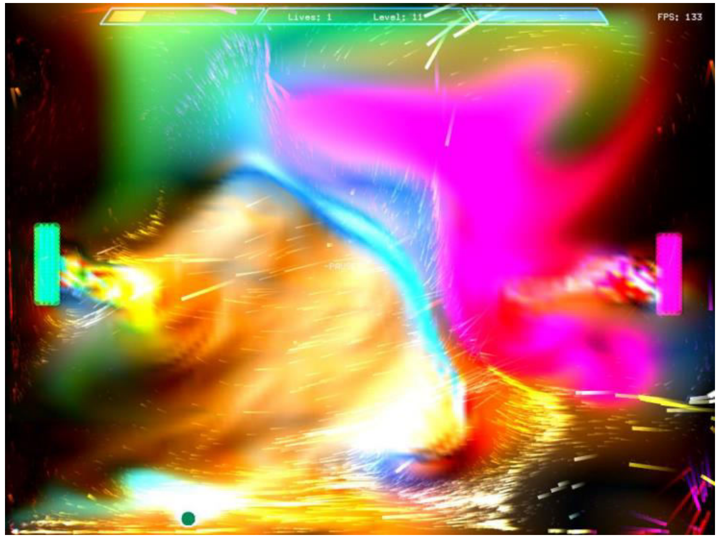
$$p = nT$$

http://www.maisondelasimulation.fr/smilei/collisions.html

Fluid Approach of Plasmas

- **Plasmas as fluids**

  - The single particle approach gets to be complicated.
  - A more statistical approach can be used because we cannot follow each particle separately.
  - Introduce the concept of an **electrically charged current-carrying fluid.**



http://www.tower.com/music-label/gulan-music-studio
Album art of music OST "Plasma Pong"

Plasmas as Fluids

$$\frac{df_\alpha}{dt} = \frac{\partial f_\alpha}{\partial t} + \vec{u}\cdot\nabla f_\alpha + \frac{q_\alpha}{m_\alpha}(\vec{E} + \vec{u}\times\vec{B})\cdot\nabla_{\vec{u}} f_\alpha = \left(\frac{\partial f_\alpha}{\partial t}\right)_c$$

- **Two fluid equations**

$$\int Q_i\left[\frac{df_\alpha}{dt} - \left(\frac{\partial f_\alpha}{\partial t}\right)_c\right]d\vec{u} = 0$$

$Q_1 = 1$ mass

$Q_2 = m_\alpha \vec{u}$ momentum

$Q_3 = m_\alpha u^2/2$ energy

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_c = \sum_\beta C_{\alpha\beta}$$

$$\vec{u} = \vec{v}_\alpha(\vec{r},t) + \vec{w},\ \langle\vec{w}\rangle = 0$$

$$n_\alpha(\vec{r},t) \equiv \int f_\alpha d\vec{u}$$

$$\vec{v}_\alpha(\vec{r},t) \equiv \frac{1}{n_\alpha}\int \vec{u} f_\alpha d\vec{u}$$

$$\langle Q\rangle \equiv \frac{1}{n_\alpha}\int Q f_\alpha d\vec{u}$$

Plasmas as Fluids

$$\frac{df_\alpha}{dt} = \frac{\partial f_\alpha}{\partial t} + \vec{u}\cdot\nabla f_\alpha + \frac{q_\alpha}{m_\alpha}(\vec{E}+\vec{u}\times\vec{B})\cdot\nabla_{\vec{u}} f_\alpha = \left(\frac{\partial f_\alpha}{\partial t}\right)_c$$

- **Continuity equation**

$Q_1 = 1$ mass

$$\int 1\times\left[\frac{df_\alpha}{dt} - \left(\frac{\partial f_\alpha}{\partial t}\right)_c\right]d\vec{u} = 0$$

$$\left(\frac{\partial f_\alpha}{\partial t}\right)_c = \sum_\beta C_{\alpha\beta}$$

$$\int \frac{\partial f_\alpha}{\partial t}d\vec{u} + \int \vec{u}\cdot\nabla f_\alpha d\vec{u} + \frac{q_\alpha}{m_\alpha}\int\left(\vec{E}+\vec{u}\times\vec{B}\right)\cdot\frac{\partial f_\alpha}{\partial \vec{u}}d\vec{u} - \int\left(\frac{\partial f_\alpha}{\partial t}\right)_c d\vec{u} = 0$$

$$\vec{u} = \vec{v}_\alpha(\vec{r},t) + \vec{w},\ \langle\vec{w}\rangle = 0$$

$$\int \frac{\partial f_\alpha}{\partial t}d\vec{u} = \frac{\partial}{\partial t}\int f_\alpha d\vec{u} = \frac{\partial n_\alpha}{\partial t}$$

$$n_\alpha(\vec{r},t) \equiv \int f_\alpha d\vec{u}$$

$$\int \vec{u}\cdot\nabla f_\alpha d\vec{u} = \int \nabla\cdot(\vec{u}f_\alpha)d\vec{u} - \int f_\alpha \nabla\cdot\vec{u}d\vec{u} = \nabla\cdot\int \vec{u}f_\alpha d\vec{u} = \nabla\cdot(n\vec{v}_\alpha)$$

$$\vec{v}_\alpha(\vec{r},t) \equiv \frac{1}{n_\alpha}\int \vec{u}f_\alpha d\vec{u}$$

$$\langle Q\rangle \equiv \frac{1}{n_\alpha}\int Qf_\alpha d\vec{u}$$

$$\frac{\partial n_\alpha}{\partial t} + \nabla\cdot\left(n_\alpha\vec{v}_\alpha\right) = 0$$

Plasmas as Fluids

$$\frac{df_\alpha}{dt} = \frac{\partial f_\alpha}{\partial t} + \vec{u}\cdot\nabla f_\alpha + \frac{q_\alpha}{m_\alpha}(\vec{E}+\vec{u}\times\vec{B})\cdot\nabla_{\vec{u}} f_\alpha = \left(\frac{\partial f_\alpha}{\partial t}\right)_c$$

- **Two fluid equations**

$$\int Q_i\left[\frac{df_\alpha}{dt} - \left(\frac{\partial f_\alpha}{\partial t}\right)_c\right]d\vec{u} = 0$$

$Q_1 = 1$ mass

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$$\left(\frac{\partial f_\alpha}{\partial t}\right)_c = \sum_\beta C_{\alpha\beta}$$

$$\vec{u} = \vec{v}_\alpha(\vec{r},t) + \vec{w},\ \langle \vec{w}\rangle = 0$$

$$\frac{dn_\alpha}{dt} + n_\alpha \nabla\cdot\vec{v}_\alpha = 0$$

$$n_\alpha m_\alpha \frac{d\vec{v}_\alpha}{dt} = q_\alpha n_\alpha(\vec{E}+\vec{v}_\alpha\times\vec{B}) - \nabla\cdot\overleftrightarrow{P}_\alpha + \vec{R}_\alpha$$

$$\frac{3}{2}n_\alpha\frac{dT_\alpha}{dt} = -\overleftrightarrow{P}_\alpha : \nabla\vec{v}_\alpha - \nabla\cdot\vec{h}_\alpha + Q_\alpha$$

$$\overleftrightarrow{P}_\alpha \equiv n_\alpha m_\alpha \langle w^2\rangle$$

$$\vec{R}_\alpha \equiv \int m_\alpha \vec{w} C_{\alpha\beta} d\vec{w}$$

$$h_\alpha \equiv \frac{1}{2} n_\alpha m_\alpha \langle w^2 \vec{w}\rangle$$

$$Q_\alpha \equiv \int \frac{1}{2} m_\alpha w_\alpha^2 C_{\alpha\beta} d\vec{w}$$

$$n_\alpha(\vec{r},t) \equiv \int f_\alpha d\vec{u}$$

$$\vec{v}_\alpha(\vec{r},t) \equiv \frac{1}{n_\alpha}\int \vec{u} f_\alpha d\vec{u}$$

$$\langle Q\rangle \equiv \frac{1}{n_\alpha}\int Q f_\alpha d\vec{u}$$

Plasmas as Fluids

- **Two fluid equations**

$$\frac{dn_\alpha}{dt} + n_\alpha \nabla \cdot \vec{v}_\alpha = 0$$

$$n_\alpha m_\alpha \frac{d\vec{v}_\alpha}{dt} = q_\alpha n_\alpha (\vec{E} + \vec{v}_\alpha \times \vec{B}) - \nabla \cdot \overleftrightarrow{P}_\alpha + \vec{R}_\alpha$$

$$\frac{3}{2} n_\alpha \frac{dT_\alpha}{dt} = -\overleftrightarrow{P}_\alpha : \nabla \vec{v}_\alpha - \nabla \cdot \vec{h}_\alpha + Q_\alpha$$

$$\overleftrightarrow{P}_\alpha \equiv n_\alpha m_\alpha \left\langle w^2 \right\rangle$$

$$\vec{R}_\alpha \equiv \int m_\alpha \vec{w} C_{\alpha\beta} d\vec{w}$$

$$h_\alpha \equiv \frac{1}{2} n_\alpha m_\alpha \left\langle w^2 \vec{w} \right\rangle$$

$$Q_\alpha \equiv \int \frac{1}{2} m_\alpha w_\alpha^2 C_{\alpha\beta} d\vec{w}$$

Closure?

$$x + 2y = 4$$
$$x - y + z = 2$$
$$2x + 3y - z + \alpha = 1$$
$$\ldots$$

Plasmas as Fluids

- **Single-fluid magnetohydrodynamics (MHDs)**

A single-fluid model of a fully ionised plasma, in which the plasma is treated as a single hydrodynamic fluid acted upon by electric and magnetic forces.

- **The magnetohydrodynamic (MHD) equation**

$$\rho = n_i M + n_e m \approx n(M+m) \approx nM$$ mass density

Hydrogen plasma, charge neutrality assumed

$$\sigma = (n_i - n_e)e$$ charge density

$$\vec{v} = (n_i M \vec{v}_i + n_e m \vec{v}_e)/\rho \approx n(M\vec{v}_i + m\vec{v}_e)/nM = \vec{v}_i + (m/M)\vec{v}_e$$ mass velocity

$$\vec{J} = e(n_i \vec{v}_i - n_e \vec{v}_e) \approx ne(\vec{v}_i - \vec{v}_e)$$

$$\vec{v}_i \approx \vec{v} + \frac{m}{M}\frac{\vec{J}}{ne}, \quad \vec{v}_e \approx \vec{v} - \frac{\vec{J}}{ne}$$

electron inertia neglected: electrons have an infinitely fast response time because of their small mass

Plasmas as Fluids

$$\rho = n_i M + n_e m$$

$$\vec{v} = (n_i M \vec{v}_i + n_e m \vec{v}_e) / \rho$$

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \vec{v}_i) = 0 \qquad M \frac{\partial n_i}{\partial t} + \nabla \cdot (M n_i \vec{v}_i) = 0$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \vec{v}_e) = 0 \qquad m \frac{\partial n_e}{\partial t} + \nabla \cdot (m n_e \vec{v}_e) = 0$$

$$\frac{\partial (M n_i + m n_e)}{\partial t} + \nabla \cdot (M n_i \vec{v}_i + m n_e \vec{v}_e) = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0$$

Plasmas as Fluids

$$Mn_i \frac{d\vec{v}_i}{dt} = en_i\left(\vec{E} + \vec{v}_i \times \vec{B}\right) - \nabla p_i + \vec{R}_{ie}$$

$$mn_e \frac{d\vec{v}_e}{dt} = -en_e\left(\vec{E} + \vec{v}_e \times \vec{B}\right) - \nabla p_e + \vec{R}_{ei}$$

$$\rho = n_i M + n_e m$$

$$\vec{v} = (n_i M\vec{v}_i + n_e m\vec{v}_e)/\rho$$

$$\vec{v}_i \approx \vec{v} + \frac{m}{M}\frac{\vec{J}}{ne}, \quad \vec{v}_e \approx \vec{v} - \frac{\vec{J}}{ne}$$

$$\vec{J} = e(n_i\vec{v}_i - n_e\vec{v}_e) \approx ne(\vec{v}_i - \vec{v}_e)$$

$$Mn_i \frac{d\vec{v}}{dt} + mn_i \frac{d}{dt}\left(\frac{\vec{J}}{ne}\right) = en_i\vec{E} + en_i\vec{v}_i \times \vec{B} - \nabla p_i + \vec{R}_{ie}$$

$$mn_e \frac{d\vec{v}}{dt} - mn_e \frac{d}{dt}\left(\frac{\vec{J}}{ne}\right) = -en_e\vec{E} - en_e\vec{v}_e \times \vec{B} - \nabla p_e + \vec{R}_{ei}$$

$$\rho \frac{d\vec{v}}{dt} = \vec{J} \times \vec{B} - \nabla p$$

Plasmas as Fluids

$$mn_e \frac{d\vec{v}_e}{dt} = -en_e\left(\vec{E} + \vec{v}_e \times \vec{B}\right) - \nabla p_e + \vec{R}_{ei}$$

Neglect electron inertia entirely

$$0 = -en_e\left(\vec{E} + \vec{v}_e \times \vec{B}\right) - \nabla p_e + \vec{R}_{ei}$$

$$\vec{R}_{ei} = mn\langle \nu_{ei}\rangle(\vec{v}_i - \vec{v}_e) = \eta n^2 e^2 (\vec{v}_i - \vec{v}_e) = \eta n e \vec{J}$$

$$\vec{E} + \vec{v}_e \times \vec{B} = \eta \vec{J} - \frac{\nabla p_e}{en_e}$$

$$\vec{E} + \vec{v} \times \vec{B} = \eta \vec{J} + \frac{\vec{J} \times \vec{B} - \nabla p_e}{en}$$

$$\rho = n_i M + n_e m$$

$$\vec{v} = (n_i M \vec{v}_i + n_e m \vec{v}_e)/\rho$$

$$\vec{v}_i \approx \vec{v} + \frac{m}{M}\frac{\vec{J}}{ne}, \quad \vec{v}_e \approx \vec{v} - \frac{\vec{J}}{ne}$$

$$\vec{J} = e(n_i \vec{v}_i - n_e \vec{v}_e) \approx ne(\vec{v}_i - \vec{v}_e)$$

Plasmas as Fluids

F. F. Chen, Ch. 5.7

- **Ideal MHD model**

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0$$

Mass continuity equation
(continuously flowing fluid)

$$\rho \frac{d\vec{v}}{dt} = \vec{J} \times \vec{B} - \nabla p$$

Single-fluid equation of motion

$$\frac{d}{dt}\left(\frac{p}{\rho^{\gamma}}\right) = 0$$

Energy equation (equation of state):
adiabatic evolution

$$\vec{E} + \vec{v} \times \vec{B} = 0$$

Ohm's law: perfect conductor → "ideal" MHD

$$\vec{E} + \vec{v} \times \vec{B} = \eta \vec{J} + \frac{\vec{J} \times \vec{B} - \nabla p_e}{en}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

$$\nabla \cdot \vec{B} = 0$$

Maxwell equations

$\varepsilon_0 \to 0$ assumed
(Full → low-frequency Maxwell's equations)
Displacement current, net charge neglected

Plasmas as Fluids

$$\frac{dn_\alpha}{dt} + n_\alpha \nabla \cdot \vec{v}_\alpha = 0$$

$$n_\alpha m_\alpha \frac{d\vec{v}_\alpha}{dt} - q_\alpha n_\alpha (\vec{E} + \vec{v}_\alpha \times \vec{B}) + \nabla \cdot \overleftrightarrow{P}_\alpha = \vec{R}_\alpha$$

$$\frac{3}{2} n_\alpha \frac{dT_\alpha}{dt} + \overleftrightarrow{P}_\alpha : \nabla \vec{v}_\alpha + \nabla \cdot \vec{h}_\alpha = Q_\alpha$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 e (n_i \vec{v}_i - n_e \vec{v}_e) + \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \cdot \vec{E} = \frac{e}{\varepsilon_0} (n_i - n_e)$$

$$\nabla \cdot \vec{B} = 0$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{v} = 0$$

$$\rho \frac{d\vec{v}}{dt} = \vec{J} \times \vec{B} - \nabla p$$

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0$$

$$\vec{E} + \vec{v} \times \vec{B} = 0$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

$$\nabla \cdot \vec{B} = 0$$

Plasmas as Fluids

- **Magnetohydrodynamics (MHD): 1946 by H. Alfvén**

STOCKHOLMS OBSERVATORIUMS ANNALER

(ASTRONOMISKA IAKTTAGELSER OCH UNDERSÖKNINGAR Å STOCKHOLMS OBSERVATORIUM)

BAND 14. N:o 9.

ON THE COSMOGONY OF THE SOLAR SYSTEM

III

BY

HANNES ALFVÉN

WIKIPEDIA

Hannes Alfvén
(1908-1995)

"Nobel prize in Physics (1970)"

Plasmas as Fluids

- **Magnetohydrodynamics (MHD): 1946 by H. Alfvén**

The laws found can be applied to the planetary as well as to the satellite systems (§§ 7—9) and also — as is pointed out in § 10 — to the non-solar planets newly discovered.

At last some remarks are made about the transfer of momentum from the Sun to the planets, which is fundamental to the theory (§ 11). The importance of the magneto-hydrodynamic waves in this respect is pointed out. It is possible that the Sun's general magnetic field was much stronger at the time of formation of the solar system than it is now. This must be assumed in order to explain the momentum transfer quantitatively.

Kungl. Tekniska Högskolan, Stockholm, November 1945.

Plasmas as Fluids

- **Ideal MHD**

  - Single-fluid model
  - Ideal:

    Perfect conductor with zero resistivity

  - MHD:

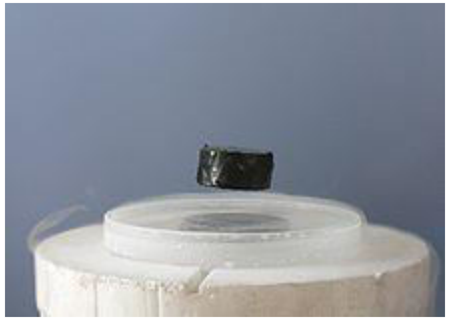
    Magnetohydrodynamic (magnetic fluid dynamic)

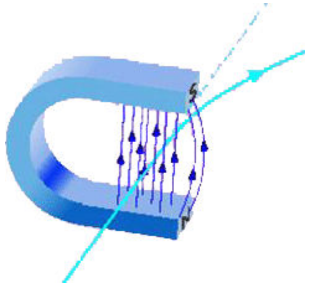
  - Assumptions:

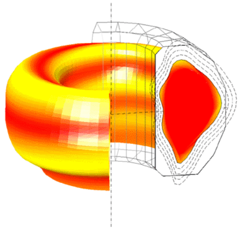
    Low-frequency, long-wavelength
    collision-dominated plasma

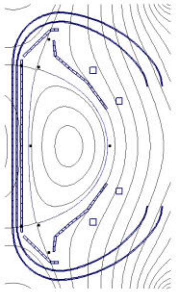
  - Applications:

    Equilibrium and stability in fusion plasmas



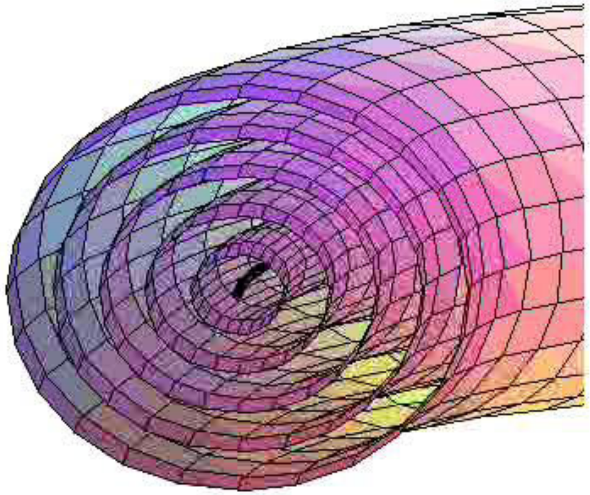




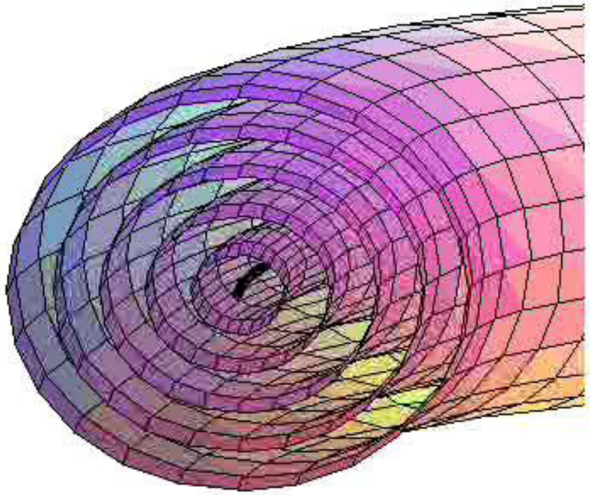


What is Ideal MHD?

- Ideal MHD: $\eta = 0$

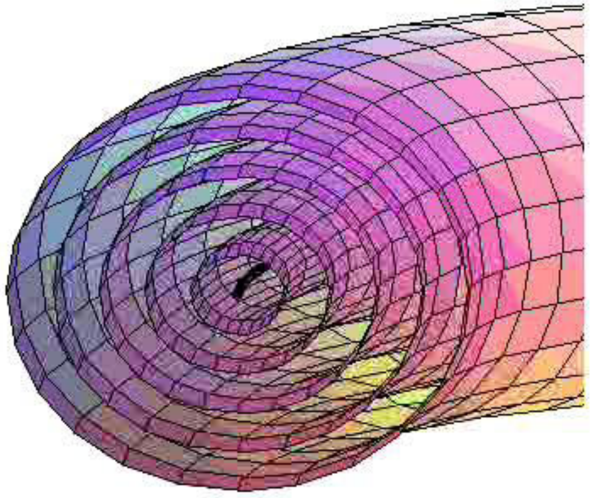


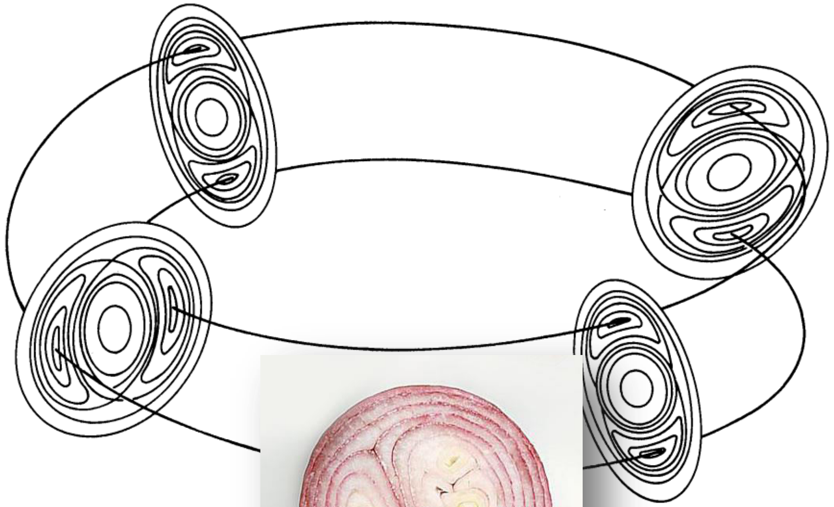
- Resistive MHD: $\eta \neq 0$



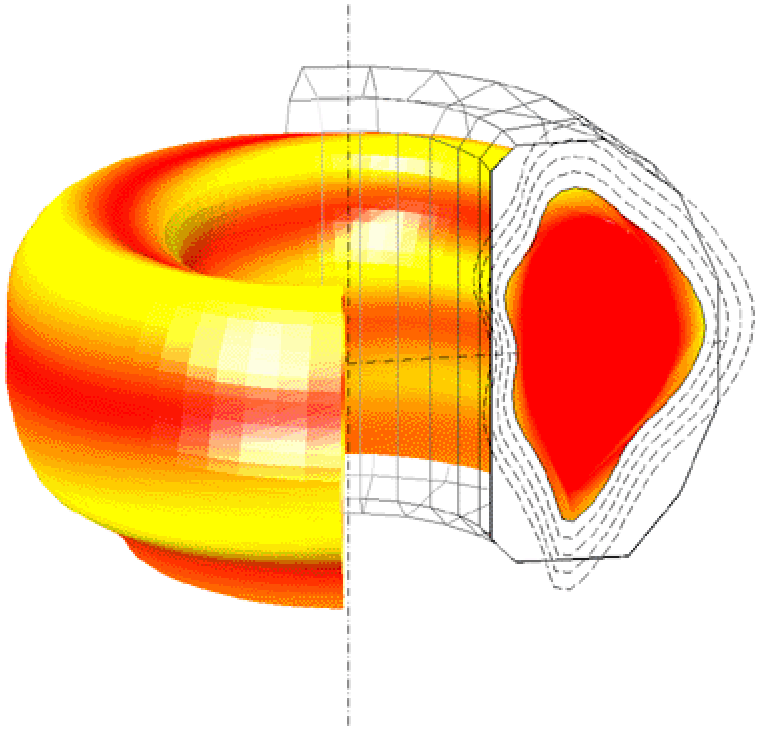
What is Ideal MHD?

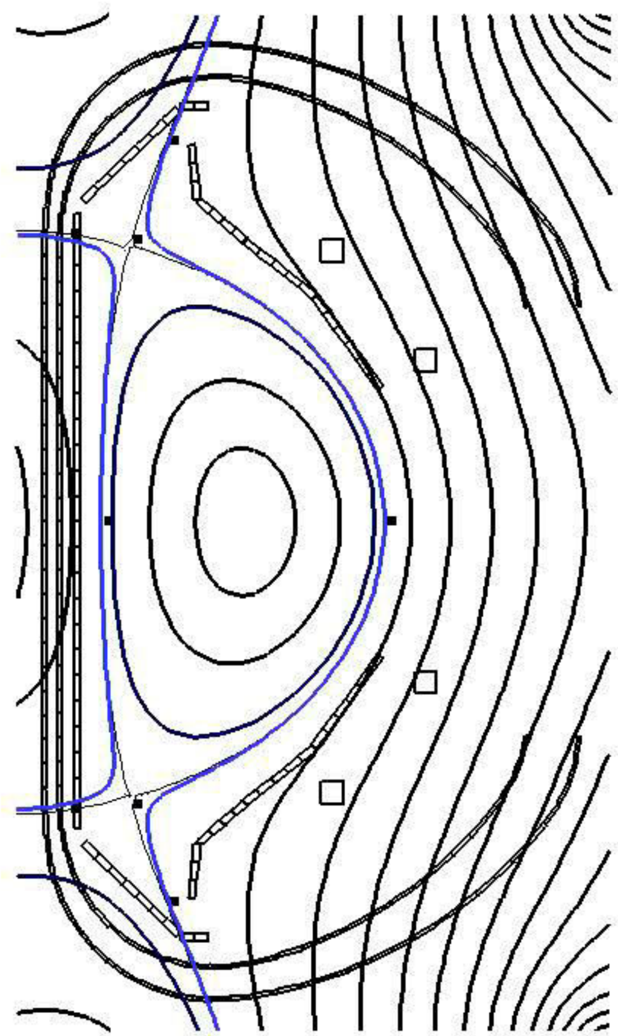
- Ideal MHD: $\eta = 0$
- Resistive MHD: $\eta \neq 0$





Applications





Plasma Eqauilibrium and Stability

https://fusion.gat.com/theory/File:Theory1.gif