INVISCID FLOW
Week 1

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Course Introduction

- Textbook
  - Main
    - Lecture Slide – please check "eTL" before the class
    - Fundamental Mechanics of Fluids (I. G. Currie), 3rd Ed., Marcel Dekker, Inc. (2003)
  - References
    - Inviscid Incompressible Flow (J. S. Marshall)
    - Physical Fluid Dynamics (D. J. Tritton),
    - Vectors, Tensors and the Basic Equations of Fluid Dynamics (R. Aris)
    - An Album of Fluid Motion (M. Van Dyke)
    - Papers from archival journals
    - ETC.

Tensor

- What is tensor?
  - Difficult to define with a simple word, but it refers to objects that have multiple indices.
    - e.g., a "scalar" is a tensor of 0^{th} order, a "vector" is a tensor of 1^{st} order
  - Adopted to describe physical phenomena irrespective of coordinate systems
  - To avoid tedious work of writing long equations with various terms, we have made some rules with tensor
  - Offers a more compact form of governing equations in various disciplines, e.g., fluid mechanics, solid mechanics, electrical engineering……

Tensor

- Index notation (in Cartesian coordinates)
  - Einstein's summation convention
    - **Free index**: If a given index appears only once in each term of a tensor equation, it is a free index and the equation holds for all possible values of that index.

      Ex) $i = 1, 2, 3$ $\quad x_i = (x_1, x_2, x_3)$
    - **Dummy index**: If an index appears twice in any term, it is understood that a summation is to be made over all possible values of that index.

      Ex) $i = 1, 2, 3$ $\quad a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$

      $a_i b_i c_j = a_1 b_1 c_j + a_2 b_2 c_j + a_3 b_3 c_j$
    - No index may appear more than twice in any term.
  - Dummy index may appear only twice in any given term. *cf)* $a_i b_i c_i$
  - Same free indices must appear in every term in an equation.

    cf) $a_i = 3 + b_i$

Tensor

- A tensor of rank r is a quantity having n^r components in n-dimensional space. The components of a tensor quantity expressed in two different coordinate system are related as follows:

$$T'_{ijk\cdots m} = \underbrace{C_{is}C_{jt}C_{ku}\cdots C_{mv}}_{\text{Directional cosine between the axes of two different coordinate systems.}} T_{stu\cdots v}$$

- Rank 0 tensor in 3-dimensional space → a scalar having 1 component
- Rank 1 tensor in 3-dimensional space → a vector having 3 components
- Rank 2 tensor is called a *dyadic* → represented by a matrix
  - r =2 and n =3 → 3 x 3 matrix (9 components)

$$A_{ij} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

(i,j = 1,2,3)

Tensor

- When do we use the tensor?
  - Governing equations such as Navier-Stokes Equation

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

  - Turbulence modeling

$$u_i u_j = \frac{2}{3}k\delta_{ij} - \nu_T\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

  - Coordinate transformation
  - Grid generation in CFD
  - Matrix calculation
  - …….

Tensor

- Tensor Algebra
  - Addition: Two tensors of equal rank may be added to yield a third tensor of the same rank.

$$C_{ij....k} = A_{ij....k} + B_{ij....k}$$

  - Multiplication: If tensor A has rank '**a**' and tensor B has rank '**b**', the multiplication of these two tensors yields a third one of rank '**c**'.

$$C_{ij....krs....t} = A_{ij....k} B_{rs....t}$$ c = a + b

  - Contraction: If any two indices of a tensor of rank r ≥2 are set equal, a tensor of rank r-2 is obtained

$$C_{ij} = A_i B_j$$ Tensor C of rank 2

$$C_{ii} = A_i B_i = A_1 B_1 + A_2 B_2 + A_3 B_3$$ If i=j, tensor of rank 0 (scalar) is obtained

Tensor

- Tensor Algebra
  - Symmetry: If the tensor A has the property below, then it is said to be symmetric in the indices j and k. As a consequence of the relation above, the tensor has $n^{r-1}(n+1)/2$ independent components in n-dimensional space. Here, r is the rank of a Tensor.

$$A_{i...j...k...l} = A_{i...k...j...l}$$

  - Anti-symmetry: If the tensor A has the property below, then it is said to be anti-symmetric in the indices j and k. As a consequence of the relation above, the tensor has $n^{r-1}(n-1)/2$ independent components in n-dimensional space. Here, r is the rank of a Tensor.

$$A_{i...j...k...l} = -A_{i...k...j...l}$$

Tensor

- Tensor Operation
  - Gradient: the gradient of tensor R (rank of r) yields a tensor T (rank of r+1).

$$T_{ij....kl} = \frac{\partial R_{ij....k}}{\partial x_t}$$

  - Divergence: the gradient of tensor R (rank of r) yields a tensor T (rank of r-1).

$$T_{i....jl....m} = \frac{\partial R_{i....jkl....m}}{\partial x_k}$$

  - Curl: If R is a tensor of rank r, the curl operation will produce an anti-symmetric tensor of rank (r+1).

$$T_{i....j....kl} = \frac{\partial R_{i....j....k}}{\partial x_l} - \frac{\partial R_{i....l....k}}{\partial x_j}$$

Tensor

- Identity tensor, I: components δ_{ij} are called the Kronecker delta.
  - Kronecker delta: $\delta_{ij} = 0$ ($i \neq j$) or 1 ($i = j$)
  - Multiplication of the components of any vector by yields a change in the index.
    - $a_i\delta_{ij} = a_1\delta_{1j}+a_2\delta_{2j}+a_3\delta_{3j} = a_3\delta_{3j} = a_j$ (if j = 3)
  - In a Cartesian coordinate system, if X denotes the position vector with components x_i, then

$$\frac{\partial x_i}{\partial x_j} = \delta_{ij}$$

- Permutation tensor: components are defined as ε_{ijk} which is 1 if (i, j, k) are cyclic, -1 if (i, j, k) are anti-cyclic, and 0 if any of (i, j, k) are the same.

- Useful identity from Kronecker delta and permutation tensor:

$$\varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

Tensor

- Isotropic Tensors
  - An isotropic tensor is one whose components are invariant with respect to all possible rotations of the coordinate system. That is, there are no preferred directions, and the quantity represented by the tensor is a function of position only and not of orientation.
  - Isotropic tensors of Rank 0: all tensors of rank 0 (i.e., scalars) are isotropic.
  - Isotropic tensors of Rank 1: there are no isotropic tensors of rank 1. That is, vectors are not isotropic, since there are preferred directions.
  - Isotropic tensors of Rank 2: the only isotropic tensors of rank 2 are of the form $\alpha\delta_{ij}$, where α is a scalar and δ_{ij} is the Kronecker delta.
  - Isotropic tensors of Rank 3: the isotropic tensors of rank 3 are of the form $\alpha\varepsilon_{ijk}$, where α is a scalar and ε_{ijk} is a permutation tensor.

Tensor

- Del Operator (∇)
  - Definition

$$\nabla \equiv e_i \frac{\partial}{\partial x_i}$$

  - Divergence of a vector **u**

$$\nabla \cdot u = \frac{\partial u_i}{\partial x_i}$$

  - Curl of **u**

$$\nabla \times u = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j} e_i$$

  - Laplacian of a scalar ϕ

$$\nabla^2 \phi = \nabla \cdot \nabla \phi = \frac{\partial^2 \phi}{\partial x_i \partial x_i}$$

Tensor

- Symmetric and Skew-symmetric Tensor
  - A second order tenor **A** is symmetric if **A** = $\mathbf{A}^T$, or $A_{ij} = A_{ji}$
  - It is said to be skew-symmetric if **A** = -$\mathbf{A}^T$, or $A_{ij} = -A_{ji}$

  - The diagonal components of a skew-symmetric tensor are zero
  - e.g., Kronecker delta (δ_{ij}) is symmetric

    Permutation tensor (ε_{ijk}) is skew-symmetric in any two of indices

  - If **A** is symmetric and **B** is skew-symmetric tensors, respectively, the scalar product of **A** and **B** is zero.

    $A_{ij}B_{ji} = A_{ji}B_{ji} = -A_{ji}B_{ij} = -A_{ij}B_{ji}, \ \therefore\ A_{ij}B_{ji} = 0$

Tensor

- Useful Vector Identities – can be proved with tensor notation
  - $\nabla \times \nabla \phi = 0$
  - $\nabla \cdot (\nabla \times \mathbf{a}) = 0$
  - $\nabla^2 \mathbf{a} = \nabla(\nabla \cdot \mathbf{a}) - \nabla \times (\nabla \times \mathbf{a})$
  - $\nabla \cdot (\phi \mathbf{a}) = \phi(\nabla \cdot \mathbf{a}) + \mathbf{a} \cdot \nabla\phi$
  - $\nabla \times (\phi \mathbf{a}) = \phi(\nabla \times \mathbf{a}) + (\nabla\phi) \times \mathbf{a}$
  - $\nabla(\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a})$
  - $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$
  - $\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}$
  - $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$

Tensor

- Example 1
  - $\nabla \cdot (\nabla \times \mathbf{a}) = 0$

$$\nabla \times a = \varepsilon_{ijk} \frac{\partial a_k}{\partial x_j} e_i$$

$$\therefore \nabla \cdot (\nabla \times a) = \varepsilon_{ijk} \frac{\partial^2 a_k}{\partial x_i \partial x_j} = 0$$

Symmetric in i and j

Skew-symmetric

Tensor

- Example 2
  - $\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b})$

$$a \times b = \varepsilon_{ijk} a_j b_k e_i$$

$$\nabla \cdot (a \times b) = \frac{\partial}{\partial x_i}(\varepsilon_{ijk} a_j b_k) = \varepsilon_{ijk} b_k \frac{\partial a_j}{\partial x_i} + \varepsilon_{ijk} a_j \frac{\partial b_k}{\partial x_i}$$

$$= \varepsilon_{kij} b_k \frac{\partial a_j}{\partial x_i} - \varepsilon_{jik} a_j \frac{\partial b_k}{\partial x_i} = b_k \varepsilon_{kij} \frac{\partial a_j}{\partial x_i} - a_j \varepsilon_{jik} \frac{\partial b_k}{\partial x_i}$$

$$= b \cdot (\nabla \times a) - a \cdot (\nabla \times b)$$

Tensor

- Example 3
  - **a** × (**b** × **c**) = (**a** · **c**)**b** - (**a** · **b**)**c**

$$b \times c = \varepsilon_{klm} b_l c_m e_k$$

$$a \times (b \times c) = \varepsilon_{ijk} a_j (\varepsilon_{klm} b_l c_m) e_i = \varepsilon_{ijk} \varepsilon_{klm} a_j b_l c_m e_i$$

$$= \varepsilon_{kij} \varepsilon_{klm} a_j b_l c_m e_i = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{lj}) a_j b_l c_m e_i$$

$$= (\delta_{il}\delta_{jm} a_j b_l c_m - \delta_{im}\delta_{lj} a_j b_l c_m) e_i$$

$$= (a_j b_i c_j - a_j b_j c_i) e_i$$

$$= a_j c_j b_i e_i - a_j b_j c_i e_i$$

$$= (a \cdot c) b - (a \cdot b) c$$