INVISCID FLOW
Week 2

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Basic Conservation Laws

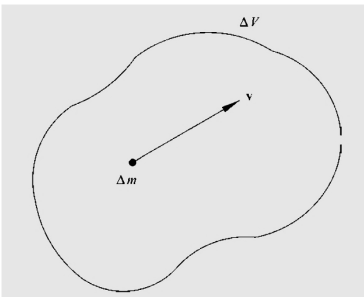
- Two ways in which the conservation laws are derived
  - Statistical approach
  - Continuum approach
- Choice of reference frame
  - Lagrangian
  - Eulerian
- Reynolds transport theorem
- Continuity equation – mass conservation
- Navier-Stokes equation – momentum conservation
- Energy equation – thermal energy conservation

Basic Conservation Laws – Approach

- Approaches to derive governing equations in fluids
  - Molecular approach (statistical method)
    - The motion of molecule follows the laws of dynamics
    - Assumption: the macroscopic phenomena arise from the molecular motion of the molecules
    - The theory attempts to predict the macroscopic behavior of the fluid from the laws of mechanics and probability theory
    - Incomplete for dense gases and for liquids
  - Continuum approach
    - Assumption: the mean-free-path of the molecule is much smaller than the smallest physical length scale of the flow phenomena
    - Most of phenomena encountered in fluid mechanics fall well within the continuum domain and may involve liquids as well as gases

Basic Conservation Laws – Continuum method

- Validity of Continuum Concept
  - Field variables, e.g., density (ρ) and velocity ($\mathbf{u}$) is a function of space and time: $\rho = \rho(\mathbf{x},t)$, $\mathbf{u} = \mathbf{u}(\mathbf{x},t)$
    - defined in terms of the properties of the various molecules that occupy a small volume (ΔV) in the neighborhood of that point



Δm: mass of individual molecule
V: velocity of individual molecule

$$\rho = \lim_{\Delta V \to \varepsilon} \left(\frac{\sum \Delta m}{\Delta V} \right)$$

$$u = \lim_{\Delta V \to \varepsilon} \left(\frac{\sum v \Delta m}{\sum \Delta m} \right)$$

- ε is a volume which is sufficiently small that $\varepsilon^{1/3}$ is small compared with the smallest significant length scale in the flow field but is sufficiently large that it contains a large number of molecules.

Basic Conservation Laws – Continuum method

- A sufficient condition, not a necessary condition, for the valid continuum approach

$$\frac{1}{n} << \varepsilon << L^3$$

  - n: # of molecules per unit volume
  - L: smallest length scale with a significant physical meaning (macroscopic length scale)

- Example
  - A cube (2 μm × 2 μm × 2 μm) of gas (at normal temperature and pressure) contains about 2×10^8 molecules and 2×10^{11} molecules for a liquid
  - Continuum condition is readily met in the vast majority of flow phenomena encountered in physics and engineering

- In continuum approach: the deformation should be proportional to the stress!

Basic Conservation Laws – Reference Frame

- Choice of the Reference Frame
  - Eulerian
  - Lagrangian

- Eulerian
  - Independent variables: x, y, z and t (time)
  - Focusing on the fluid which passes through a **control volume**, fixed in space
- Lagrangian
  - Independent variables: x_0, y_0, z_0 and t (location of a fluid element at t_0)
  - Attention is fixed on a particular mass (**material volume**) of fluid as it flows
- Lagrangian coordinate system tends to be used to derive the basic conservation equations; but the eulerian system is the preferred for solving the majority of problems

Basic Conservation Laws – Reference Frame

- Material Derivative (Total Derivative)
  - Let α be any variable in a fluid field;
  - For a short time (δt), the change in α is

$$\delta\alpha = \frac{\partial\alpha}{\partial t}\delta t + \frac{\partial\alpha}{\partial x}\delta x + \frac{\partial\alpha}{\partial y}\delta y + \frac{\partial\alpha}{\partial z}\delta z$$

  - In Lagrangian coordinate, where x, y, z is the function of time

$$\frac{\delta\alpha}{\delta t} = \frac{\partial\alpha}{\partial t} + \frac{\partial\alpha}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial\alpha}{\partial y}\frac{\delta y}{\delta t} + \frac{\partial\alpha}{\partial z}\frac{\delta z}{\delta t}$$

  - As $\delta t \rightarrow 0$, we have

$$\frac{D\alpha}{Dt} = \frac{\partial\alpha}{\partial t} + u\frac{\partial\alpha}{\partial x} + v\frac{\partial\alpha}{\partial y} + w\frac{\partial\alpha}{\partial z}$$

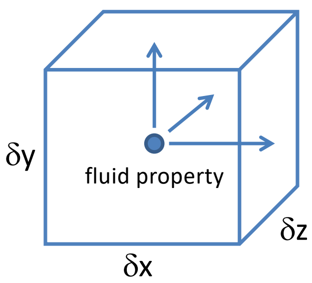
$$= \frac{\partial\alpha}{\partial t} + (u\cdot\nabla)\alpha = \frac{\partial\alpha}{\partial t} + u_k\frac{\partial\alpha}{\partial x_k}$$

→ Expresses the lagrangian rate of change $D\alpha/Dt$ of α for a given fluid element in terms of the eulerian derivatives $\partial\alpha/\partial t$ and $\partial\alpha/\partial x_k$.

Local derivative + Convective derivative

Basic Conservation Laws – Control Volume

- Control Volumes



- Fluid property is expanded in a Taylor series to give expressions for that property at each face of the control volume

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \ldots.$$

- Then, the conservation law is applied with (δx, δy and $\delta z \rightarrow 0$): differential equations

- Arbitrary Shaped CV
  - Each conservation principle is applied to an integral over the control volume

$$\int_V L\alpha dV = 0$$

L: differential operator
α: fluid property
V: volume of arbitrary CV

Basic Conservation Laws – Control Volume

- Since the CV is arbitrary, the only way to solve this equation is to set $L\alpha = 0$, which gives the differential equation of the conservation law
- Needless to say that the results obtained by the two methods are identical

Basic Conservation Laws – Reynolds Transport Theorem

- What we want to do is
  - Derive the basic equations from the conservation laws by using the continuum concept and following an arbitrarily shaped control volume in a lagrangian frame of reference
  - Material derivatives of volume integrals
  - It is necessary to transform such terms into equivalent expressions involving volume integrals of eulerian derivatives: Reynolds transport theorem.
- Consider a specific mass of fluid and follow it for a short period of time (δt) as it flows. And, let α be any property of the fluid.
  - quantity α will be a function of t only as the control volume moves with the fluid: $\alpha = \alpha(t)$

Basic Conservation Laws – Reynolds Transport Theorem

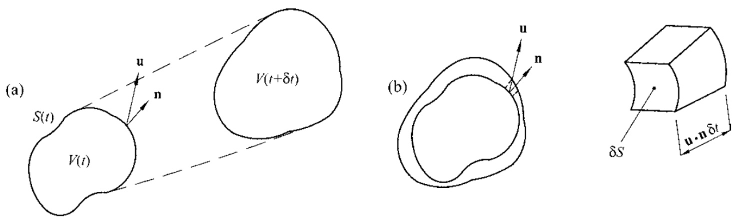
- Rate of change of the integral of α

$$\frac{D}{Dt}\int_{V(t)}\alpha(t)dV = \lim_{\delta t\to 0}\left\{\frac{1}{\delta t}\left[\int_{V(t+\delta t)}\alpha(t+\delta t)dV - \int_{V(t)}\alpha(t)dV\right]\right\}$$

$$= \lim_{\delta t\to 0}\left\{\frac{1}{\delta t}\left[\int_{V(t+\delta t)}\alpha(t+\delta t)dV - \int_{V(t)}\alpha(t+\delta t)dV + \int_{V(t)}\alpha(t+\delta t)dV - \int_{V(t)}\alpha(t)dV\right]\right\}$$

$$= \lim_{\delta t\to 0}\left\{\frac{1}{\delta t}\left[\int_{V(t+\delta t)-V(t)}\alpha(t+\delta t)dV\right]\right\} + \int_{V(t)}\frac{\partial\alpha}{\partial t}dV$$

Let's look into this term!



- The perpendicular distance from any point on the inner surface to the outer surface is u·nδt, so that an element of surface area δS will correspond to an element of volume change δV
- δV = u·nδtδS.

Basic Conservation Laws – Reynolds Transport Theorem

$$\lim_{\delta t\to 0}\left\{\frac{1}{\delta t}\left[\int_{V(t+\delta t)-V(t)}\alpha(t+\delta t)dV\right]\right\} = \lim_{\delta t\to 0}\left\{\left[\int_{S(t)}\alpha(t+\delta t)u\cdot n dS\right]\right\}$$

$$\frac{D}{Dt}\int_{V(t)}\alpha(t)dV = \int_{S(t)}\alpha(t)u\cdot n dS + \int_{V(t)}\frac{\partial\alpha}{\partial t}dV$$

- Now, the lagrangian derivative of a volume integral has been converted into a surface integral and a volume integral in which the integrands contain only eulerian derivatives

- On the other hand, from Gauss Theorem (or divergence Theorem),

$$\int_{S(t)}\alpha(t)u\cdot n dS = \int_{V(t)}\nabla\cdot(\alpha u)dV$$

Gauss Theorem: outward flux of a vector field through a closed surface is equal to the volume integral of the divergence over the region inside the surface. Intuitively, it states that *the sum of all sources minus the sum of all sinks gives the net flow out of a region*.

Basic Conservation Laws – Reynolds Transport Theorem

$$\frac{D}{Dt}\int_V \alpha dV = \int_V \left[\frac{\partial \alpha}{\partial t} + \nabla\cdot(\alpha u)\right] dV = \int_V \left[\frac{\partial \alpha}{\partial t} + \frac{\partial}{\partial x_k}(\alpha u_k)\right] dV$$

- Now, the lagrangian derivative of a volume integral of a given mass has been related to a volume integral in which the integrand has eulerian derivatives only.

Basic Conservation Laws – Conservation of Mass

- Consider an arbitrarily chosen, specific mass of fluid (volume V)
- If this given fluid mass is followed as it flows, its size and shape will be observed to change but its mass will remain unchanged: Mass Conservation
- Mathematically, lagrangian derivative D/Dt of the mass of fluid contained in V is equal to zero

$$\frac{D}{Dt}\int_V \rho dV = 0$$

- Using Reynolds Transport Theorem,

$$\int_V \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k)\right] dV = 0$$

- Since V is arbitrarily chosen,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k) = 0$$ **Continuity Equation**

Basic Conservation Laws – Conservation of Mass

- In incompressible flow, where the variation of density of the fluid is ignored, the density will remain constant as well as the mass

$$\frac{D\rho}{Dt} = 0$$

- To use this,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k) = \frac{\partial \rho}{\partial t} + u_k \frac{\partial \rho}{\partial x_k} + \rho \frac{\partial u_k}{\partial x_k} = 0$$

$$\therefore \frac{D\rho}{Dt} + \rho \frac{\partial u_k}{\partial x_k} = 0$$

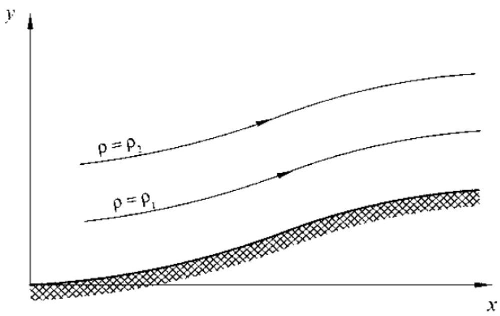
Lagrangian+Eulerian → not useful for solving fluid-mechanics problem, but frequently used form due to its simplicity

- In incompressible flow, the continuity equation becomes

$$\frac{\partial u_k}{\partial x_k} = 0$$

Also valid for stratified fluid in ocean or atmosphere

Basic Conservation Laws – Conservation of Mass



- In a stratified fluid, ρ is not constant everywhere, so that ∂ρ/∂x ≠ 0 and ∂ρ/∂y ≠ 0.
- A fluid particle that follows the lines $\rho = \rho_1$ or $\rho = \rho_2$ will have its density remain fixed so that Dρ/Dt = 0, in the Lagrangian viewpoint.

- Most of the time, however, we deal with the incompressible flow

Basic Conservation Laws – Conservation of Momentum

- Newton's Second Law
  - the rate at which the momentum of the fluid mass is changing is equal to the net external force acting on the mass
    - Body force: gravitational, electromagnetic

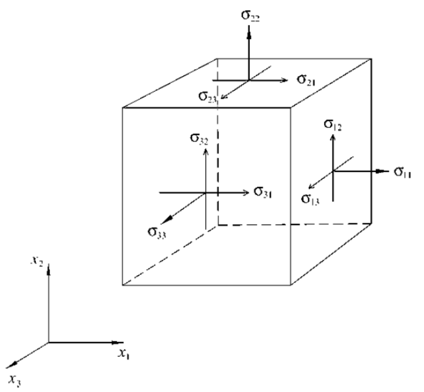
$$\int_V \rho f dV \quad \text{(f: body force per unit mass)}$$

    - Surface force: pressure, viscous stress

$$\int_S P dS \quad \text{(P: pressure force per unit area)}$$

  - $$\frac{D}{Dt}\int_V \rho u dV = \int_S P dS + \int_V \rho f dV$$ General form of momentum conservation

Basic Conservation Laws – Conservation of Momentum



- The stress acting on any point has nine components and can be represented by σ_{ij} (i, j = 1, 2, 3). That is, it is acting on the x_i-plane and the second subscript indicates that it acts in the x_j-direction.
- Rank 2 tensor
- Consider surface pressure force, P
- At x_1-plane, for example, $P_1 = \sigma_{11}n_1$, $P_2 = \sigma_{12}n_1$, $P_3 = \sigma_{13}n_1$ (n_1 is unit normal vectors)
- Then, $P_j = \sigma_{ij}n_i$

$$\therefore \frac{D}{Dt}\int_V \rho u_j dV = \int_S \sigma_{ij} n_i dS + \int_V \rho f_j V$$

Basic Conservation Laws – Conservation of Momentum

- Using Reynolds Transfer Theorem,

$$\frac{D}{Dt}\int_V \rho u_j dV = \int_V \left[\frac{\partial}{\partial t}(\rho u_j) + \frac{\partial}{\partial x_k}(\rho u_j u_k)\right]$$

- Using Gauss Theorem,

$$\int_S \sigma_{ij} n_i dS = \int_V \frac{\partial \sigma_{ij}}{\partial x_i} dV$$

- Therefore, in the form of tensor, the momentum conservation equation becomes

$$\frac{\partial}{\partial t}(\rho u_j) + \frac{\partial}{\partial x_k}(\rho u_j u_k) = \frac{\partial \sigma_{ij}}{\partial x_i} + \rho f_j$$

$$\rho \frac{\partial u_j}{\partial t} + \underline{u_j \frac{\partial \rho}{\partial t} + u_j \frac{\partial}{\partial x_k}(\rho u_k)} + \rho u_k \frac{\partial u_j}{\partial x_k} = \frac{\partial \sigma_{ij}}{\partial x_i} + \rho f_j$$

Zero from a continuity equation

Basic Conservation Laws – Conservation of Momentum

- So, we have

$$\boxed{\boxed{\rho \frac{\partial u_j}{\partial t}} + \boxed{\rho u_k \frac{\partial u_j}{\partial x_k}}} = \frac{\partial \sigma_{ij}}{\partial x_i} + \rho f_j$$

rate of change of momentum of a unit volume of the fluid (or the inertia force per unit volume)

temporal acceleration term

convective acceleration (nonlinear)

Gradient of surface shear stress

Body force

Convection?