INVISCID FLOW
Week 3

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Basic Conservation Laws – Conservation of Momentum

- Conservation of Angular Momentum
  - the rate at which the angular momentum of the fluid mass is changing is equal to the net external moment acting on the mass

$$\frac{D}{Dt}\int_V \{r\times(\rho u)\}dV = \int_S (r\times P)dS + \int_V \{r\times(\rho f)\}dV \quad (*)$$

- Another representation of Reynolds Transport Theorem

$$\frac{D}{Dt}\int_V \rho\beta dV = \int_V \left[\frac{\partial(\rho\beta)}{\partial t}+\frac{\partial(\rho\beta u_k)}{\partial x_k}\right]dV = \int_V \left[\beta\frac{\partial\rho}{\partial t}+\rho\frac{\partial\beta}{\partial t}+\beta u_k\frac{\partial\rho}{\partial x_k}+\rho u_k\frac{\partial\beta}{\partial x_k}+\rho\beta\frac{\partial u_k}{\partial x_k}\right]dV$$

$$= \int_V \left[\beta\left(\frac{\partial\rho}{\partial t}+u_k\frac{\partial\rho}{\partial x_k}+\rho\frac{\partial u_k}{\partial x_k}\right)+\rho\frac{\partial\beta}{\partial t}+\rho u_k\frac{\partial\beta}{\partial x_k}\right]dV = \int_V \left[\rho\left(\frac{\partial\beta}{\partial t}+u_k\frac{\partial\beta}{\partial x_k}\right)\right]dV$$

$$= \int_V \left(\rho\frac{D\beta}{Dt}\right)dV$$

Basic Conservation Laws – Conservation of Momentum

- LHS of angular mtm conservation eqn

$$\frac{D}{Dt}\int_V \{r\times(\rho u)\}dV = \int_V \rho\frac{D(r\times u)}{Dt}dV = \int_V \rho\left[\frac{\partial(r\times u)}{\partial t}+u_k\frac{\partial(r\times u)}{\partial x_k}\right]dV$$

$$=\int_V \rho\left[\frac{\partial r}{\partial t}\times u + r\times\frac{\partial u}{\partial t}+u_k\frac{\partial r}{\partial x_k}\times u + u_k r\times\frac{\partial u}{\partial x_k}\right]dV$$

$$=\int_V \rho\left[r\times\left(\frac{\partial u}{\partial t}+u_k\frac{\partial u}{\partial x_k}\right)+u_k\delta_{ik}e_i\times u\right]dV$$

$$=\int_V \left[r\times\rho\left(\frac{\partial u}{\partial t}+u_k\frac{\partial u}{\partial x_k}\right)+\rho u_i e_i\times u\right]dV$$

$$=\int_V \left[r\times\rho\frac{Du}{Dt}\right]dV$$

Basic Conservation Laws – Conservation of Momentum

- First term in RHS of angular mtm conservation eqn

$$\int_S (r\times P)dS = \int_S \varepsilon_{ijk}r_j P_k e_i dS = \int_S \varepsilon_{ijk}r_j\sigma_{pk}n_p e_i dS$$

$$\Big\downarrow \text{Divergence Theorem}$$

$$=\int_V \frac{\partial(\varepsilon_{ijk}r_j\sigma_{pk}e_i)}{\partial x_p}dV$$

$$=\int_V \varepsilon_{ijk}\left(\frac{\partial r_j}{\partial x_p}\sigma_{pk}+r_j\frac{\partial\sigma_{pk}}{\partial x_p}\right)e_i dV = \int_V \varepsilon_{ijk}\left(\delta_{jp}\sigma_{pk}+r_j\frac{\partial\sigma_{pk}}{\partial x_p}\right)e_i dV$$

- Clear the eqn (*), then we get

$$\int_V \left(r\times\rho f+\varepsilon_{ijk}r_j\frac{\partial\sigma_{pk}}{\partial x_p}e_i - r\times\rho\frac{Du}{Dt}\right)dV+\int_V \varepsilon_{ijk}\sigma_{jk}e_i dV = 0$$

$$\int_V \left(\varepsilon_{ijk}r_j\rho f_k+\varepsilon_{ijk}r_j\frac{\partial\sigma_{pk}}{\partial x_p}-\varepsilon_{ijk}r_j\times\rho\frac{Du_k}{Dt}\right)e_i dV+\int_V \varepsilon_{ijk}\sigma_{jk}e_i dV = 0$$

Basic Conservation Laws – Conservation of Momentum

$$\int_V \varepsilon_{ijk}\sigma_{jk}e_i dV = 0$$

$$\varepsilon_{ijk}\sigma_{jk}e_i = 0$$

$$\varepsilon_{ipq}\varepsilon_{ijk}\sigma_{jk} = 0$$

$$(\delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj})\sigma_{jk} = 0$$

$$\sigma_{jk} = \sigma_{kj}$$

- So, we can prove the symmetry of shear stress at a point using the conservation of angular momentum

Basic Conservation Laws – Conservation of Energy

- 1st Law of Thermodynamics
  - dE = δW + δQ (change in internal energy = total work done on the system + heat added to the system)
  - Valid for equilibrium (fixed) state

- In case of fluid flow, not in equilibrium
  - $e_T = e + 1/2 u \cdot u$ (total energy per unit mass = internal energy per unit mass + kinetic energy per unit mass)

- Modified 1st Law of Thermodynamics
  - rate of change of the total energy (internal+ kinetic) of the fluid as it flows is equal to the sum of the rate at which work is being done on the fluid by external forces and the rate at which heat is being added by conduction

Basic Conservation Laws – Conservation of Energy

$$\underbrace{\frac{D}{Dt}\int_V\left(\rho e+\frac{1}{2}\rho u\cdot u\right)dV}_{\text{Rate of change of total energy of the system}}=\underbrace{\int_S u\cdot P dS+\int_V u\cdot\rho f dV}_{\text{Rate of work done on the fluid}}-\underbrace{\int_S q\cdot n dS}_{\text{Rate of heat leaving the fluid}}$$

$$\frac{D}{Dt}\int_V\left(\rho e+\frac{1}{2}\rho u\cdot u\right)dV=\int_V\left\{\frac{\partial}{\partial t}\left(\rho e+\frac{1}{2}\rho u\cdot u\right)+\frac{\partial}{\partial x_k}\left[\left(\rho e+\frac{1}{2}\rho u\cdot u\right)u_k\right]\right\}dV$$

Reynolds Transport Theorem

$$\int_S u\cdot P dS=\int_S u_j\sigma_{ij}n_i dS=\int_V\frac{\partial}{\partial x_i}(u_j\sigma_{ij})dV$$ Green's Theorem

$$\int_S q\cdot n dS=\int_S q_j n_j dS=\int_V\frac{\partial q_j}{\partial x_j}dV$$ Green's Theorem

$$\therefore\frac{\partial}{\partial t}\left(\rho e+\frac{1}{2}\rho u_j u_j\right)+\frac{\partial}{\partial x_k}\left[\left(\rho e+\frac{1}{2}\rho u_j u_j\right)u_k\right]=\frac{\partial(u_j\sigma_{ij})}{\partial x_i}+u_j\rho f_j-\frac{\partial q_j}{\partial x_j}$$

Basic Conservation Laws – Conservation of Energy

$$\frac{\partial}{\partial t}\left(\rho e+\frac{1}{2}\rho u_j u_j\right)+\frac{\partial}{\partial x_k}\left[\left(\rho e+\frac{1}{2}\rho u_j u_j\right)u_k\right]=\frac{\partial(u_j\sigma_{ij})}{\partial x_i}+u_j\rho f_j-\frac{\partial q_j}{\partial x_j}$$

- Let's simply this eqn further (LHS first)

$$\frac{\partial}{\partial t}\left(\rho e+\frac{1}{2}\rho u_j u_j\right)=\rho\frac{\partial e}{\partial t}+e\frac{\partial\rho}{\partial t}+\rho\frac{\partial}{\partial t}(\tfrac{1}{2}u_j u_j)+\tfrac{1}{2}u_j u_j\frac{\partial\rho}{\partial t}$$

$$\frac{\partial}{\partial x_k}\left[\left(\rho e+\frac{1}{2}\rho u_j u_j\right)u_k\right]=e\frac{\partial}{\partial x_k}(\rho u_k)+\rho u_k\frac{\partial e}{\partial x_k}+\tfrac{1}{2}u_j u_j\frac{\partial}{\partial x_k}(\rho u_k)+\rho u_k\frac{\partial}{\partial x_k}(\tfrac{1}{2}u_j u_j)$$

$$=-e\frac{\partial\rho}{\partial t}+\rho u_k\frac{\partial e}{\partial x_k}-\tfrac{1}{2}u_j u_j\frac{\partial\rho}{\partial t}+\rho u_k\frac{\partial}{\partial x_k}(\tfrac{1}{2}u_j u_j)$$

$$\boxed{\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x_k}(\rho u_k)=0}$$

Continuity Eqn

- Combining the two above, then we have

$$\frac{\partial}{\partial t}\left(\rho e+\frac{1}{2}\rho u_j u_j\right)+\frac{\partial}{\partial x_k}\left[\left(\rho e+\frac{1}{2}\rho u_j u_j\right)u_k\right]=\rho\frac{\partial e}{\partial t}+\rho u_k\frac{\partial e}{\partial x_k}+\rho u_j\frac{\partial u_j}{\partial t}+\rho u_j u_k\frac{\partial u_j}{\partial x_k}$$

Basic Conservation Laws – Conservation of Energy

$$\frac{\partial}{\partial t}\left(\rho e+\frac{1}{2}\rho u_j u_j\right)+\frac{\partial}{\partial x_k}\left[\left(\rho e+\frac{1}{2}\rho u_j u_j\right)u_k\right]=\frac{\partial(u_j\sigma_{ij})}{\partial x_i}+u_j\rho f_j-\frac{\partial q_j}{\partial x_j}$$

- Now, the first term at RHS becomes

$$\frac{\partial}{\partial x_i}(u_j\sigma_{ij})=u_j\frac{\partial \sigma_{ij}}{\partial x_i}+\sigma_{ij}\frac{\partial u_j}{\partial x_i}$$

- So, the basic energy conservation eqn becomes

$$\rho\frac{\partial e}{\partial t}+\rho u_k\frac{\partial e}{\partial x_k}+\underline{\rho u_j\frac{\partial u_j}{\partial t}+\rho u_j u_k\frac{\partial u_j}{\partial x_k}}=\underline{u_j\frac{\partial \sigma_{ij}}{\partial x_i}}+\sigma_{ij}\frac{\partial u_j}{\partial x_i}+\underline{u_j\rho f_j}-\frac{\partial q_j}{\partial x_j}$$

Deleted terms denote the mechanical energy balance (rate of work done)

$$\boxed{\rho\frac{\partial u_j}{\partial t}+\rho u_k\frac{\partial u_j}{\partial x_k}=\frac{\partial \sigma_{ij}}{\partial x_i}+\rho f_j}$$

mtm conserv. Eqn

$$\rho\frac{\partial e}{\partial t}+\rho u_k\frac{\partial e}{\partial x_k}=\sigma_{ij}\frac{\partial u_j}{\partial x_i}-\frac{\partial q_j}{\partial x_j}$$

Balance of thermal energy
Or, simply energy eqn

Basic Conservation Laws – Conservation of Energy

$$\rho\frac{\partial e}{\partial t}+\rho u_k\frac{\partial e}{\partial x_k}=\sigma_{ij}\frac{\partial u_j}{\partial x_i}-\frac{\partial q_j}{\partial x_j}$$

- The LHS: the rate of change of internal energy
  - the first term: temporal change
  - the second term: local convective changes, caused by the fluid flowing from one area to another.
- The RHS: the cause of the change in internal energy
  - the first term: reversible and/or irreversible conversion of mechanical energy into thermal energy due to surface stresses
  - the second term: the rate at which heat is being added by conduction from outside

Basic Conservation Laws – Further Discussion

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k) = 0$$

$$\rho \frac{\partial u_j}{\partial t} + \rho u_k \frac{\partial u_j}{\partial x_k} = \frac{\partial \sigma_{ij}}{\partial x_i} + \rho f_j$$

$$\rho \frac{\partial e}{\partial t} + \rho u_k \frac{\partial e}{\partial x_k} = \sigma_{ij} \frac{\partial u_j}{\partial x_i} - \frac{\partial q_j}{\partial x_j}$$

- The continuity and the energy equations are scalar equations
- The momentum equation is a vector equation which represents three scalar equations
- Two state equations may be added, i.e., $\rho = \rho(P, T)$ and $e = e(P, T)$
- Unknowns: ρ, e, u_j, q_j, σ_{ij}, total 17 (14 if we consider the symmetry of σ_{ij})
- To obtain a complete set of equations, σ_{ij} and q_j must be further specified. This leads to the so-called constitutive equations in which the stress tensor is related to the deformation tensor (Navier-Stoke Eqn) and the heat-flux vector is related to temperature gradients (Fourier's Law).

Last Lecture

- Proved the symmetry of stress tensor from angular momentum conservation
- From the modified 1st law of thermodynamics, thermal energy conservation equation was derived
- Brief summary of governing equations derived so far

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k) = 0$$

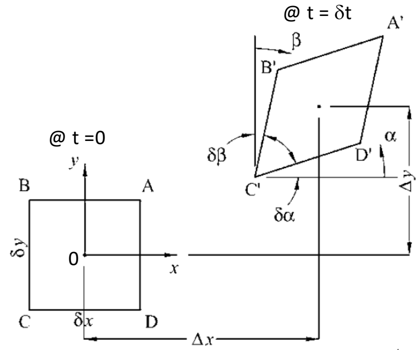
$$\rho \frac{\partial u_j}{\partial t} + \rho u_k \frac{\partial u_j}{\partial x_k} = \frac{\partial \sigma_{ij}}{\partial x_i} + \rho f_j$$

$$\rho \frac{\partial e}{\partial t} + \rho u_k \frac{\partial e}{\partial x_k} = \sigma_{ij} \frac{\partial u_j}{\partial x_i} - \frac{\partial q_j}{\partial x_j}$$

- Solution for homework #1 has been uploaded.

Basic Conservation Laws – Rotation and Rate of Shear

- Let's identify the tensor quantities that represent rotation and shearing of the fluid element



$$\Delta x = \int_0^{\delta t} u[x(t), y(t)]dt$$

$$= \int_0^{\delta t} \left[u(0,0) + x(t)\frac{\partial u(0,0)}{\partial x} + y(t)\frac{\partial u(0,0)}{\partial y} + ... \right] dt$$

$$= u(0,0)\delta t + \int_0^{\delta t} \left[x(t)\frac{\partial u(0,0)}{\partial x} + y(t)\frac{\partial u(0,0)}{\partial y} + ... \right] dt$$

$$\Delta y = v(0,0)\delta t + \int_0^{\delta t} \left[x(t)\frac{\partial v(0,0)}{\partial x} + y(t)\frac{\partial v(0,0)}{\partial y} + ... \right] dt$$

$$\delta\alpha = \tan^{-1}\left(\frac{y \text{ component of } D'C'}{x \text{ component of } D'C'}\right)$$

$$= \tan^{-1}\left(\frac{[v(\tfrac{1}{2}\delta x, -\tfrac{1}{2}\delta y)\delta t + ...] - [v(-\tfrac{1}{2}\delta x, -\tfrac{1}{2}\delta y)\delta t + ...]}{\delta x + ...}\right)$$

Basic Conservation Laws – Rotation and Rate of Shear

- Expanding v in a Taylor series about (0,0);

$$\delta\alpha = \tan^{-1}\left\{ \frac{[v(0,0) + \tfrac{1}{2}\delta x(\partial v/\partial x)(0,0) - \tfrac{1}{2}\delta y(\partial v/\partial y)(0,0) + ...]\delta t}{\delta x + ...} + \frac{-[v(0,0) - \tfrac{1}{2}\delta x(\partial v/\partial x)(0,0) - \tfrac{1}{2}\delta y(\partial v/\partial y)(0,0) + ...]\delta t}{\delta x + ...} \right\}$$

$$= \tan^{-1}\left\{ \frac{[\delta x(\partial v/\partial x)(0,0) + ...]\delta t}{\delta x + ...} \right\}$$

$$= \tan^{-1}\left\{ \frac{[(\partial v/\partial x)(0,0) + ...]\delta t}{1 + ...} \right\}$$

$$= \tan^{-1}\left\{ \left[\frac{\partial v}{\partial x}(0,0) + ... \right] \delta t \right\} \sim \left[\frac{\partial v}{\partial x}(0,0) + ... \right] \delta t$$

$$\therefore \frac{\delta\alpha}{\delta t} = \frac{\partial\alpha}{\partial t} = \dot{\alpha} = \frac{\partial v}{\partial x}(0,0)$$

$$\dot{\beta} = \frac{\partial u}{\partial y}(0,0)$$

Note that α is measured counterclockwise and β is measured clockwise!

Basic Conservation Laws – Rotation and Rate of Shear

- Then, the rate of clockwise rotation of the fluid element about its centroid is given by

$$\frac{1}{2}(\dot{\beta} - \dot{\alpha}) = \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$$

- Likewise, the shearing can be analyzed by the motion of B'C' and D'C' (approaching each other)

$$\frac{1}{2}(\dot{\beta} + \dot{\alpha}) = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

- Extending to the general coordinate system

$$\text{Rate of rotation} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$ Anti-symmetric

$$\text{Rate of shearing} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$ Symmetric

Basic Conservation Laws – Rotation and Rate of Shear

- These two tensors are actually the anti-symmetric and symmetric parts of another tensor called the deformation-rate tensor, e_{ij}

$$e_{ij} = \frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Basic Conservation Laws – Constitutive Equations

- Now, we will find a relation between σ_{ij} and e_{kl}
- In a Newtonian Fluid (air, water….. most of fluid),
  - (1) When the fluid is at rest, the stress is hydrostatic and the pressure exerted by the fluid is a thermodynamic pressure

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$$

thermodynamic pressure   Shear-stress tensor

  - (2) Stress tensor σ_{ij} is linearly related to the deformation-rate tensor e_{kl} and depends only on it
  - (3) Since there is no shearing action in a solid-body rotation of the fluid, no shear stresses will act during such a motion
  - (4) There are not preferred directions in the fluid, so that the fluid properties are point functions

Basic Conservation Laws – Constitutive Equations

- From condition (2), $\tau_{ij} \propto e_{kl}$
- Each nine components in τ_{ij} should be a linear combination of the nine elements of e_{kl}
- So, we need a tensor of rank 4 to have a general form of

$$\tau_{ij} = \alpha_{ijkl}\frac{\partial u_k}{\partial x_l}$$

- Now, to satisfy condition (3), i.e., solid body rotation, $\tau_{ij} = 0$

$$\tau_{ij} = \beta'_{ijkl}\frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} - \frac{\partial u_l}{\partial x_k}\right) + \beta_{ijkl}\frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) = 0$$

  - For solid-body rotation, first term is not zero definitely. Therefore we have

$$\beta'_{ijkl} = 0, \beta_{ijkl}\frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) = 0$$

Basic Conservation Laws – Constitutive Equations

$$\tau_{ij} = \beta_{ijkl} \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right)$$

- Still, 81 components of β_{ijkl} is still unknown.
- Let's consider condition (4), i.e., condition of isotropy: results obtained should be independent of the orientation of the coordinate system
- $\beta_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) + \gamma(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})$: most general form of isotropic tensor of rank 4 (Appendix B, Currie)
- From condition (3), β_{ijkl} should be symmetric → $\gamma = 0$

$$\tau_{ij} = \frac{1}{2}[\lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right)$$

Basic Conservation Laws – Constitutive Equations

$$\frac{1}{2}\lambda\delta_{ij}\delta_{kl}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right) = \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k}$$

$$\frac{1}{2}\mu\delta_{ik}\delta_{jl}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right) = \frac{1}{2}\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

$$\frac{1}{2}\mu\delta_{il}\delta_{jk}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right) = \frac{1}{2}\mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right)$$

$$\tau_{ij} = \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right)$$

$$\therefore \sigma_{ij} = -p\delta_{ij} + \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right)$$

- Thus, the nine elements of the stress tensor σ_{ij} have now been expressed in terms of the pressure and the velocity gradients and two coefficients λ and μ.
- λ and μ must be determined empirically

- The second constitutive relation involves q_j and conduction: Fourier's law

$$q_j = -k\frac{\partial T}{\partial x_j}$$ k: thermal conductivity