INVISCID FLOW
Week 6

Prof. Hyungmin Park
Multiphase Flow and Flow Visualization Lab.

Department of Mechanical and Aerospace Engineering
Seoul National University

Ideal-Fluid Flow: Introduction

- Ideal-Fluid Flow
  - Incompressible
  - Inviscid
  - That is, the effect of inertia is dominant.
  - The study of ideal-fluid flows is frequently referred to as hydrodynamics.

Governing Equations

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial u_j}{\partial t} + u_k \frac{\partial u_j}{\partial x_k} = -\frac{1}{\rho}\frac{\partial p}{\partial x_j} + f_j$$

In tensor forms

or

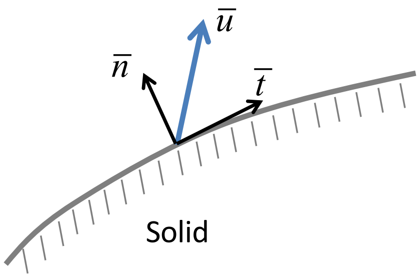
$$\nabla \cdot u = 0$$

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho}\nabla p + f$$

In vector forms

Ideal-Fluid Flow: Introduction

- Boundary conditions (B.C's) for Euler equation
  - Euler equation is one order lower than the Navier-Stokes eqn
    → B.C's for N-S equation should be relaxed for Euler eqn.
  - The effects of viscosity is manifest by no-slip at solid boundary.
    → the condition of no tangential slip at boundaries is dropped.



$$\overline{u}\cdot\overline{n} = \overline{U}\cdot\overline{n}$$
$$\overline{u}\cdot\overline{t}: \text{ unspecified}$$

No-slip condition on a solid boundary → the surface of the body must be a streamline
How about the B.C far away from the body?

$$\overline{u} = 0 \text{ as } \overline{x} \to \infty \text{ (same as viscous flow)}$$

Ideal-Fluid Flow: Introduction

- Potential Flow
  - If an ideal-fluid flow around a body is **irrotational**,
  - the flow will remain irrotational even near the body (Kelvin's Theorem)

$$\omega = \nabla \times u = 0 \quad \text{(everywhere)}$$
$$\nabla \times \nabla\phi = 0 \quad (\phi: \text{ any scalar function})$$
$$\therefore u = \nabla\phi \quad (\phi: \text{ velocity potential})$$

  - Equation for velocity potential
    - From the continuity equation

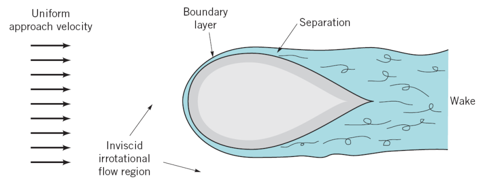
$$\nabla \cdot u = \nabla^2\phi = 0 \qquad \underline{\text{Laplace eqn}}$$

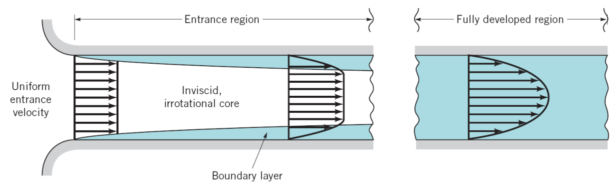
    - We can obtain velocity fields without solving equations of motion (i.e., Euler equation)
    - However, to get the information of pressure, we should solve equation of motion: e.g., Bernoulli equation

Ideal-Fluid Flow: Introduction

- Linear equation: principle of superposition is satisfied
  - If ϕ_1 and ϕ_2 are solutions of $\nabla^2\phi = 0$, their linear combinations (e.g., $a\phi_1 + b\phi_2$) are also the solutions of the equation.

- Obviously, irrotational fluids differ from real fluids in certain important respects —“Dry water” and “Wet water” (Richard Feynman)
- Inviscid flow solutions are still useful for modeling many flow phenomena but at the same time, their deficiencies help us to understand the importance of viscosity in real “wet” fluids.





2D Potential Flows – Stream Function

- Velocity potential (ϕ)
  - automatically satisfies the condition of irrotationality
  - solution of a Laplace equation

- A second function (ψ) may be defined in a complementary way
  - automatically satisfies the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$
in a 2D, Cartesian coordinates

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$
stream function, valid for rotational and irrotational flows

  - and satisfies the condition of irrotationality as well
    - in two-dimensional flow

$$\omega_1 = \omega_2 = 0, \ \omega_3 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \qquad \therefore \ \omega_3 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

2D Potential Flows – Stream Function

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=\frac{\partial}{\partial x}(-\frac{\partial \psi}{\partial x})-\frac{\partial}{\partial y}(\frac{\partial \psi}{\partial y})=\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}=\nabla^2\psi=0$$

- Stream function, like the velocity potential, should satisfy Laplace equation.

1. Flow lines of constant ψ are streamlines
   – (Proof)

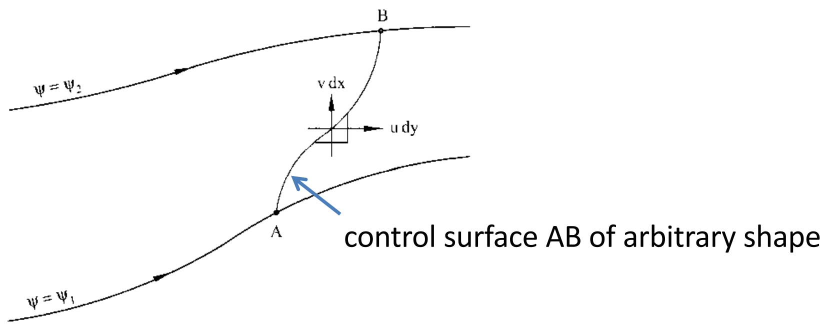
$$\psi=\psi(x,y)$$

$$d\psi=\frac{\partial \psi}{\partial x}dx+\frac{\partial \psi}{\partial y}dy=-vdx+udy=0\ (\because \psi=\ \text{constant})$$

$$\therefore \left.\frac{dy}{dx}\right)_{\psi}=\frac{v}{u}$$

2D Potential Flows – Stream Function

2. Difference of values between two streamlines gives the volume of fluid which is flowing between these two streamlines



total volume flow rate between streamlines per unit depth $Q=\int_A^B udy-\int_A^B vdx$

integrating $d\psi=-vdx+udy$

$$\psi_2-\psi_1=-\int_A^B vdx+\int_A^B udy=Q$$

2D Potential Flows – Stream Function

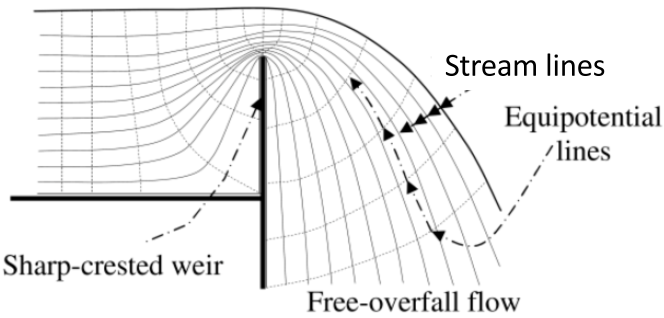
3. Streamlines (ψ = constant) and the lines of ϕ = constant (equipotential lines) are orthogonal to each other

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = udx + vdy = 0 \qquad \text{(constant } \phi\text{)}$$

$$\left.\frac{dy}{dx}\right)_{\phi} = -\frac{u}{v}$$

$$\therefore \left.\frac{dy}{dx}\right)_{\phi}\left.\frac{dy}{dx}\right)_{\psi} = -1$$

orthogonal



example: flow over a free over-fall weir
(Kabiri-Samani *et al*. Int. J. Hydraulic Eng., 2012)

Weir – kind of a small dam

2D Potential Flows – Complex Potential/Velocity

- Complex variables theory
  - Analytic Function
    - A function F(z) of the complex variable z = x + iy is said to be analytic if the derivative dF/dz exists at a point z_0 and in some neighborhood of z_0 and if the value of dF/dz is independent of the direction in which it is calculated.
  - Singular Points
    - A singular point is any point at which F(z) is not analytic. If F(z) is analytic in some neighborhood of the point z_0, but not at z_0 itself, then z_0 is called an isolated singular point of F(z).
  - Derivative of Analytic Function
    - If F(z) is analytic, then dF/dz will exist and may be calculated in any direction, so that

$$\frac{dF}{dz} = \frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$$

2D Potential Flows – Complex Potential/Velocity

- The velocity components *u* and *v* may be expressed in terms of either the velocity potential or the stream function

$$\begin{aligned} u &= \boxed{\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}} \\ v &= \boxed{\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}} \end{aligned}$$

ϕ and ψ automatically satisfy the Cauchy-Riemann equations

- Let's define **complex potential** F(z) to make it easier to analyze potential flow

$$F(z) = \phi(x, y) + i\psi(x, y), \text{ where } z = x + iy$$

  - If F(z) is an analytic function, ϕ and ψ will automatically satisfy the Cauchy-Riemann equations
  - for every analytic function F(z), the **real part** is automatically a valid **velocity potential** and the **imaginary part** is a valid **stream function**

2D Potential Flows – Complex Potential/Velocity

- First, solve Laplace equations for ψ and ϕ
- Second, velocity field is obtained from the definition of ψ and ϕ
- Third, pressure field is obtained from Bernoulli eqn

- Disadvantage
  - inverse: first get the solution and then find the original physical problem (it's ok as an educational purpose)
  - can't be generalized to three-dimensional flow
- Advantage
  - Powerful
  - Need not to solve complex PDEs
  - Since F(z) is analytic, dF/dz is a point function (independent of the direction)

2D Potential Flows – Complex Potential/Velocity

$$W(z) = \frac{dF}{dz} = \frac{\partial F}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = u - iv$$

Complex Velocity

$$\text{or} \quad -i\frac{\partial F}{\partial y} = -\left(\frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y}\right) = u - iv$$

$$W\bar{W} = (u - iv)(u + iv) = u^2 + v^2 \ \text{(always scalar)}$$

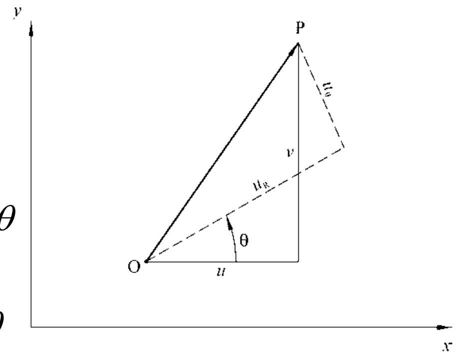
Recall, $u \cdot u = \nabla\phi \cdot \nabla\phi = u^2 + v^2$

2D Potential Flows – Complex Potential/Velocity

- Sometimes, we may need to work with cylindrical coordinates

$$u_R = \frac{1}{R}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial R},\ u_\theta = -\frac{\partial \psi}{\partial R} = \frac{1}{R}\frac{\partial \phi}{\partial R}$$

$$F(z) = \phi(z) + i\psi(z),\ z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$



$$u = u_R\cos\theta + u_\theta\cos\left(\frac{\pi}{2}-\theta\right) = u_R\cos\theta - u_\theta\sin\theta$$

$$v = u_R\sin\theta + u_\theta\sin\left(\frac{\pi}{2}-\theta\right) = u_R\sin\theta + u_\theta\cos\theta$$

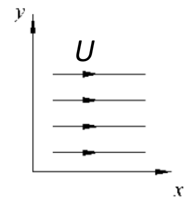
$$W = (u_R - iu_\theta)e^{-i\theta}$$

Uniform Flow

- $F(z) = Uz$ (*U*: real number)

  Velocity potential: $W(z) = u - iv = U$

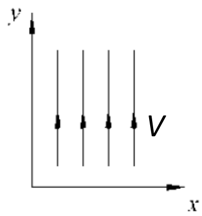
  $u = U,\ v = 0$ ⟶ Uniform rectilinear flow



- $F(z) = -iVz$

  Velocity potential: $W(z) = u - iv = -iV$

  $u = 0,\ v = V$ ⟶ Uniform vertical flow

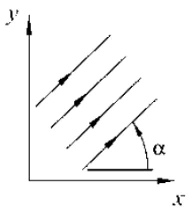


- $F(z) = Ve^{-i\alpha}z$

  Velocity potential: $W(z) = u - iv = V\cos\alpha - iV\sin\alpha$

  $u = V\cos\alpha,\ v = V\sin\alpha$

  ⟶ uniform flow inclined at an angle α to the x axis



Source, Sink, and Vortex Flows

- Complex potentials that correspond to the flow fields generated by sources, sinks, and vortices are obtained by considering F(z) to be proportional to 'lnz'.
- $F(z) = c\ln z = c\ln(R\,e^{i\theta}) = c\ln R + ic\theta = \phi + i\psi$

$\phi = c\ln R$

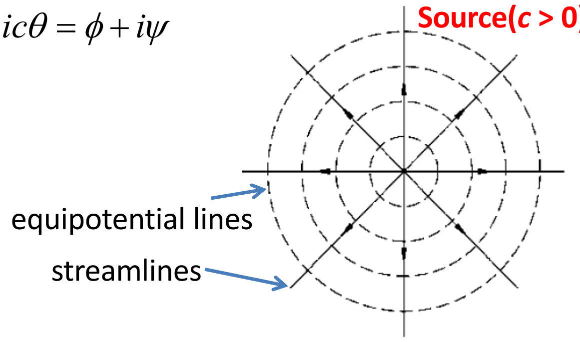
$\psi = c\theta$

In polar coordinates

$$W(z) = \frac{c}{z} = \frac{c}{R}e^{-i\theta} = (u_R - iu_\theta)e^{-i\theta}$$

$$u_R = \frac{c}{R},\ u_\theta = 0$$

Source($c > 0$)



- Strength of a source

$$m = \int_0^{2\pi} u_R R d\theta = \int_0^{2\pi} c d\theta = 2\pi c$$

Volume of fluid leaving the source per unit time per unit depth of the flow field

$$F(z) = \frac{m}{2\pi}\ln z,\ F(z) = \frac{m}{2\pi}\ln(z - z_0)$$

Source, Sink, and Vortex Flows

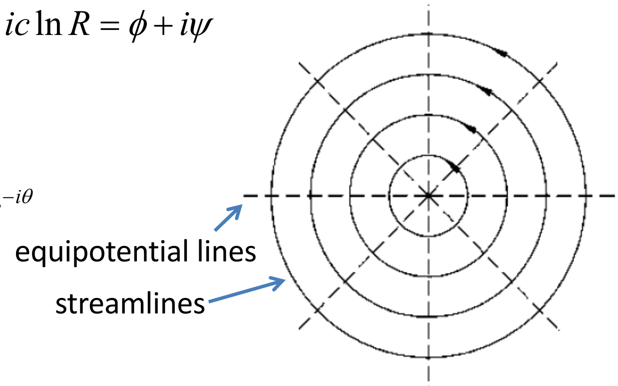
- Clearly, the complex potential for a sink, which is a negative source, is obtained by replacing m by –m.
- $F(z) = -ic\ln z = -ic\ln(R\,e^{i\theta}) = c\theta - ic\ln R = \phi + i\psi$

$\phi = c\theta$

$\psi = -c\ln R$

$$W(z) = -i\frac{c}{z} = -i\frac{c}{R}e^{-i\theta} = (u_R - iu_\theta)e^{-i\theta}$$

$$u_R = 0,\ u_\theta = \frac{c}{R}$$



- Direction of the flow is positive (counterclockwise) for c > 0, and the resulting flow field is a vortex; clockwise for c < 0
- Strength of a vortex, circulation

$$\Gamma = \int u dl = \int_0^{2\pi} u_\theta R d\theta = \int_0^{2\pi} c d\theta = 2\pi c \longrightarrow F(z) = -i\frac{\Gamma}{2\pi}\ln(z - z_0)$$

Source, Sink, and Vortex Flows

- The flow field for $z_0 = 0$ is called free vortex.
- That is, for any closed contour that does not include the singularity, the circulation will be zero and the flow will be irrotational.
- All the circulation and vorticity associated with this type of vortex is concentrated at the singularity.

Flow in a Sector

- Flows in sharp bends or sectors are represented by complex potentials that are proportional to z^n ($n \geq 1$).
- $F(z) = Uz^n = U(R\mathrm{e}^{i\theta})^n = UR^n e^{in\theta} = UR^n(\cos n\theta + i\sin n\theta) = \phi + i\psi$

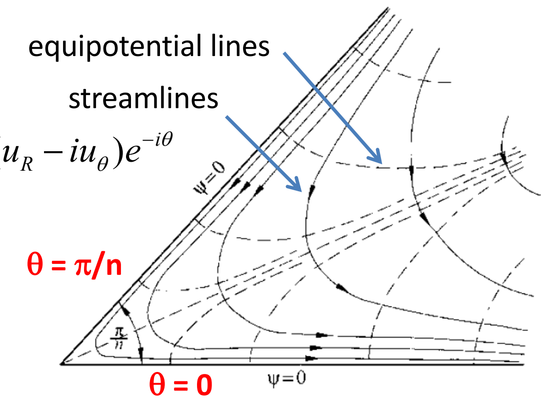
$$\phi = UR^n \cos n\theta$$

$$\psi = UR^n \sin n\theta$$

$$W(z) = nUz^{n-1} = nUR^{n-1}e^{i(n-1)\theta}$$

$$= (nUR^{n-1}\cos n\theta + inUR^{n-1}\sin n\theta)e^{-i\theta} = (u_R - iu_\theta)e^{-i\theta}$$

$$u_R = nUR^{n-1}\cos n\theta,\ u_\theta = -nUR^{n-1}\sin n\theta$$



$$\begin{cases} 0 < \theta < \dfrac{\pi}{2n}:\ u_R > 0,\ u_\theta < 0 \\ \dfrac{\pi}{2n} < \theta < \dfrac{\pi}{n}:\ u_R < 0,\ u_\theta < 0 \end{cases}$$

Flow around a Sharp Edge

- The complex potential for the flow around a sharp edge, such as the edge of a flat plate, is obtained from the function $z^{1/2}$.
- $F(z) = cz^{1/2} = c(R e^{i\theta})^{1/2} = cR^{1/2} e^{i\theta/2} = cR^{1/2}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}) = \phi + i\psi$

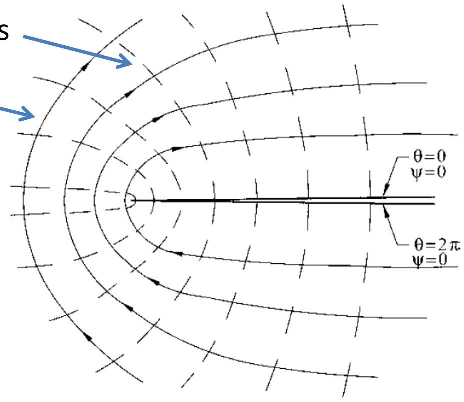
$$\phi = cR^{1/2}\cos\frac{\theta}{2}$$

$$\psi = cR^{1/2}\sin\frac{\theta}{2}$$

$$W(z) = \frac{c}{2z^{1/2}} = \frac{c}{2R^{1/2}} e^{-i\theta/2} = \frac{c}{2R^{1/2}} e^{i\theta/2} e^{-i\theta}$$

$$= \frac{c}{2R^{1/2}}(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}) e^{-i\theta} = (u_R - iu_\theta) e^{-i\theta}$$

$$u_R = \frac{c}{2R^{1/2}}\cos\frac{\theta}{2},\ u_\theta = -\frac{c}{2R^{1/2}}\sin\frac{\theta}{2}$$



$$\begin{cases} 0 < \theta < \pi:\ u_R > 0,\ u_\theta < 0 \\ \pi < \theta < 2\pi:\ u_R < 0,\ u_\theta < 0 \end{cases}$$

Flow due to Doublet

- The function 1/z has a singularity at z = 0, and in the context of complex potentials, this singularity is called a **doublet**.
- Physically, doublet may be considered to be the coalescing of a source and a sink.

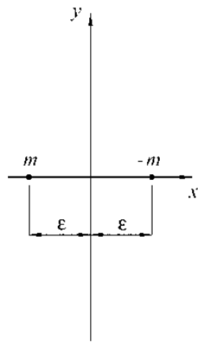
$$F(z) = \frac{m}{2\pi}\ln(z+\varepsilon) - \frac{m}{2\pi}\ln(z-\varepsilon)$$

$$= \frac{m}{2\pi}\ln\frac{z+\varepsilon}{z-\varepsilon} = \frac{m}{2\pi}\ln\frac{1+\varepsilon/z}{1-\varepsilon/z}$$

$$\frac{1}{1-\frac{\varepsilon}{z}} = 1 + \frac{\varepsilon}{z} + \left(\frac{\varepsilon}{z}\right)^2 + \left(\frac{\varepsilon}{z}\right)^3 + \cdots$$

$$F(z) = \frac{m}{2\pi}\ln\left[\left(1+\frac{\varepsilon}{z}\right)\left(1+\frac{\varepsilon}{z}+\left(\frac{\varepsilon}{z}\right)^2+\left(\frac{\varepsilon}{z}\right)^3+\cdots\right)\right]$$

$$= \frac{m}{2\pi}\ln\left[1+2\frac{\varepsilon}{z}+\left(\frac{\varepsilon}{z}\right)^2+\left(\frac{\varepsilon}{z}\right)^3+\cdots\right] = \frac{m}{2\pi}\ln\left[1+2\frac{\varepsilon}{z}\right] = \frac{m}{2\pi}\left(2\frac{\varepsilon}{z}\right)$$



Flow due to Doublet

- Let $\varepsilon \to 0$ and $m \to \infty$ such as $\lim_{\varepsilon \to 0}(m\varepsilon) = \pi\mu$ (μ: constant)

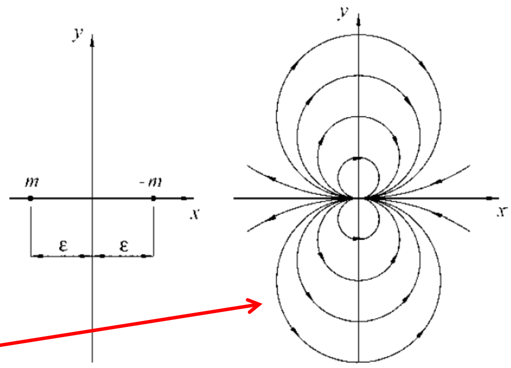
  Then, $F(z) = \dfrac{\mu}{z}$
- complex potential μ/z may be thought of as being the equivalent of the superposition of a very strong source and a very strong sink that are very close together.

$$F(z) = \frac{\mu}{z} = \frac{\mu}{x+iy} = \mu\frac{x-iy}{x^2+y^2} = \phi + i\psi$$

$$\phi = \mu\frac{x}{x^2+y^2}, \psi = -\mu\frac{y}{x^2+y^2}$$

$$x^2 + y^2 + \frac{\mu}{\psi}y = 0$$

$$x^2 + \left(y + \frac{\mu}{2\psi}\right)^2 = \left(\frac{\mu}{2\psi}\right)^2$$



Flow due to Doublet

- $W(z) = -\dfrac{\mu}{z^2} = -\dfrac{\mu}{R^2}e^{-i2\theta} = -\dfrac{\mu}{R^2}e^{-i\theta}e^{-i\theta}$

$$= -\frac{\mu}{R^2}(\cos\theta - i\sin\theta)e^{-i\theta}$$

$$u_R = -\frac{\mu}{R^2}\cos\theta, u_\theta = -\frac{\mu}{R^2}\sin\theta$$

$$F(z) = \frac{\mu}{z - z_0}$$ Doublet flow whose strength μ is located at z_0