INVISCID FLOW
Week 7

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Flow around a Circular Cylinder (w/o circulation)

- Consider the superposition of a uniform rectilinear flow and a doublet at the origin, using a superposition principle

$$F(z) = Uz + \frac{\mu}{z}$$

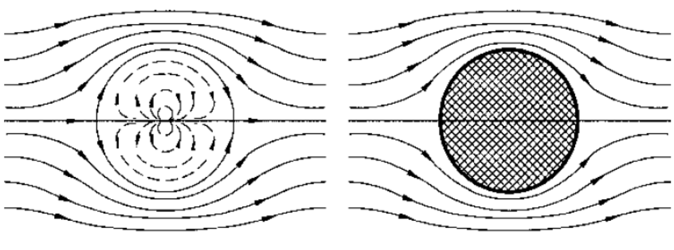
- For a certain choice of μ, the circle R = a becomes a streamline. On the circle R = a, z = ae$^{i\theta}$, so that the complex potential on this circle is

$$F(z) = Uae^{i\theta} + \frac{\mu}{a}e^{-i\theta} = (Ua + \frac{\mu}{a})\cos\theta + i(Ua - \frac{\mu}{a})\sin\theta$$

$$\therefore \psi = (Ua - \frac{\mu}{a})\sin\theta$$

If $\mu = Ua^2$, $\psi = 0$ on $R = a$.

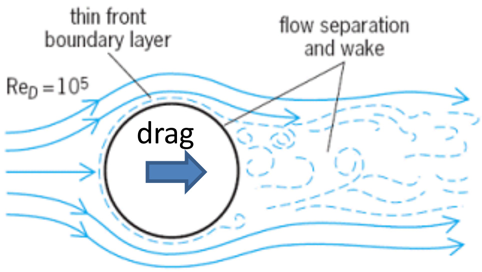
$$F(z) = U\left(z + \frac{a^2}{z}\right)$$



(a) (b)

Flow around a Circular Cylinder (w/o circulation)

- Here, the flow around a circular cylinder predicts no hydrodynamic force acting on the cylinder.
  - the flow is symmetric about the x axis; pressure is same for upper and lower region; No lift force
  - Similarly, the symmetry of the flow about the y axis; No drag force (D'Alembert Paradox)



White (1986)

  - Idealized flow situation that would be approached if viscous effects are minimized
  - For more streamlined bodies, e.g., airfoils, the potential-flow solution is approached over the entire length of the body

Flow around a Circular Cylinder (w/ circulation)

- $$F(z) = U\left(z + \frac{a^2}{z}\right) + i\frac{\Gamma}{2\pi}\ln z + c$$

A vortex

By adding the vortex, ψ will no longer be zero on R = a, another constant value. It is useful to make ψ = 0 on R = a.

$$F(z) = U\left(ae^{i\theta} + ae^{-i\theta}\right) + i\frac{\Gamma}{2\pi}\ln ae^{i\theta} + c$$

$$= 2Ua\cos\theta - \frac{\Gamma}{2\pi}\theta + i\frac{\Gamma}{2\pi}\ln a + c$$

$$\text{Then, choose } c = -i\frac{\Gamma}{2\pi}\ln a \rightarrow \psi = 0 \text{ on } R = a$$

$$F(z) = U\left(z + \frac{a^2}{z}\right) + i\frac{\Gamma}{2\pi}\ln\frac{z}{a}$$

uniform rectilinear flow of magnitude U approaching a circular cylinder of radius a that has a negative vortex of strength Γ around it

Flow around a Circular Cylinder (w/ circulation)

- $W(z) = U\left(1-\frac{a^2}{z^2}\right) + i\frac{\Gamma}{2\pi}\frac{1}{z} = U\left(1-\frac{a^2}{R^2}e^{-i2\theta}\right) + i\frac{\Gamma}{2\pi R}e^{-i\theta}$

$$= \left[U\left(e^{i\theta}-\frac{a^2}{R^2}e^{-i\theta}\right)+\frac{i\Gamma}{2\pi R}\right]e^{-i\theta}$$

$$= \left\{U\left(1-\frac{a^2}{R^2}\right)\cos\theta + i\left[U\left(1+\frac{a^2}{R^2}\right)\sin\theta+\frac{\Gamma}{2\pi R}\right]\right\}e^{-i\theta}$$

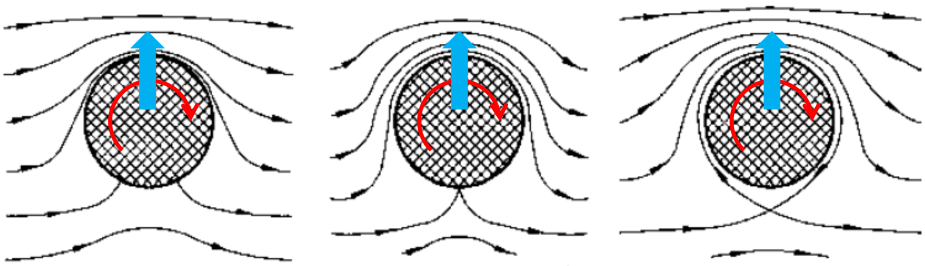
$$u_R = U\left(1-\frac{a^2}{R^2}\right)\cos\theta,\ u_\theta = -U\left(1+\frac{a^2}{R^2}\right)\sin\theta - \frac{\Gamma}{2\pi R}$$

On the surface of a cylinder $(R=a)$

$$u_R = 0,\ u_\theta = -2U\sin\theta - \frac{\Gamma}{2\pi a}$$

At stagnation point, all velocity components are zero → $\sin\theta_s = -\frac{\Gamma}{4\pi Ua}$

Flow around a Circular Cylinder (w/ circulation)



$0 < \frac{\Gamma}{4\pi Ua} < 1,\ \theta_s < 0$ $\quad \frac{\Gamma}{4\pi Ua} = 1,\ \theta_s = \frac{3}{2}\pi$ $\quad \frac{\Gamma}{4\pi Ua} > 1$

- No stagnation on the surface of a cylinder

$$U\left(1-\frac{a^2}{R_s^2}\right)\cos\theta_s = 0,\ U\left(1+\frac{a^2}{R_s^2}\right)\sin\theta_s = -\frac{\Gamma}{2\pi R_s}$$

$$R_s \neq a$$

$$\theta_s = \frac{\pi}{2}, \frac{3}{2}\pi \ \rightarrow U\left(1+\frac{a^2}{R_s^2}\right) = \pm\frac{\Gamma}{2\pi R_s}$$

Flow around a Circular Cylinder (w/ circulation)

- No stagnation on the surface of a cylinder

$$R_s^2 - \frac{\Gamma}{2\pi U} R_s + a^2 = 0$$

$$R_s = \frac{\Gamma}{4\pi U} \pm \sqrt{\left(\frac{\Gamma}{4\pi U}\right)^2 - a^2}, \frac{R_s}{a} = \frac{\Gamma}{4\pi U a}\left[1 \pm \sqrt{1 - \left(\frac{4\pi U a}{\Gamma}\right)^2}\right]$$

$$\begin{cases} \theta_s = \dfrac{\pi}{2}, \dfrac{R_s}{a} = \dfrac{\Gamma}{4\pi U a}\left[1 - \sqrt{1 - \left(\dfrac{4\pi U a}{\Gamma}\right)^2}\right] \\ \boxed{\theta_s = \dfrac{3}{2}\pi, \dfrac{R_s}{a} = \dfrac{\Gamma}{4\pi U a}\left[1 + \sqrt{1 - \left(\dfrac{4\pi U a}{\Gamma}\right)^2}\right]} \end{cases}$$

$R_s < a$ as $\Gamma \to \infty$: reject

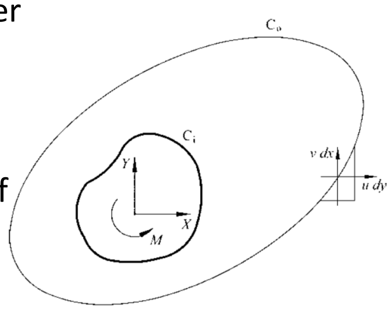
Blasius' Integral Laws

- The obvious way to evaluate the magnitude of force
  - velocity components from the complex potential → pressure distribution around the body surface from Bernoulli equation → integration of this pressure distribution

- The Blasius laws
  - a convenient alternative
  - complex potential for the flow around a body → evaluate the forces and the turning moment acting on the body by means of simple contour integrals

Blasius' Integral Laws

- Let's consider a body (C_i) enclosed by the contour C_0.
  - X, Y, M: forces and moment acting on the center of gravity
  - Force balance for the fluid between C_0 and C_i: net external force acting on the positive x direction must equal the net rate of increase of the x component momentum



$$-X - \int_{C_0} p dy = \int_{C_0} \rho u(udy - vdx)$$

  - Since C_i (body surface) is a streamline, there is no momentum transfer through it.
  - Force "X" is coming from the integration of pressure acting on C_i.
  - Similarly in y-direction: $-Y + \int_{C_0} p dx = \int_{C_0} \rho v(udy - vdx)$

Blasius' Integral Laws

- $$\begin{cases} X = \int_{C_0} (-pdy - \rho u^2 dy - \rho uvdx) \\ Y = \int_{C_0} (pdx - \rho uvdy + \rho v^2 dx) \end{cases}$$

$$p + \frac{1}{2}\rho(u^2 + v^2) = B \text{ (constant)}$$ **Bernoulli equation**

$$\begin{cases} X = \rho \int_{C_0} \left[uvdx - \frac{1}{2}(u^2 - v^2)dy \right] \\ Y = -\rho \int_{C_0} \left[uvdy + \frac{1}{2}(u^2 - v^2)dx \right] \end{cases} \longleftarrow \int_{C_0} Bdx = \int_{C_0} Bdy = 0$$

Eliminate the pressure!

- Evaluation of the complex integral of complex velocity, W

$$i\frac{\rho}{2}\int_{C_0} W^2 dz$$

Blasius' Integral Laws

- $$i\frac{\rho}{2}\int_{C_0} W^2 dz = i\frac{\rho}{2}\int_{C_0}(u-iv)^2(dx+idy)$$

$$= i\frac{\rho}{2}\int_{C_0}\left\{\left[(u^2-v^2)dx+2uvdy\right]+i\left[(u^2-v^2)dy-2uvdx\right]\right\}$$

$$= \rho\int_{C_0}\left\{\left[uvdx-\frac{1}{2}(u^2-v^2)dy\right]+i\left[uvdy+\frac{1}{2}(u^2-v^2)dx\right]\right\}$$

$$= X - iY$$

$$X - iY = i\frac{\rho}{2}\int_{C_0} W^2 dz$$

- C_0 is any closed contour that encloses the body under consideration
- **Force can be calculated directly from the velocity**
- **First Blasius' integral law**
- **Residue Theorem** will be used to apply this actually (will see next chaper)

Blasius' Integral Laws

- Moment balance

$$-M + \int_{C_0}\left[pxdx + pydy + \rho(udy - vdx)uy - \rho(udy - vdx)vx\right] = 0$$

Bernoulli equation was used to remove pressure!

$$M = \int_{C_0}\left[pxdx + pydy + \rho(u^2 ydy + v^2 xdx - uvydx - uvxdy)\right]$$

$$= \rho\int_{C_0}\left[-\frac{1}{2}(u^2+v^2)(xdx+ydy) + (u^2 ydy + v^2 xdx) - (uvydx + uvxdy)\right]$$

$$= -\frac{\rho}{2}\int_{C_0}\left[(u^2-v^2)(xdx-ydy) + 2uv(xdy+ydx)\right]$$

- Similarly, it can be shown that

$$\mathrm{Re}\left(\frac{\rho}{2}\int_{C_0} zW^2 dz\right) = \mathrm{Re}\left[\frac{\rho}{2}\int_{C_0}(x+iy)(u-iv)^2(dx+idy)\right] = -M$$

$$M = -\frac{\rho}{2}\mathrm{Re}\left(\int_{C_0} zW^2 dz\right)$$

- M is hydrodynamic moment acting on the body (clockwise direction is positive!)
- **Second Blasius' integral law**

Laurent Series and Residue Theorem

- Laurent Series
  - If $F(z)$ is analytic at all points within the annular region $r_0 < r < r_1$ whose center is at z_0, then $F(z)$ may be represented by

$$F(z) = \cdots + \frac{b_2}{(z-z_0)^2} + \frac{b_1}{(z-z_0)^1} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$a_n = \frac{1}{2\pi i}\int_c \frac{F(\xi)}{(\xi - z_0)^{n+1}} d\xi, \quad n = 0,1,2,\ldots$$

$$b_n = \frac{1}{2\pi i}\int_{c_0} \frac{F(\xi)}{(\xi - z_0)^{-n+1}} d\xi, \quad n = 0,1,2,\ldots$$

  - The contour C: $r = r_0$, contour C_0: $r = r_1$
  - Part of the series that contains b_n: principal part

Laurent Series and Residue Theorem

- Residues
  - The residues of a function at z_0: coefficient b_1 (i.e., $1/z$ term) in its Laurent Series about the point z_0.
- Residue Theorem
  - If F(z) is analytic within and on a closed curve C, except for a finite number of singular points z_1, z_2,.....z_n, then,

$$\int_C F(z)dz = 2\pi i(R_1 + R_2 + \cdots R_n)$$

  - R_n is the residue at z_n.

Force and Moment on a Circular Cylinder

- Complex potential for the flow around a circular cylinder w/ circulation

$$F(z) = U\left(z + \frac{a^2}{z}\right) + \frac{i\Gamma}{2\pi}\ln\frac{z}{a}$$

- Velocity potential

$$W(z) = U\left(1 - \frac{a^2}{z^2}\right) + \frac{i\Gamma}{2\pi z}$$

$$W^2(z) = U^2 - \frac{2U^2a^2}{z^2} + \frac{U^2a^4}{z^4} + \frac{iU\Gamma}{\pi z} - \frac{iU\Gamma a^2}{\pi z^3} - \frac{\Gamma^2}{4\pi^2 z^2}$$

$$X - iY = i\frac{\rho}{2}\int_{C_0} W^2 dz$$

$$= i\frac{\rho}{2}\left[2\pi i \sum (\text{residues of } W^2 \text{ inside } C_0)\right]$$

Force and Moment on a Circular Cylinder

- Singular point of W^2 is only $z_0 = 0$, where the centers of doublet and vortex are located together.
- Furthermore, W^2 is already written as a form of Laurent Series about $z = 0$

$$W^2(z) = U^2 - \frac{2U^2a^2}{z^2} + \frac{U^2a^4}{z^4} + \frac{iU\Gamma}{\pi z} - \frac{iU\Gamma a^2}{\pi z^3} - \frac{\Gamma^2}{4\pi^2 z^2}$$

$$R_0 = \frac{iU\Gamma}{\pi}$$

$$X - iY = i\frac{\rho}{2}\left[2\pi i\left(\frac{iU\Gamma}{\pi}\right)\right] = -i\rho U\Gamma$$

$X = 0$ Drag force: d'Alembert Paradox

$Y = \rho U\Gamma$ Lift force: Kutta-Joukowski Law

Force and Moment on a Circular Cylinder

- Evaluation of the moment

$$zW^2(z) = U^2 z - \frac{2U^2a^2}{z} + \frac{U^2a^4}{z^3} + \frac{iU\Gamma}{\pi} - \frac{iU\Gamma a^2}{\pi z^2} - \frac{\Gamma^2}{4\pi^2 z}$$

$$M = -\frac{\rho}{2}\mathrm{Re}\left(\int_{C_0} zW^2 dz\right)$$

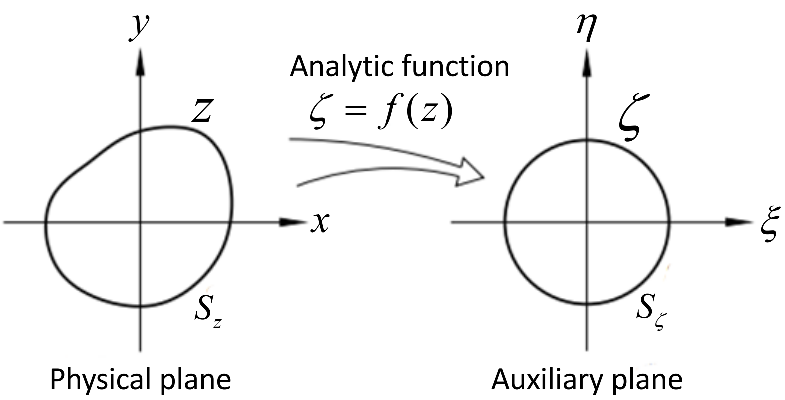
$$= -\frac{\rho}{2}\left[2\pi i \sum (\text{residues of } zW^2 \text{ inside } C_0)\right]$$

$$R_0 = -2U^2a^2 - \frac{\Gamma^2}{4\pi^2}$$

$$M = -\frac{\rho}{2}\mathrm{Re}\left[2\pi i\left(-2U^2a^2 - \frac{\Gamma^2}{4\pi^2}\right)\right] = 0$$ As expected!

Conformal Transformation

- Let’s consider 2D potential flow.
- Flow may be analyzed via the complex (physical) plane (*z*-plane), where the body contour is denoted by S_z, usually in a complicated shape.
- Corresponding to this flow, we can introduce an auxiliary complex plane (ζ-plane) with the body contour S_ζ simple enough, say, a circle, such that the complex potential F(ζ) for this flow may be easily found.



Conformal Transformation

- $\phi(x,y) \rightarrow \phi(\xi,\eta)$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \rightarrow \frac{\partial^2\phi}{\partial \xi^2} + \frac{\partial^2\phi}{\partial \eta^2} = 0$$

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial\xi}\boxed{\frac{\partial\xi}{\partial x}} + \frac{\partial\phi}{\partial\eta}\boxed{\frac{\partial\eta}{\partial x}}$$

can be found from $\zeta = f(z)$

$$\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial\xi}\frac{\partial\xi}{\partial x}\right) = \left(\frac{\partial^2\phi}{\partial\xi^2}\frac{\partial\xi}{\partial x} + \frac{\partial^2\phi}{\partial\xi\partial\eta}\frac{\partial\eta}{\partial x}\right)\frac{\partial\xi}{\partial x} + \frac{\partial\phi}{\partial\xi}\frac{\partial^2\xi}{\partial x^2}$$

$\phi = \phi(\xi,\eta)$

$$\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial\eta}\frac{\partial\eta}{\partial x}\right) = \left(\frac{\partial^2\phi}{\partial\xi\partial\eta}\frac{\partial\xi}{\partial x} + \frac{\partial^2\phi}{\partial\eta^2}\frac{\partial\eta}{\partial x}\right)\frac{\partial\eta}{\partial x} + \frac{\partial^2\eta}{\partial x^2}\frac{\partial\phi}{\partial\eta}$$

$$\therefore \frac{\partial^2\phi}{\partial x^2} = \frac{\partial^2\phi}{\partial\xi^2}\left(\frac{\partial\xi}{\partial x}\right)^2 + \frac{\partial^2\phi}{\partial\eta^2}\left(\frac{\partial\eta}{\partial x}\right)^2 + 2\frac{\partial^2\phi}{\partial\xi\partial\eta}\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\phi}{\partial\xi}\frac{\partial^2\xi}{\partial x^2} + \frac{\partial\phi}{\partial\eta}\frac{\partial^2\eta}{\partial x^2}$$

$$\frac{\partial^2\phi}{\partial y^2} = \frac{\partial^2\phi}{\partial\xi^2}\left(\frac{\partial\xi}{\partial y}\right)^2 + \frac{\partial^2\phi}{\partial\eta^2}\left(\frac{\partial\eta}{\partial y}\right)^2 + 2\frac{\partial^2\phi}{\partial\xi\partial\eta}\frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y} + \frac{\partial\phi}{\partial\xi}\frac{\partial^2\xi}{\partial y^2} + \frac{\partial\phi}{\partial\eta}\frac{\partial^2\eta}{\partial y^2}$$

Conformal Transformation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$$

$$\left[\left(\frac{\partial\xi}{\partial x}\right)^2 + \left(\frac{\partial\xi}{\partial y}\right)^2\right]\left(\frac{\partial^2\phi}{\partial\xi^2}\right) + \left[\left(\frac{\partial\eta}{\partial x}\right)^2 + \left(\frac{\partial\eta}{\partial y}\right)^2\right]\left(\frac{\partial^2\phi}{\partial\eta^2}\right) + 2\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y}\right)\frac{\partial^2\phi}{\partial\xi\partial\eta}$$

$$+\left(\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2}\right)\frac{\partial\phi}{\partial\xi} + \left(\frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2}\right)\frac{\partial\phi}{\partial\eta} = 0$$

- For conformal transformation, *f(z)* is analytic and ξ, η are harmonic.
  - ξ, η are solution of Cauchy-Riemann equation and of Laplace equation, as well.

$$\frac{\partial^2\xi}{\partial x^2} + \frac{\partial^2\xi}{\partial y^2} = 0, \quad \frac{\partial^2\eta}{\partial x^2} + \frac{\partial^2\eta}{\partial y^2} = 0$$

$$\begin{cases} \dfrac{\partial\xi}{\partial x} = \dfrac{\partial\eta}{\partial y} \\ \dfrac{\partial\xi}{\partial y} = -\dfrac{\partial\eta}{\partial x} \end{cases} \longrightarrow \frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x} + \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial y} = \frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial x} - \frac{\partial\eta}{\partial x}\frac{\partial\xi}{\partial x} = 0$$

Conformal Transformation

$$\begin{cases} \left[\left(\frac{\partial \eta}{\partial x}\right)^2 + \left(\frac{\partial \eta}{\partial y}\right)^2\right]\left(\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2}\right) = 0 \\ \left[\left(\frac{\partial \xi}{\partial x}\right)^2 + \left(\frac{\partial \xi}{\partial y}\right)^2\right]\left(\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2}\right) = 0 \end{cases}$$

should be satisfied for all *f(z)* ➡

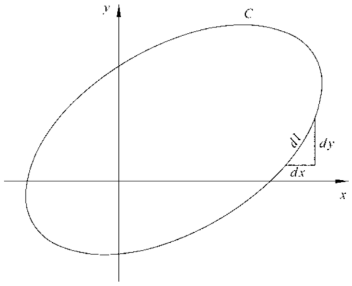
- Laplace equation in the *z*-plane transforms into Laplace equation in the ζ-plane, provided these two planes are related by a conformal transformation.
- Complex potential in the *z*-plane is also a valid complex potential in the ζ-plane, and vice versa: $\phi(x,y) + i\psi(x,y) = \phi(\xi,\eta) + i\psi(\xi,\eta)$
- Then, how about the complex velocity?

$$W(z) = \frac{dF(z)}{dz} = \frac{dF(\zeta)}{d\zeta}\frac{d\zeta}{dz} = \frac{d\zeta}{dz}W(\zeta)$$

→ Not mapped one to one, but they are proportional to each other

Conformal Transformation

- Strength of a singular point is maintained after the transformation



Net strength of all the sources/sinks inside C: *m*
Net strength of all the vortices inside C: Γ

$$m = \int_C u \cdot n dl = \int_C (udy - vdx)$$

$$\Gamma = \int_C u dl = \int_C (udx + vdy)$$

$$\int_C W(z)dz = \int_C (u - iv)(dx + idy)$$
$$= \int_C (udx + vdy) + i\int_C (udy - vdx)$$
$$= \Gamma + im$$

$$\Gamma_z + im_z = \int_{C_z} W(z)dz = \int_{C_\zeta} W(\zeta)\frac{d\zeta}{dz}dz$$
$$= \int_{C_z} W(\zeta)d\zeta = \Gamma_\zeta + im_\zeta$$

Sources, sinks, and vortices map into sources, sinks, and vortices of the same strength under a conformal transformation.

Conformal Transformation

- In summary,
  - complex potential for the flow around some body in the *z*-plane → complex potential for the body corresponding to the conformal mapping via substituting $\zeta = f(z)$ into the complex potential F(z).
  - Complex velocities, on the other hand, do not transform one to one but are proportional to each other.
  - Sources, sinks, and vortices maintain the same strength under conformal transformations.