INVISCID FLOW
Week 8

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Joukowski Transformation

- Solutions for the flow around ellipses and a family of airfoils

$$z = \zeta + \frac{c^2}{\zeta} \qquad (c^2\text{: real number})$$

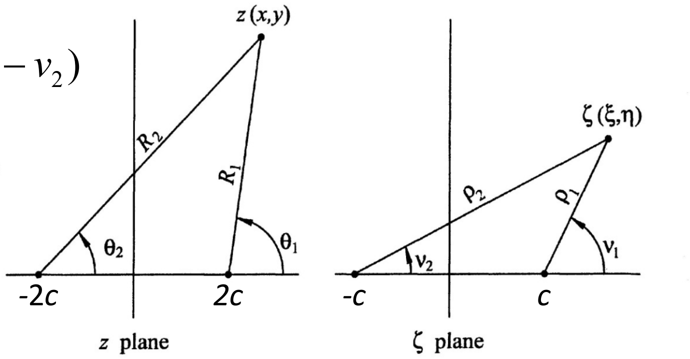
  - For $\zeta \rightarrow \pm\infty$, $z = \zeta$: the complex velocity in the two planes is the same far from the origin (**identity mapping**)
    - if a uniform flow of a certain magnitude is approaching a body in the *z*-plane at some angle of attack, a uniform flow of the same magnitude and angle of attack will approach the corresponding body in the ζ-plane.
    - $dz/d\zeta = 1 \rightarrow W(z) = W(\zeta)$
  - Singularity point of transformation @ $\zeta = 0$ (normally, inside the body and of no consequence)
  - Critical points of transformation: $dz/d\zeta = 0;\ \zeta = \pm c$
    - Smooth curves passing through critical points may become corners in the transformed plane.

Joukowski Transformation

$$z+2c=\frac{(\zeta+c)^2}{\zeta},\ z-2c=\frac{(\zeta-c)^2}{\zeta}\rightarrow\frac{z-2c}{z+2c}=\left(\frac{\zeta-c}{\zeta+c}\right)^2$$

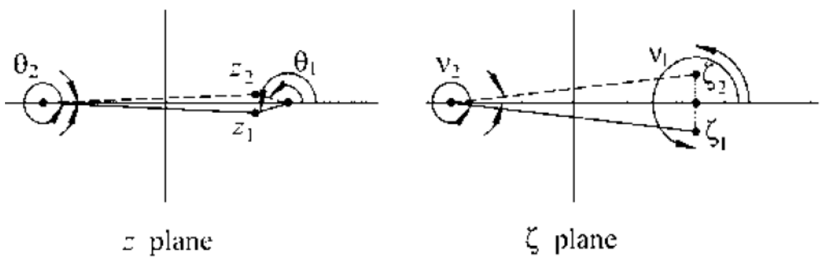
$$\frac{R_1e^{i\theta_1}}{R_2e^{i\theta_2}}=\left(\frac{\rho_1e^{iv_1}}{\rho_2e^{iv_2}}\right)^2,\ \frac{R_1}{R_2}e^{i(\theta_1-\theta_2)}=\left(\frac{\rho_1}{\rho_2}\right)^2e^{i2(v_1-v_2)}$$

$$\rightarrow\frac{R_1}{R_2}=\left(\frac{\rho_1}{\rho_2}\right)^2,\theta_1-\theta_2=2(v_1-v_2)$$



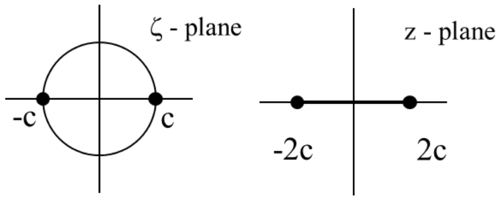
Joukowski Transformation

- Therefore, a smooth curve passes through the point $\zeta = c$, the corresponding curve in the *z* plane will form a knife-edge or cusp.



- Example: a circle centered at the origin of the ζ-plane (radius is *c*)

$$\zeta=ce^{iv},\ z=ce^{iv}+ce^{-iv}=2c\cos v=x+iy$$



Application of Joukowski Transformation

- Flow around Ellipses
  - simple geometry of the circle, which we know details, will be placed in the ζ-plane, and the corresponding body in the *z*-plane will be investigated
    - *c*: real and positive
    - Radius: *a* (> *c*), center at the origin

$$\zeta = ae^{iv} \quad \rightarrow \quad z = \zeta + \frac{c^2}{\zeta}$$

$$z = ae^{iv} + \frac{c^2}{a} ae^{-iv} = \left(a + \frac{c^2}{a}\right)\cos v + i\left(a - \frac{c^2}{a}\right)\sin v$$

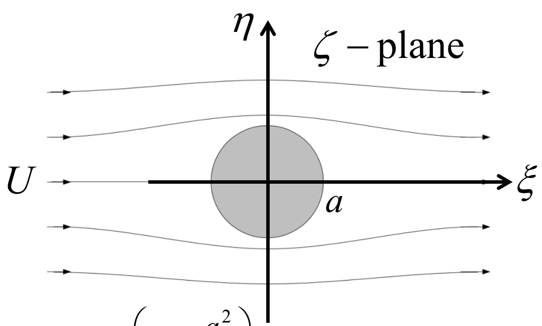
$$x = \left(a + \frac{c^2}{a}\right)\cos v,\ y = \left(a - \frac{c^2}{a}\right)\sin v$$

Application of Joukowski Transformation

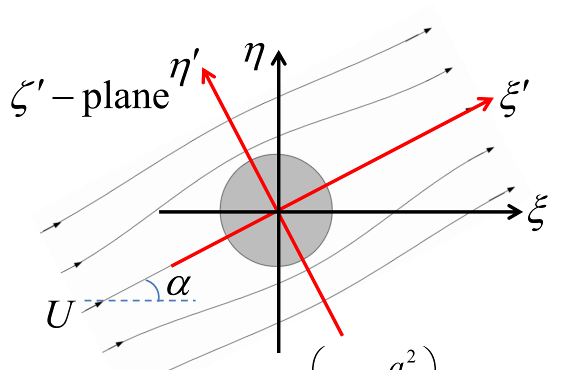
- Equation of ellipses in *z*-plane is obtained.

$$\therefore \left(\frac{x}{a + c^2/a}\right)^2 + \left(\frac{y}{a - c^2/a}\right)^2 = 1$$

1) Complex potential for a uniform flow (*U*) approaching this ellipse at an angle of attack α ← the same flow to approach the circular cylinder in the *z*-plane



$$F(\zeta) = U\left(\zeta + \frac{a^2}{\xi}\right)$$



$$F(\zeta') = U\left(\zeta' + \frac{a^2}{\zeta'}\right), \zeta' = \zeta e^{-i\alpha}$$

Application of Joukowski Transformation

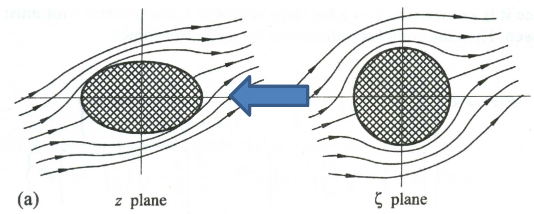
$$\therefore F(\zeta) = U\left(\zeta e^{-i\alpha} + \frac{a^2}{\zeta} e^{i\alpha}\right)$$

$$z = \zeta + \frac{c^2}{\zeta} \rightarrow \zeta^2 - z\zeta + c^2 = 0: \ \zeta = \frac{z}{2} \pm \sqrt{\left(\frac{z}{2}\right)^2 - c^2}$$

As $z \rightarrow \infty$, $\zeta \rightarrow z$: $\zeta = \dfrac{z}{2} + \sqrt{\left(\dfrac{z}{2}\right)^2 - c^2}$

$$F(z) = U\left[z e^{-i\alpha} + \left(\frac{a^2}{c^2} e^{i\alpha} - e^{-i\alpha}\right)\left(\frac{z}{2} - \sqrt{\left(\frac{z}{2}\right)^2 - c^2}\right)\right]$$

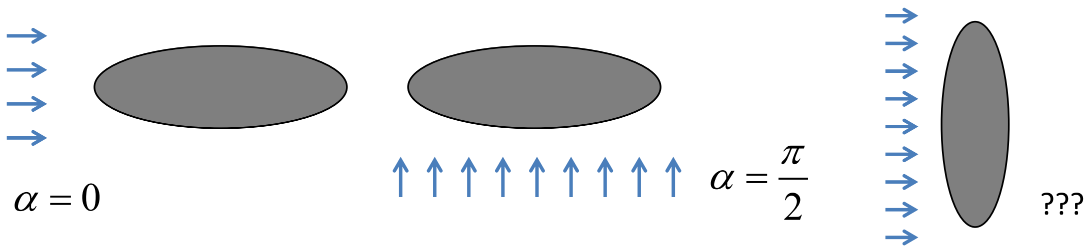
Application of Joukowski Transformation



2) Position of stagnation point

- In ζ-plane: $\zeta = \pm a e^{i\alpha}$
- Corresponding points in z-plane

$$z = \pm a e^{i\alpha} \pm \frac{c^2}{a} e^{-i\alpha} = \pm\left(a + \frac{c^2}{a}\right)\cos\alpha \pm i\left(a - \frac{c^2}{a}\right)\sin\alpha$$



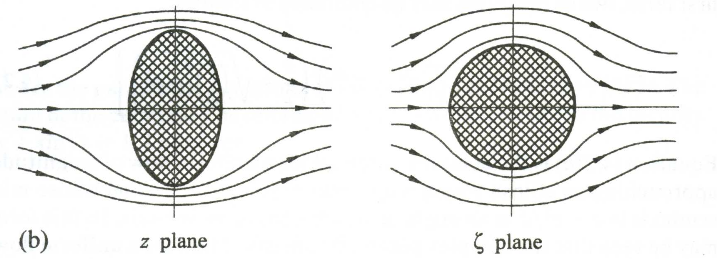
Application of Joukowski Transformation

– Modified Joukouski Transformation

$$z=\zeta+\frac{c^2}{\zeta},\ c=ib\rightarrow z=\zeta-\frac{b^2}{\zeta}$$

$$\left(\frac{x}{a-b^2/a}\right)^2+\left(\frac{y}{a+b^2/a}\right)^2=1$$ Equation of ellipses in *z*-plane

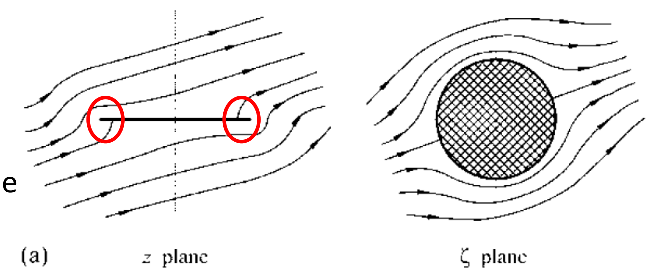
$$F(z)=U\left[z-\left(1+\frac{a^2}{b^2}\right)\left(\frac{z}{2}-\sqrt{\left(\frac{z}{2}\right)^2+b^2}\right)\right]$$



Kutta Condition and Flat-plate Airfoil

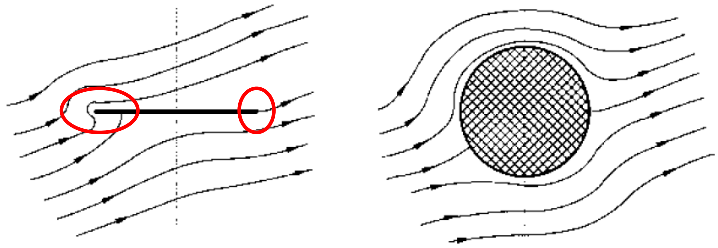
- As a reminder,
  – potential flow solution for flow around a sharp edge has a singularity (infinite velocity) at the edge itself → physically impossible
- Two ways to correct this
  – Place separation point at the edge, i.e., *V* = finite value
  – **Place stagnation point at the edge, i.e., *V* = 0 (Kutta Condition)**
- From the flow around an ellipse, if *c* → *a, then* resulting ellipse in the *z*-plane degenerates to a flat plate defined by the strip (-2*a* ≤ x ≤ 2*a*).

Physically impossible to realize



Kutta Condition and Flat-plate Airfoil

- Leading-edge: real airfoils have a finite thickness and so have a finite radius of curvature at the leading edge → will not be a matter
- Trailing-edge: usually quite sharp
  - if a circulation exists around the flat-plate and the magnitude of this circulation is enough to rotate the rear stagnation point to the trailing edge, this problem will be solved.
- **Kutta condition**
  - For bodies with sharp trailing edges that are at small angles of attack to the free-stream, the flow will adjust itself in such a way that the rear stagnation point coincides with the trailing edge

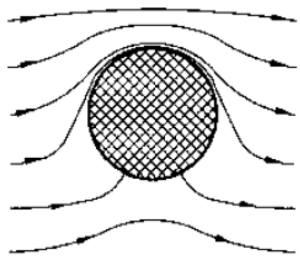


Kutta Condition and Flat-plate Airfoil

- How to determine the amount of circulation?

$$\Gamma = 4\pi U a \sin\alpha \text{ (clockwise direction)}$$

Recall



$$\sin\theta_s = -\frac{\Gamma}{4\pi U a}$$

$$F(\zeta) = U\left(\zeta e^{-i\alpha} + \frac{a^2}{\zeta}e^{i\alpha}\right) + i2Ua\sin\alpha \ln\frac{\zeta}{a}$$

$$z = \zeta + \frac{a^2}{\zeta}, \quad \zeta = \frac{z}{2} + \sqrt{\left(\frac{z}{2}\right)^2 - a^2}$$

$$F(z) = U\left\{\left[\frac{z}{2} + \sqrt{\left(\frac{z}{2}\right)^2 - a^2}\right]e^{-i\alpha} + \frac{a^2 e^{i\alpha}}{z/2 + \sqrt{(z/2)^2 - a^2}} + i2a\sin\alpha a \ln\left[\frac{1}{a}\left(\frac{z}{2} + \sqrt{\left(\frac{z}{2}\right)^2 - a^2}\right)\right]\right\}$$

Kutta Condition and Flat-plate Airfoil

- Kutta-Joukowski Law (calculation of the lift force)

$$L = 4\pi\rho U^2 a \sin\alpha$$

  – Lift coefficient, C_L

$$C_L = \frac{L}{0.5\rho U^2 l} = \frac{L}{0.5\rho U^2 (4a)} = 2\pi \sin\alpha \sim 2\pi\alpha$$

(for small α)

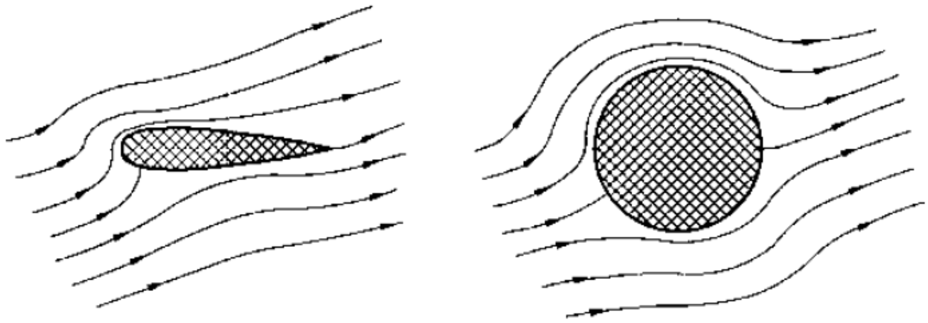
Symmetrical Joukowski Airfoil

- A family of airfoils (in z-plane) obtained by Joukowski transformation from a series of circles in ζ-plane whose centers are slightly displaced from the origin → Joukowski Airfoils
  – Variable: center and radius of the circle
- Let's consider the displacement of the circle in real axis.
- As a reminder,
  – if the circle passes through the two critical points of the Joukowski transformation, $\zeta = \pm c$, then a sharp edge is obtained in *z*-plane
- For the leading edge of the airfoil to have a finite radius of curvature and if there should be no singularities in the flow field, the point $\zeta = -c$ should be inside the circle in *z*-plane.
- But, the circle should pass $\zeta = c$ for the trailing edge to be sharp.

⇨ Center: $\zeta = -m$
Radius: $a = c + m = c(1+\varepsilon),\ \varepsilon = m/c\ (\ll 1)$

Symmetrical Joukowski Airfoil

(a) z plane ζ plane



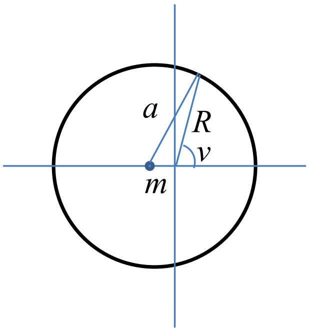
(b) z plane ζ plane

Symmetrical Joukowski Airfoil

- Trailing edge / Leading edge $\begin{cases} \zeta = c \\ \zeta = -c-2m \end{cases} \rightarrow z = \zeta + \dfrac{c^2}{\zeta} \rightarrow \begin{cases} z = 2c \\ z = -(c+2\varepsilon) - \dfrac{c}{1+2\varepsilon} \end{cases}$

$$z = -(c+2\varepsilon) - c\left[1 - 2\varepsilon + O(\varepsilon^2)\right] = -2c + O(\varepsilon^2) \sim -2c$$

- The chord length, $l = 4c$ (chord length is not affected by the shift of circle)



$$a^2 = R^2 + m^2 - 2Rm\cos(\pi - v) = R^2 + m^2 + 2Rm\cos v$$

$$(c+m)^2 = R^2\left(1 + \frac{m^2}{R^2} + 2\frac{m}{R}\cos v\right)$$

$$(c+m) = R\left(1 + 2\frac{m}{R}\cos v\right)^{1/2} = R\left(1 + \frac{m}{R}\cos v + O(\varepsilon^2)\right)$$

$$\therefore R = c\left[1 + \varepsilon(1 - \cos v)\right]$$

Symmetrical Joukowski Airfoil

- Then, from the Joukowski transformation

$$z = \zeta + \frac{c^2}{\zeta} = Re^{iv} + \frac{c^2}{Re^{iv}}$$

$$= c\left[1+\varepsilon(1-\cos v)\right]e^{iv} + \frac{ce^{-iv}}{1+\varepsilon(1-\cos v)}$$

$$= c\left[1+\varepsilon(1-\cos v)\right]e^{iv} + c\left[1-\varepsilon(1-\cos v)+O(\varepsilon^2)\right]e^{-iv}$$

$$= c\left[2\cos v + i2\varepsilon(1-\cos v)\sin v + O(\varepsilon^2)\right]$$

$$\begin{cases} x = 2c\cos v \\ y = 2c\varepsilon(1-\cos v)\sin v \end{cases} \longrightarrow y = \pm 2c\varepsilon\left(1-\frac{x}{2c}\right)\sqrt{1-\left(\frac{x}{2c}\right)^2}$$

– To find the maximum thickness, $\frac{dy}{dv} = 0 \rightarrow v = \frac{2}{3}\pi, \frac{4}{3}\pi \rightarrow y = \pm\frac{3\sqrt{3}}{2}\varepsilon c$

$$t_{\max} = 3\sqrt{3}\varepsilon c$$

Symmetrical Joukowski Airfoil

- Ratio of the thickness to chord: $\frac{t_{\max}}{l} = \frac{3\sqrt{3}\varepsilon}{4}$ or $\varepsilon = 0.77\frac{t_{\max}}{l}$

- Equation for airfoil

$$\frac{y}{t_{\max}} = \pm 0.385\left(1-2\frac{x}{l}\right)\sqrt{1-\left(2\frac{x}{l}\right)^2}$$

- Circulation for Kutta condition

$$\Gamma = 4\pi U a\sin\alpha = \pi U l\left(1+0.77\frac{t_{\max}}{l}\right)\sin\alpha$$

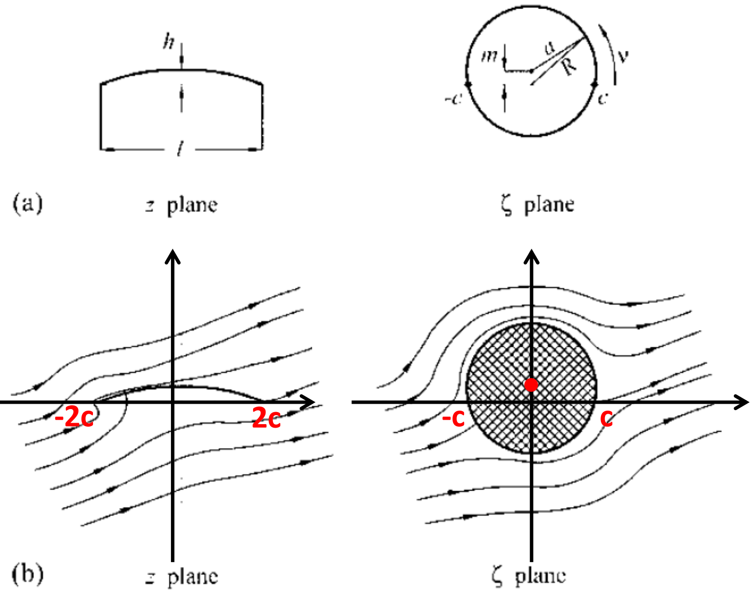
- Lift force and lift coefficient

$$L = \pi\rho U^2 l\left(1+0.77\frac{t_{\max}}{l}\right)\sin\alpha$$

$$C_L = 2\pi\left(1+0.77\frac{t_{\max}}{l}\right)\sin\alpha$$

Circular-Arc Airfoil

- We will show that a circle whose radius is slightly larger than *c* and whose center is displaced along the imaginary axis of the ζ-plane produces an airfoil that has no thickness but has a camber.

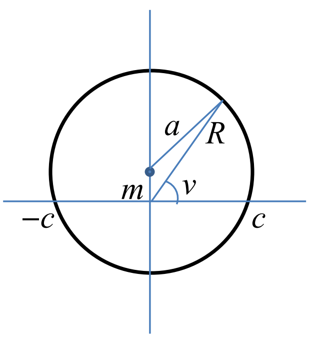


Circular-Arc Airfoil

- $z = \zeta + \dfrac{c^2}{\zeta} = Re^{iv} + \dfrac{c^2}{R}e^{-iv} = \left(R + \dfrac{c^2}{R}\right)\cos v + i\left(R - \dfrac{c^2}{R}\right)\sin v$

$$x^2\sin^2 v - y^2\cos^2 v = \left(R + \frac{c^2}{R}\right)^2 \sin^2 v\cos^2 v - \left(R - \frac{c^2}{R}\right)^2 \sin^2 v\cos^2 v$$

$= 4c^2\sin^2 v\cos^2 v$ equation of the airfoil surface in the z-plane



$$a^2 = R^2 + m^2 - 2Rm\cos(\frac{\pi}{2} - v) = R^2 + m^2 - 2Rm\sin v$$

$$\sin v = \frac{R^2 - c^2}{2Rm} = \frac{y}{2m\sin v}, \quad \cos^2 v = 1 - \frac{y}{2m}$$

$$x^2 + y^2 + 2\left(\frac{c^2}{m} - m\right)y = 4c^2$$

$$x^2 + \left[y + c\left(\frac{c}{m} - \frac{m}{c}\right)\right]^2 = c^2\left[4 + \left(\frac{c}{m} - \frac{m}{c}\right)^2\right]$$

Circular-Arc Airfoil

- Using $\varepsilon = \frac{m}{c} \ll 1$ and linearize the equation, we get

$$x^2 + \left(y + \frac{c^2}{m} \right)^2 = c^2 \left(4 + \frac{c^2}{m^2} \right)$$

- Chord length: $l = 4c$
- Camber height: $h = 2m$

$$\longrightarrow \quad x^2 + \left(y + \frac{l^2}{8h} \right)^2 = \frac{l^2}{4} \left(1 + \frac{l^2}{16h^2} \right)$$

- Circulation to satisfy Kutta condition

$$\Gamma = 4\pi U a \sin\left(\alpha + \frac{m}{c} \right) = 4\pi U c \sin\left(\alpha + \frac{m}{c} \right)$$

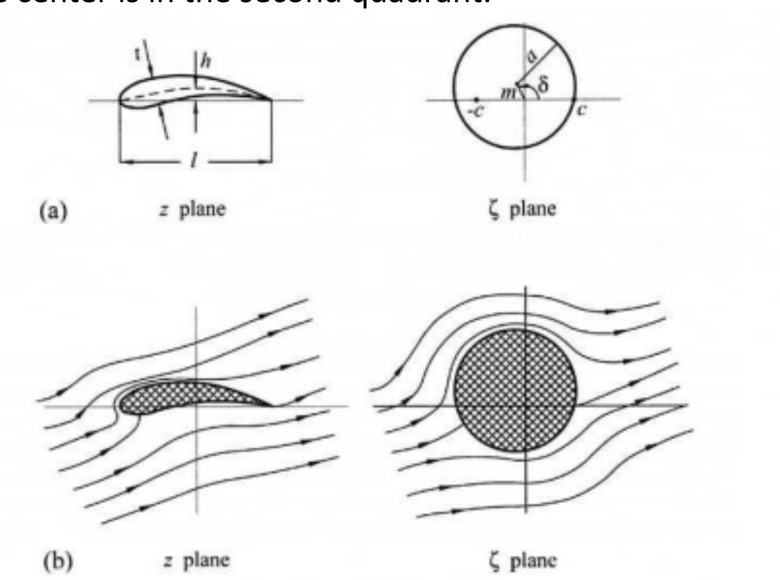
- Lift force and lift coefficient

$$L = 4\pi \rho U^2 c \sin\left(\alpha + \frac{m}{c} \right)$$

$$C_L = 8\pi \frac{c}{l} \sin\left(\alpha + \frac{m}{c} \right) = 2\pi \sin\left(\alpha + \frac{2h}{l} \right)$$

Joukowski Airfoil

- Cambered airfoil of finite thickness
  - Joukowski transformation in conjunction with a circle in the z plane whose center is in the second quadrant.



Joukowski Airfoil

- Equation for the upper and lower surfaces of the airfoil

$$y=\sqrt{\frac{l^2}{4}\left(1+\frac{l^2}{16h^2}\right)-x^2}-\frac{l^2}{8h}\pm 0.385t_{\max}\left(1-2\frac{x}{l}\right)\sqrt{1-\left(2\frac{x}{l}\right)^2}$$

Circular-arc centerline — Thickness effect

- Lift coefficient

$$C_L=2\pi\left(1+0.77\frac{t_{\max}}{l}\right)\sin\left(\alpha+\frac{2h}{l}\right)$$

Thickness effect — Camber effect

- Complex potential

$$F(\zeta)=U\left[(\zeta-me^{i\delta})e^{-i\alpha}+\frac{a^2e^{i\alpha}}{\zeta-me^{i\delta}}\right]+\frac{i\Gamma}{2\pi}\ln\left(\frac{\zeta-me^{i\delta}}{a}\right)$$

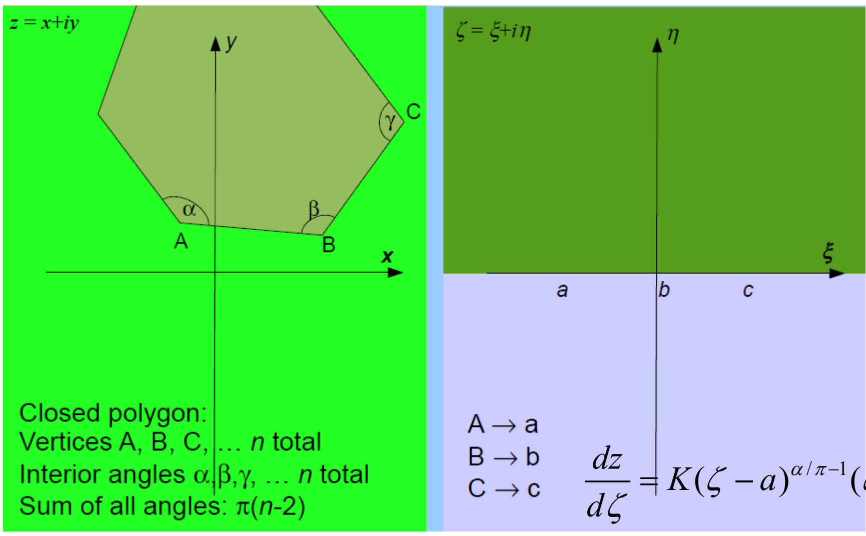
- Circulation for Kutta condition

$$\Gamma=\pi Ul\left(1+0.77\frac{t_{\max}}{l}\right)\sin\left(\alpha+\frac{2h}{l}\right)$$

* Limit in increase of thickness and camber → stall

Schwarz-Christoffel Transformation

- maps the interior of a closed polygon in the z plane onto the upper half of the ζ-plane, while the boundary of the polygon maps onto the real axis of the z plane.

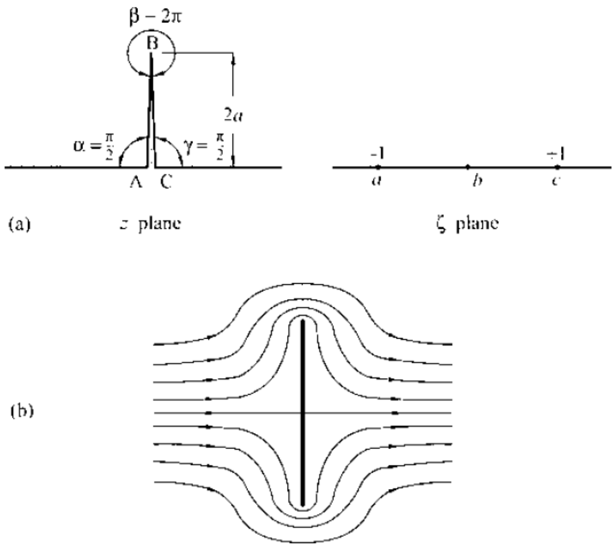


$$\frac{dz}{d\zeta}=K(\zeta-a)^{\alpha/\pi-1}(\zeta-b)^{\beta/\pi-1}(\zeta-c)^{\gamma/\pi-1}\ldots$$

$$\alpha+\beta+\gamma+\ldots=(n-2)\pi$$

Schwarz-Christoffel Transformation

- Example flow field
  - Flow around a flat plate of finite length oriented vertically (90 ° angle of attack)



$$\frac{dz}{d\zeta} = K(\zeta+1)^{-1/2}(\zeta-0)^{1}(\zeta-1)^{-1/2}$$

$$= K\frac{\zeta}{\sqrt{\zeta^2-1}}$$

$$z = K\sqrt{\zeta^2-1} + D$$

(D: constant of integral, complex)

$$\begin{cases} \zeta = 1 \leftrightarrow z = 0 \\ \zeta = -1 \leftrightarrow z = 0 \\ \zeta = 0 \leftrightarrow z = i2a \end{cases}$$

$$\therefore z = 2a\sqrt{\zeta^2-1}$$

Schwarz-Christoffel Transformation

- To find a magnitude of uniform velocity,
  - As $\zeta \rightarrow \infty$, $z \rightarrow 2a\zeta$ and $W(\zeta) \rightarrow 2aW(z) = 2aU$

    $\therefore F(\zeta) = 2aU\zeta$ Complex potential for horizontal uniform flow

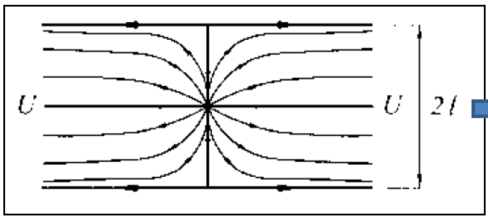
From mapping function $\zeta = \pm\sqrt{\left(\frac{z}{2a}\right)^2 + 1}$ $z \rightarrow \infty$, $\zeta \rightarrow \infty$

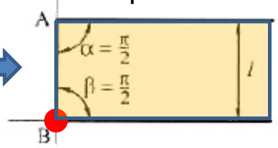
$$\therefore \zeta = \frac{1}{2a}\sqrt{z^2+4a^2},\ F(z) = U\sqrt{z^2+4a^2}$$

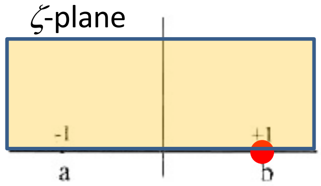
This result can be confirmed by the flow around an vertical ellipse using a modified Joukowski transformation

Source in a Channel

- Flow generated by a line source located in a two-dimensional channel







first quadrant only,
source is located at $z = 0$

$$\frac{dz}{d\zeta} = K(\zeta+1)^{-1/2}(\zeta-1)^{-1/2} = \frac{K}{\sqrt{\zeta^2-1}} \qquad \begin{cases} \zeta = 1 \leftrightarrow z = 0 \\ \zeta = -1 \leftrightarrow z = il \end{cases}$$

$$z = K\cosh^{-1}\zeta + D \qquad \therefore z = \frac{l}{\pi}\cosh^{-1}\zeta,\ \zeta = \cosh\frac{\pi z}{l}$$

– Complex potential in ζ-plane

$$F(\zeta) = \frac{m}{2\pi}\ln(\zeta-1),\ F(z) = \frac{m}{2\pi}\ln\left(\cosh\frac{\pi z}{l} - 1\right)$$

Source in a Channel

- Using $\cosh(X+Y) - \cosh(X-Y) = 2\sinh X \sinh Y$

$$\cosh\frac{\pi z}{l} - 1 = 2\sinh^2\frac{\pi z}{2l}$$

$$F(z) = \frac{m}{2\pi}\ln\left(2\sinh^2\frac{\pi z}{2l}\right) = \frac{m}{\pi}\ln\left(\sinh\frac{\pi z}{2l}\right) + \frac{m}{2\pi}\ln 2$$

this term does not affect the velocity

$$= \frac{m}{\pi}\ln\left(\sinh\frac{\pi z}{2l}\right)$$

$m = 4Ul$