INVISCID FLOW
Week 11

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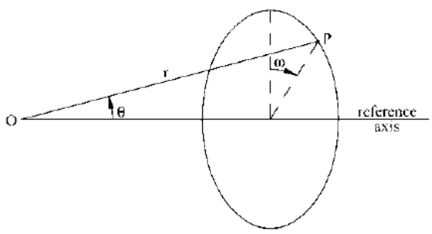
Three-dimensional Potential Flow

- Physically, there are not specific differences in 3D flow, compared to 2D flow except it has one more dimension.
- Methodologically, however, it is completely different
  - We cannot use analytic functions of complex variables
  - Should solve PDE, instead
- Since bodies of interest, such as airship hulls and submarine vehicle hulls, have an axis of symmetry, this chapter will consider only bodies that are **axisymmetric**. In so doing, it will be found convenient to work in the **spherical** coordinates (r, θ, ω).



Velocity Potential and Stoke's Stream Function

- Velocity potential exists irrespective of the dimensionality of the flow field
  → Laplace equation should be satisfied.

$$\begin{cases} \dfrac{\partial}{\partial \omega}(\bullet) = 0 \qquad \textbf{axisymmetric} \\ \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \dfrac{\partial \phi}{\partial r}\right) + \dfrac{1}{r^2 \sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta \dfrac{\partial \phi}{\partial \theta}\right) = 0 \end{cases} \rightarrow \begin{cases} u_r = \dfrac{\partial \phi}{\partial r} \\ u_\theta = \dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta} \qquad u_\omega = 0 \end{cases}$$

- In three dimensions it is not possible, in general, to satisfy the continuity equation by a single scalar function (like a stream function in 2D potential flow)
- However, in axisymmetric flow, such a function exist!

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(u_\theta \sin\theta\right) = 0, \qquad \begin{cases} u_r = \dfrac{1}{r^2\sin\theta}\dfrac{\partial \psi}{\partial \theta} \\ u_\theta = -\dfrac{1}{r\sin\theta}\dfrac{\partial \psi}{\partial r} \end{cases} \quad \textbf{Stoke's Stream Function}$$

Identically satisfied by this

Velocity Potential and Stoke's Stream Function

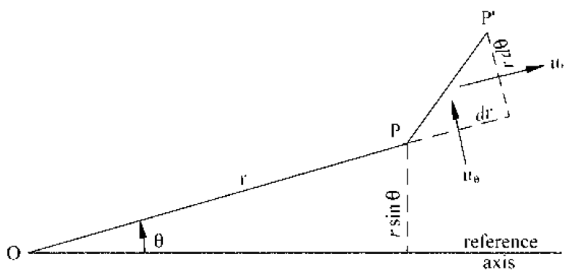
- $2\pi\psi$: **volume of fluid** crossing the surface of revolution which is formed by rotating the position vector OP around the reference axis.
  - ψ: defined such that if the position vector OP is rotated around the reference axis → **ω is varied from 0 to 2π while r and θ fixed**
  - First assume that the above statement is true
  - Consider PP' which is rotated about the reference axis, the resulting surface will have a quantity of fluid $2\pi d\psi$ crossing it per unit time

$$2\pi d\psi = 2\pi r \sin\theta (u_r r d\theta - u_\theta dr)$$

$$d\psi = u_r r^2 \sin\theta d\theta - u_\theta r \sin\theta dr$$

$$d\psi = \frac{\partial \psi}{\partial \theta} d\theta + \frac{\partial \psi}{\partial r} dr$$

$$\rightarrow \begin{cases} u_r = \dfrac{1}{r^2\sin\theta}\dfrac{\partial \psi}{\partial \theta} \\ u_\theta = -\dfrac{1}{r\sin\theta}\dfrac{\partial \psi}{\partial r} \end{cases}$$



Solution of the Potential Equation

- General form of solution by separation of variables
- The fundamental solutions so obtained will subsequently be superimposed to produce more complex solutions in a manner similar to that which was used in 2D potential flow.

$$\phi(r,\theta) = R(r)T(\theta) \qquad \text{separable solution is sought for Laplace eq;}$$

$$\frac{T}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{R}{r^2\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) = 0$$

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{T\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right)$$

- The only way the equation can remain valid is for each side to be equal to a constant, $l(l+1)$

for conveniences

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1)$$

$$-\frac{1}{T\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) = l(l+1)$$

Solution of the Potential Equation

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - l(l+1)R = 0 \rightarrow R(r) = Kr^\alpha$$

$$\alpha(\alpha+1)Kr^\alpha - l(l+1)Kr^\alpha = 0 \rightarrow \alpha = l,\ -(l+1)$$

$$R_l(r) = A_l r^l + \frac{B_l}{r^{l+1}} \qquad A_l, B_l:\ \text{arbitrary constant depending on } l$$

- Similarly for T(θ),

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{dT}{d\theta}\right) + l(l+1)T = 0 \quad \text{Legendre's eq}$$

$$x = \cos\theta$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dT}{dx}\right] + l(l+1)T = 0$$

$$T_l(\theta) = C_l P_l(\cos\theta) + D_l Q_l(\cos\theta)$$

$P_l(x)$: Legendre's function of the first kind
$Q_l(x)$: Legendre's function of the second kind

Solution of the Potential Equation

- $Q_l(\cos\theta)$ diverge for $\cos\theta = \pm 1$ for all values of l. The coefficient D_l must then be specified as being zero, since there should be no singularities in the flow field.
- Also, $P_l(\cos\theta)$ diverges for $\cos\theta = \pm 1$ if l is not an integer. Then it must be specified that the quantity l be an integer.

- Combining above results,

$$\phi_l(r,\theta) = \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta) \quad (l: \text{ integer})$$

$P_l(x)$: Legendre's function of the first kind

$$\phi(r,\theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l}(x^2 - 1)^l \qquad P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \tfrac{1}{2}(3x^2 - 1)$$

Legendre's polynomials

Uniform Flow

- By setting $B_l = 0$ for all l, we get $\phi(r,\theta) = Ur\cos\theta$.

$$A_l = \begin{cases} 0 & \text{for } l \neq 1 \\ U & \text{for } l = 1 \end{cases}$$

$$u_r = \frac{\partial\phi}{\partial r} = U\cos\theta$$

$$\frac{1}{r^2\sin\theta}\frac{d\psi}{d\theta} = U\cos\theta \rightarrow \psi = \frac{1}{2}Ur^2\sin^2\theta + f(r)$$

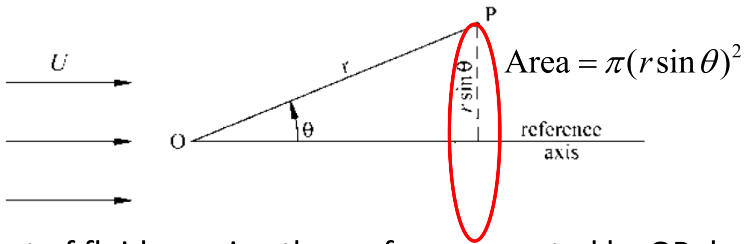
$$u_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\sin\theta$$

$$-\frac{1}{r\sin\theta}\frac{d\psi}{dr} = -U\sin\theta \rightarrow \psi = \frac{1}{2}Ur^2\sin^2\theta + g(\theta)$$

$$\therefore \psi = \frac{1}{2}Ur^2\sin^2\theta$$

Uniform Flow

- Otherwise, there is an alternative way of evaluating the stream function.



  – the amount of fluid crossing the surface generated by OP due to the uniform flow will be $2\pi\psi$.

$$2\pi\psi = U\pi(r\sin\theta)^2$$

$$\psi(r,\theta) = \frac{1}{2}Ur^2\sin^2\theta$$

Source and Sink

- By setting $A_l = 0$ for all l , we get $\phi(r,\theta) = \dfrac{B_0}{r}$. $\leftarrow P_0(\cos\theta) = 1$

$$B_l = \begin{cases} 0 & \text{for } l \neq 0 \\ B_0 \neq 0 & \text{for } l = 0 \end{cases} \qquad \boxed{u_r = -\frac{B_0}{r^2},\ u_\theta = 0}$$

  – Let Q the volume of the fluid leaving the control surface per unit time.

$$Q = \int_s \bar{u}\cdot\bar{n}ds = \int_0^{2\pi} d\omega \int_0^{\pi}\left(\frac{B_0}{r^2}\right) r^2\sin\theta d\theta = -4\pi B_0$$

$$\begin{pmatrix} \bar{u}\cdot\bar{n} = B_0/r^2 \\ ds = r^2\sin\theta d\theta d\omega \end{pmatrix}$$

  – Then, for a source of strength Q, the constant B_0 should be set equal to $-Q/(4\pi)$. That is, the velocity potential for a source of strength Q located at r = 0 is

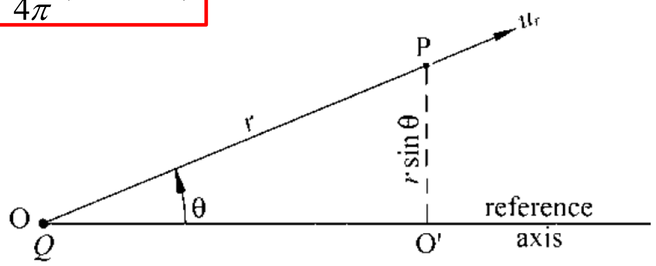
$$\boxed{\phi(r,\theta) = -\frac{Q}{4\pi r}}$$ ('+' for a sink)

Source and Sink

- The quantity of fluid that crosses the surface generated by revolving the line OP about the reference axis will depend upon whether the source Q is considered to be slightly to the left of the origin or slightly to the right of it.
- Here the source Q will be considered to be slightly to the right of O, so that the quantity of fluid crossing the surface generated by OP will be

$$2\pi\psi + Q = \int_0^\theta u_r \cos\theta 2\pi r \sin\theta \frac{r d\theta}{\cos\theta}$$

$$\boxed{\psi(r,\theta) = -\frac{Q}{4\pi}(1+\cos\theta)}$$



Flow due to a Doublet

- As was the case in two dimensions, the flow due to a doublet may be obtained by superimposing a source and sink of equal strength and letting the distance separating the source and the sink shrink to zero.

$$\phi(r,\theta) = -\frac{Q}{4\pi r} + \frac{Q}{4\pi(r-\delta r)} = -\frac{Q}{4\pi r}\left(1 - \frac{1}{1-\delta r/r}\right) = \frac{Q}{4\pi r}\left[\frac{\delta r}{r} + O\left(\frac{\delta r}{r}\right)^2\right]$$

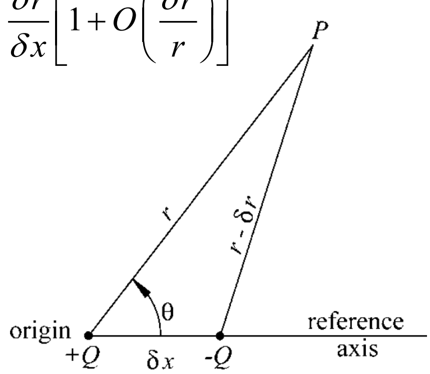
$$(r-\delta r)^2 = r^2 + (\delta x)^2 - 2r\delta x\cos\theta$$

$$\cos\theta = \frac{r^2 + (\delta x)^2 - (r-\delta r)^2}{2r\delta x} = \frac{\delta r}{\delta x} - \frac{\delta r}{2r}\frac{\delta r}{\delta x} + \frac{\delta x}{2r} = \frac{\delta r}{\delta x}\left[1 + O\left(\frac{\delta r}{r}\right)\right]$$

$$\delta r = \delta x \cos\theta\left[1 - O\left(\frac{\delta r}{r}\right)\right]$$

$$\phi(r,\theta) = \frac{Q}{4\pi r}\left\{\frac{\delta x}{r}\cos\theta\left[1 + O\left(\frac{\delta r}{r}\right)\right]\right\}$$

$\delta x \to 0,\ Q \to \infty : Q\delta x \to \mu$ $\quad \boxed{\therefore \phi(r,\theta) = \frac{\mu}{4\pi r^2}\cos\theta}$



Flow due to a Doublet

$$u_r = \frac{\partial \phi}{\partial r} = -\frac{\mu}{2\pi r^3}\cos\theta = \frac{1}{r^2 \sin\theta}\frac{\partial \psi}{\partial \theta}$$

$$\frac{d\psi}{d\theta} = -\frac{\mu}{2\pi r}\sin\theta\cos\theta \rightarrow \psi = -\frac{\mu}{4\pi r}\sin^2\theta + f(r)$$

$$u_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\mu}{4\pi r^3}\sin\theta = -\frac{1}{r\sin\theta}\frac{d\psi}{dr}$$

$$\frac{d\psi}{dr} = \frac{\mu}{4\pi r^2}\sin^2\theta \rightarrow \psi = -\frac{\mu}{4\pi r}\sin^2\theta + g(\theta)$$

$$\boxed{\therefore \psi = -\frac{\mu}{4\pi r}\sin^2\theta}$$

Flow near a blunt nose

- By superimposing the solutions for a uniform flow and a source, the solution corresponding to a long cylinder with a blunt nose is obtained.

$$\psi(r,\theta) = \frac{1}{2}Ur^2\sin^2\theta - \frac{Q}{4\pi}(1+\cos\theta)$$

  - consider ψ to be constant and solve the preceding equation for r in terms of θ:

$$r = \sqrt{\frac{2\psi}{U\sin^2\theta} + \frac{Q}{2\pi U}\frac{1+\cos\theta}{\sin^2\theta}} = \sqrt{\frac{2\psi}{U\sin^2\theta} + \frac{Q}{4\pi U\sin^2(\theta/2)}}$$

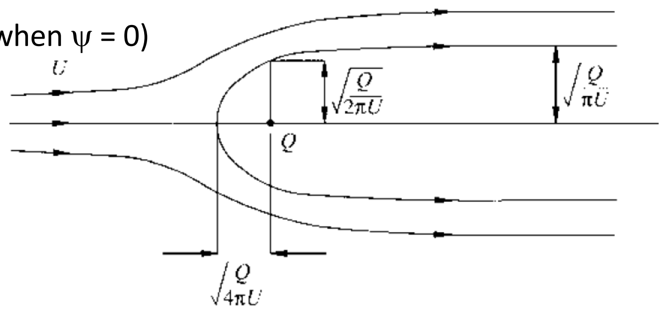
$$r_0 = \sqrt{\frac{Q}{4\pi U}}\frac{1}{\sin(\theta/2)}$$

(when ψ = 0)

$$\theta = 0,\ r_0 = \infty$$

$$\theta = \pi/2,\ r_0 = \sqrt{Q/2\pi U}$$

$$\theta = \pi,\ r_0 = \sqrt{Q/4\pi U}$$



Flow near a blunt nose

- Although the polar radius r_0 is infinite for $\theta = 0$, the cylindrical radius R_0 is finite.

$$R = r\sin\theta,\ R_0 = \sqrt{\frac{Q}{4\pi U}\frac{\sin\theta}{\sin(\theta/2)}}$$

$$\text{As } \theta \to 0;\ \frac{\sin\theta}{\sin(\theta/2)} = \frac{\theta - \theta^3/3! + \cdots}{\theta/2 - 1/3!(\theta/2)^3 + \cdots} \to 2$$

$$\therefore R_0 = \sqrt{\frac{Q}{\pi U}}$$

$$\boxed{\begin{aligned}\psi(r,\theta) &= \frac{1}{2}Ur^2\sin^2\theta - \frac{Q}{4\pi}(1+\cos\theta)\\ \phi(r,\theta) &= Ur\cos\theta - \frac{Q}{4\pi r}\end{aligned}}$$

velocity and pressure distribution in the vicinity of the nose of a blunt axisymmetric body such as an aircraft fuselage or a submarine hull

Flow around a sphere

- The stream function for a uniform flow past a sphere may be obtained by superimposing the solution for a **uniform flow** and that for a **doublet**.

$$\psi(r,\theta) = \frac{1}{2}Ur^2\sin^2\theta - \frac{\mu}{4\pi r}\sin^2\theta$$

$$0 = \frac{1}{2}Ur_0^2\sin^2\theta - \frac{\mu}{4\pi r_0}\sin^2\theta \to r_0 = \left(\frac{\mu}{2\pi U}\right)^{1/3}$$

Equation for surface of a sphere ($\psi = 0$)

If we choose

$$\mu = 2\pi Ua^3,\ r_0 = a$$

$$\boxed{\begin{aligned}\psi(r,\theta) &= \frac{1}{2}U\left(r^2 - \frac{a^3}{r}\right)\sin^2\theta\\ \theta(r,\theta) &= U\left(r + \frac{1}{2}\frac{a^3}{r^2}\right)\cos\theta\end{aligned}}$$