INVISCID FLOW
Week 12

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Line-distributed Sources

- The stream function and the velocity potential for a source that is distributed over a finite strip will be established
  - *qL*: total volume of fluid that emanates from the source per unit time

$$\psi = -\int_0^L \frac{q\,d\xi}{4\pi}(1+\cos\nu)$$

$R = r\sin\theta = \eta\sin\alpha$ remains constant throughout the integration

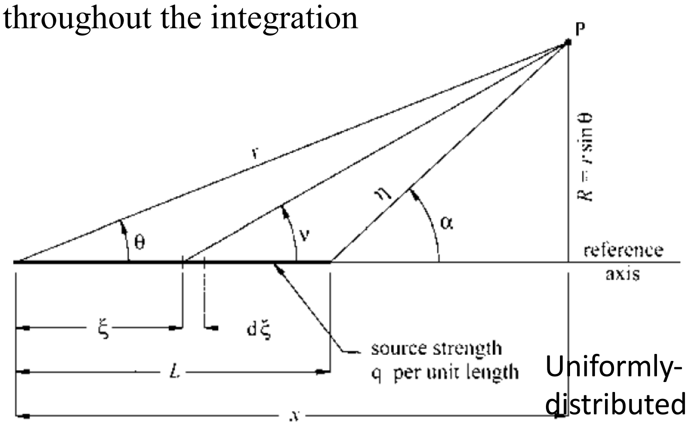
$$x-\xi = R\cot\nu \rightarrow -d\xi = -R\csc^2\nu\, d\nu$$

$$\therefore \psi(r,\theta) = -\frac{qR}{4\pi}\int_\theta^\alpha \csc^2\nu(1+\cos\nu)d\nu$$

$$= -\frac{qR}{4\pi}\left(\cot\theta - \cot\alpha + \frac{1}{\sin\theta} - \frac{1}{\sin\alpha}\right)$$

$$x = R\cot\theta, x-L = R\cot\alpha$$

$$r = \frac{R}{\sin\theta}, \eta = \frac{R}{\sin\alpha}$$



Uniformly-distributed sources

Line-distributed Sources

$$\therefore \psi = -\frac{q}{4\pi}(L + r - \eta)$$

$$\phi = -\int_0^L \frac{q d\xi}{4\pi(R/\sin\nu)}, \quad x - \xi = R\cot\nu, -d\xi = -R\csc^2\nu d\nu$$

$$\phi = -\frac{q}{4\pi}\int_\theta^\alpha \sin\nu \csc^2\nu d\nu = -\frac{q}{4\pi}\int_\theta^\alpha \frac{d\nu}{\sin\nu} = -\frac{q}{4\pi}\ln\left(\frac{\tan\alpha/2}{\tan\theta/2}\right)$$

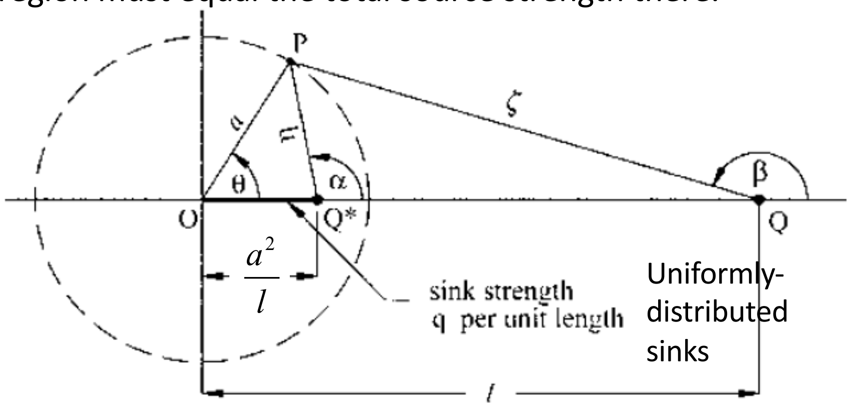
- Note that stream function was more compact when expressed in terms of lengths, the result for the velocity potential is more compact in terms of angles.

Sphere in the flow field of a source

- Source Q: located at a distance l along the reference axis
- Source Q*: located at the image point a²/l
- Uniformly distributed line sink of strength q per unit length along OQ*

- If the spherical surface r = a is to be a stream surface, the total sink strength inside this region must equal the total source strength there.

$$\frac{qa^2}{l} = Q^*$$



Sphere in the flow field of a source

$$\psi(r,\theta) = -\frac{Q}{4\pi}(1+\cos\beta) - \frac{Q^*}{4\pi}(1+\cos\alpha) + \frac{Q^*}{4\pi}\frac{l}{a^2}\left(\frac{a^2}{l} + r - \eta\right)$$

$$\psi(a,\theta) = -\frac{Q}{4\pi}(1+\cos\beta) - \frac{Q^*}{4\pi}(1+\cos\alpha) + \frac{Q^*}{4\pi}\left(1 + \frac{1}{a} - \frac{l\eta}{a^2}\right)$$ On the sphere surface

– if the point P lies on the spherical surface r = a,

$$\frac{a^2/l}{a} = \frac{a}{l} \rightarrow \frac{OQ^*}{OP} = \frac{OP}{OQ} \rightarrow \Delta OPQ^* \sim \Delta OQP$$

$$\eta = \frac{a^2}{l}\cos(\pi-\alpha) + a\cos(\pi-\beta) = -\frac{a^2}{l}\cos\alpha - a\cos\beta$$

$$\psi(a,\theta) = -\frac{Q}{4\pi}(1+\cos\beta) - \frac{Q^*}{4\pi}(1+\cos\alpha) + \frac{Q^*}{4\pi}(1 + \frac{l}{a} + \cos\alpha + \frac{l}{a}\cos\beta)$$

$$= (1+\cos\beta)\left(-\frac{Q}{4\pi} + \frac{Q^*}{4\pi}\frac{l}{a}\right)$$

Sphere in the flow field of a source

– Thus by choosing the source strength Q* to be equal to aQ/l, the surface r = a corresponds to the stream surface ψ = 0.

– Then the stream function for a sphere of radius a whose center is at the origin and that is exposed to a point source of strength Q located a distance l along the positive reference axis is

$$\psi(r,\theta) = -\frac{Q}{4\pi}(1+\cos\beta) - \frac{Q}{4\pi}\frac{l}{a}(1+\cos\alpha) + \frac{Q}{4\pi}(\frac{l}{a} + \frac{r}{a} - \frac{\eta}{a})$$

$$\phi(r,\theta) = -\frac{Q}{4\pi\zeta} - \frac{Qa}{4\pi\eta l} + \frac{Q}{4\pi a}\ln\left(\frac{\tan\alpha/2}{\tan\theta/2}\right)$$

ζ is the distance from the field point P to the source Q

Rankine Solids

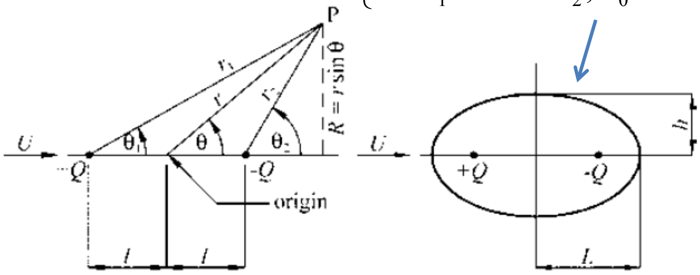
- The solution for the flow around a family of bodies (known as Rankine solids) is obtained by superimposing a source and a sink of equal strength in a uniform flow field.

$$\psi(r,\theta) = \frac{1}{2}Ur^2\sin^2\theta - \frac{Q}{4\pi}(\cos\theta_1 - \cos\theta_2)$$

$$0 = \frac{1}{2}Ur_0^2\sin^2\theta - \frac{Q}{4\pi}(\cos\theta_1 - \cos\theta_2)$$

$$R = r\sin\theta \qquad \text{(cylindrical radius)}$$

$$R_0^2 = \frac{Q}{2\pi U}(\cos\theta_1 - \cos\theta_2) \rightarrow \begin{cases} \theta_1 = \theta_2 = 0, \theta_1 = \theta_2 = \pi; R_0 = 0 \\ \cos\theta_1 = -\cos\theta_2; R_0 \text{ maximum} \end{cases}$$



Rankine Solids

- The principal dimensions of this body are the **half width L** and the **half height h**. Both these parameters depend upon the free-stream velocity U, the source and sink strength Q, and the distance l.
  - velocity at the downstream stagnation point is zero

$$U + \frac{Q}{4\pi(L+l)^2} - \frac{Q}{4\pi(L-l)^2} = 0$$

$$(L^2 - l^2)^2 - \frac{Ql}{\pi U}L = 0$$

  - value of the cylindrical radius R_0 = h, when $\cos\theta_1 = -\cos\theta_2$

$$h^2 = \frac{Q}{2\pi U}\left(\frac{l}{\sqrt{h^2+l^2}} + \frac{l}{\sqrt{h^2+l^2}}\right)$$

$$h^2\sqrt{h^2+l^2} - \frac{Ql}{\pi U} = 0$$

For various values of the parameters U, Q, and l, a family of bodies of revolution is defined.

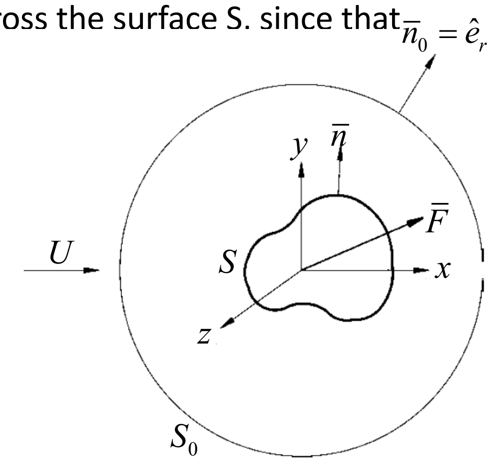
D'Alembert Paradox

- If an arbitrary three-dimensional body is immersed in a uniform flow, the equations of hydrodynamics predict that there will be no force exerted on the body by the fluid.
  - equation of force equilibrium for the body of fluid contained between the surfaces S and S_0
  - There is no transfer of momentum across the surface S. since that surface is a stream surface.

$$0 = -\bar{F} - \int_{S_0} \left[p n_0 + \rho \bar{u} (\bar{u} \cdot \bar{n}_0) \right] dS$$

$$p + \frac{1}{2} \rho \bar{u} \cdot \bar{u} = B, \quad \int_{S_0} B \bar{n}_0 dS = 0$$

$$\bar{F} = \rho \int_{S_0} \left[\tfrac{1}{2} (\bar{u} \cdot \bar{u}) n_0 - \bar{u} (\bar{u} \cdot \bar{n}_0) \right] dS$$



D'Alembert Paradox

$$\bar{u} = \bar{U} + \bar{u}' \quad \text{(perturbation velocity } u'\text{)}$$

$$F = \rho \int_{S_0} \left\{ \left(\tfrac{1}{2} U^2 + \bar{U} \cdot \bar{u}' + \tfrac{1}{2} \bar{u}' \cdot \bar{u}' \right) \bar{n}_0 - \left(\bar{U} + \bar{u}' \right) \left[\left(\bar{U} + \bar{u}' \right) \cdot \bar{n}_0 \right] \right\} dS$$

$$= \rho \int_{S_0} \left\{ \left(\bar{U} \cdot \bar{u}' + \tfrac{1}{2} \bar{u}' \cdot \bar{u}' \right) \bar{n}_0 - \left[\bar{u}' (\bar{U} \cdot \bar{n}_0) + \bar{u}' (\bar{u}' \cdot \bar{n}_0) \right] \right\} dS$$

$$= \rho \int_{S_0} \left[-\bar{U} \times (\bar{u}' \times \bar{n}_0) \bar{n}_0 + \tfrac{1}{2} (\bar{u}' \cdot \bar{u}') \bar{n}_0 - \bar{u}' (\bar{u}' \cdot \bar{n}_0) \right] dS$$

- It will now be shown that each of these terms is zero.
- Let φ' be the velocity potential corresponding to the perturbation velocity u'.

$$\phi' = \sum_{l=0}^{\infty} A_l \frac{P_l(\cos\theta)}{r^{l+1}} = -\frac{Q}{4\pi r} + \frac{\mu \cos\theta}{4\pi r^2} + O\left(\frac{1}{r^3} \right)$$

$$\bar{u}' = \nabla \phi' \rightarrow |\bar{u}'| = O\left(\frac{1}{r^2} \right)$$

$$\bar{u}' \times \bar{n}_0 = \nabla \left[\frac{Q}{4\pi r} - \frac{\mu \cos\theta}{4\pi r^2} + O\left(\frac{1}{r^3} \right) \right] \times \hat{e}_r = \left[\frac{Q}{4\pi r^2} \hat{e}_r + O\left(\frac{1}{r^3} \right) \right] \times \hat{e}_r$$

D'Alembert Paradox

$$\left|\bar{u}' \times \bar{n}_0\right| = O\left(\frac{1}{r^3}\right)$$

$$dS = r^2 \sin\theta d\theta d\omega \rightarrow dS = O(r^2)$$

$$\therefore \int_{S_0} \bar{U} \times (\bar{u}' \times \bar{n}_0) dS = O\left(\frac{1}{r}\right)$$

$$\int_{S_0} (\bar{u}' \cdot \bar{u}') \bar{n}_0 dS = O\left(\frac{1}{r^2}\right)$$

$$\int_{S_0} \bar{u}' (\bar{u}' \cdot \bar{n}_0) dS = O\left(\frac{1}{r^2}\right)$$

– if the radius of the spherical surface S_0 is taken to be very large, each of these integrals will be vanishingly small → ***F*** = 0