INVISCID FLOW
Week 13

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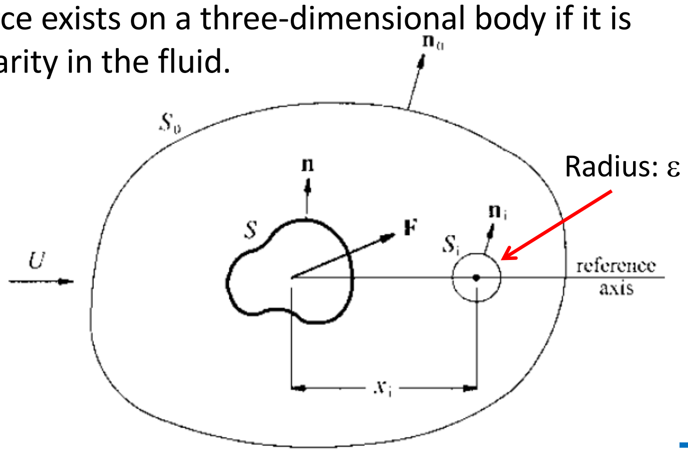
Forces induced by singularities

- Previously, no force exists on a body that is in a uniform flow field.
- This agrees with our previous results: the Kutta-Joukowski law shows that in the absence of circulation around a body there are no forces acting on two-dimensional bodies.
- It is very difficult to establish an appreciable circulation around short bodies - that is, around three-dimensional bodies.
- It will be shown that a force exists on a three-dimensional body if it is exposed to a point singularity in the fluid.



Forces induced by singularities

- For equilibrium of the forces that act on the body of fluid that is inside S_0 but outside S and S_i, the sum of the forces must be zero

$$\frac{D}{Dt}\int_V \rho\bar{u}\,dV = \int_V \frac{\partial}{\partial t}(\rho\bar{u})\,dV + \int_{S+S_i+S_o} \rho\bar{u}\,(\bar{u}\cdot\bar{n})\,dS$$

$$= \int_{S+S_i+S_o} \bar{P}\,dS + \int_V \rho\bar{f}\,dV$$

$$\int_{S+S_i+S_o} \bar{P}\,dS = \int_{S+S_i+S_o} \rho\bar{u}\,(\bar{u}\cdot\bar{n})\,dS$$

$$\int_S \bar{P}\,dS + \int_{S_i} \bar{P}\,dS + \int_{S_o} \bar{P}\,dS = \cancel{\int_S \rho\bar{u}\,(\bar{u}\cdot\bar{n})\,dS} + \int_{S_i} \rho\bar{u}\,(\bar{u}\cdot\bar{n})\,dS + \int_{S_o} \rho\bar{u}\,(\bar{u}\cdot\bar{n})\,dS$$

$$-\bar{F} - \int_{S_o} p\bar{n}_o\,dS - \int_{S_i} p(-\bar{n}_i)\,dS = \int_{S_o} \rho\bar{u}\,(\bar{u}\cdot\bar{n}_o)\,dS + \int_{S_i} \rho\bar{u}\,(\bar{u}\cdot(-\bar{n}_i))\,dS$$

$$0 = -F - \cancel{\int_{S_o}\left[p\bar{n}_o + \rho\bar{u}\,(\bar{u}\cdot\bar{n}_o)\right]dS} + \int_{S_i}\left[p\bar{n}_i + \rho\bar{u}\,(\bar{u}\cdot\bar{n}_i)\right]dS$$

$$F = \int_{S_i}\left[p\bar{n}_i + \rho\bar{u}(\bar{u}\cdot\bar{n}_i)\right]dS$$

$$p = B - \rho(\bar{u}\cdot\bar{u})/2$$

$$F = \rho\int_{S_i}\left[-\tfrac{1}{2}(\bar{u}\cdot\bar{u})\bar{n}_i + \bar{u}(\bar{u}\cdot\bar{n}_i)\right]dS$$

Forces induced by singularities

- In order to further reduce the integral, it is necessary to specify the nature of the singularity located inside the surface S_i. The case of a source, or sink, and that of a doublet will be examined.

  - First, the singularity at x = x_i to be a source of strength Q

$$\bar{u} = \frac{Q}{4\pi\varepsilon^2}\hat{e}_\varepsilon + \bar{u}_i$$

where $\hat{e}_\varepsilon$ is the unit vector radial from the point $x = x_i$ and u_i is the velocity induced by all means other than the source under consideration.

$$\bar{u}\cdot\bar{u} = \frac{Q^2}{16\pi^2\varepsilon^4} + \frac{Q}{2\pi\varepsilon^2}\hat{e}_\varepsilon\cdot\bar{u}_i + \bar{u}_i\cdot\bar{u}_i$$

$$\bar{u}\cdot\bar{n}_i = \bar{u}\cdot\hat{e}_\varepsilon = \frac{Q}{4\pi\varepsilon^2} + \bar{u}_i\cdot\hat{e}_\varepsilon$$

Forces induced by singularities

$$\bar{F} = \rho \int_{S_i} \left[-\frac{1}{2}\left(\frac{Q^2}{16\pi^2\varepsilon^4} + \frac{Q}{2\pi\varepsilon^2}\hat{e}_\varepsilon \cdot \bar{u}_i + \bar{u}_i \cdot \bar{u}_i \right)\hat{e}_\varepsilon + \left(\frac{Q}{4\pi\varepsilon^2}\hat{e}_\varepsilon + \bar{u}_i \right)\left(\frac{Q}{4\pi\varepsilon^2} + \bar{u}_i \cdot \hat{e}_\varepsilon \right) \right] dS$$

$$= \rho \int_{S_i} \left[\overset{0}{\cancel{\frac{Q^2}{32\pi^2\varepsilon^4}}}\hat{e}_\varepsilon - \frac{1}{2}\overset{0 \text{ as } \varepsilon \to 0}{\cancel{(\bar{u}_i \cdot \bar{u}_i)}}\hat{e}_\varepsilon + \frac{Q}{4\pi\varepsilon^4}\bar{u}_i + \overset{0 \text{ as } \varepsilon \to 0}{\cancel{(\bar{u}_i \cdot \hat{e}_\varepsilon)}}\bar{u}_i \right] dS$$

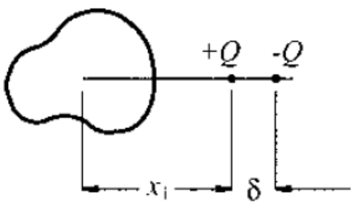
$$\bar{F} = \rho \int_{S_i} \frac{Q}{4\pi\varepsilon^2}\bar{u}_i dS$$

$$= \frac{\rho Q \bar{u}_i}{4\pi\varepsilon^2}\int_{S_i} dS = \rho Q \bar{u}_i$$

- For a sink, $\bar{F} = -\rho Q \bar{u}_i$

Forces induced by singularities

– Second, the singularity is a doublet ($\delta \to 0$).



- Then if u_i is the fluid velocity at $x = x_i$ due to all components of the flow except the source and the sink under consideration, the velocity at $x = x_i$, less that due to the source itself, will be

$$\frac{Q}{4\pi\delta^2}\hat{e}_x + \bar{u}_i$$

- The velocity at $x = x_i + \delta$, less that due to the sink, will be

$$\frac{Q}{4\pi\delta^2}\hat{e}_x + \bar{u}_i + \delta\frac{\partial \bar{u}_i}{\partial x} + \cdots$$

Forces induced by singularities

– Then,

$$\bar{F} = \frac{\rho Q}{4\pi}\bar{u}_i \int_0^{2\pi} d\omega \int_0^{\pi} \sin\theta d\theta \rightarrow \begin{cases} \rho Q\left(\dfrac{Q}{4\pi\delta^2}\hat{e}_x + \bar{u}_i\right) & \text{due to a source} \\ -\rho Q\left(\dfrac{Q}{4\pi\delta^2}\hat{e}_x + \bar{u}_i + \delta\dfrac{\partial \bar{u}_i}{\partial x} + \cdots\right) & \text{due to a sink} \end{cases}$$

$$\therefore F = -\rho Q\delta \frac{\partial \bar{u}_i}{\partial x}$$

As $\delta \rightarrow 0$ and $Q \rightarrow \infty$, set $Q\delta \rightarrow \mu$,

then, $F = -\rho\mu \dfrac{\partial \bar{u}_i}{\partial x}$

Forces induced by singularities

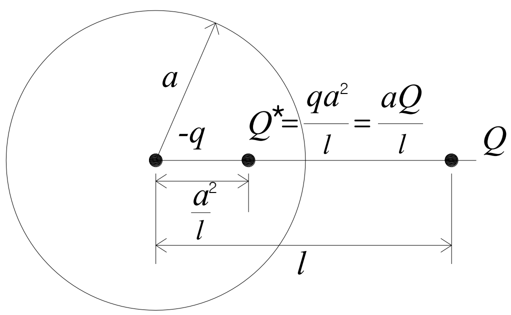
- Example: consider a sphere in the presence of a source

$$\bar{u}_i = \frac{Qa/l}{4\pi}\frac{1}{(1-a^2/l)^2}\hat{e}_x - \int_0^{a^2/l}\frac{Q/a}{4\pi}\frac{\hat{e}_x}{(l-x)^2}dx$$

$$= \frac{Qa/l}{4\pi}\frac{1}{(1-a^2/l)^2}\hat{e}_x - \frac{Q/a}{4\pi}\left[\frac{1}{(l-a^2/l)} - \frac{1}{l}\right]\hat{e}_x$$

$$= \frac{Qa^3}{4\pi l(l^2-a^2)^2}\hat{e}_x$$

$$\therefore \bar{F} = \frac{\rho Q^2 a^3}{4\pi l(l^2-a^2)^2}\hat{e}_x$$



Sphere is attracted to the source with a force that is proportional to Q^2.

Kinetic energy of a moving fluid

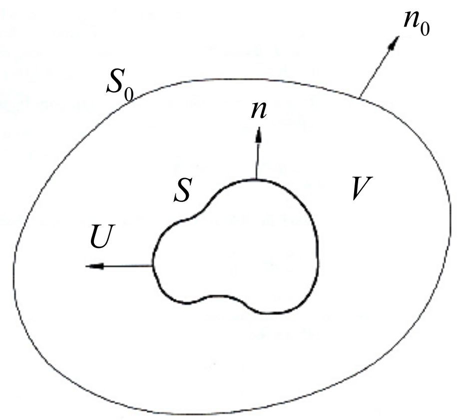
- The kinetic energy associated with the fluid in the uniform flow around a stationary body will be infinite if the flow field is infinite in extent.
- However, the kinetic energy induced in a quiescent fluid by the passage of a body through it will be finite, even if the flow field is infinite in extent.
- For this reason, discussions of kinetic-energy considerations are based on a frame of reference in which the fluid far from the body is at rest and the body is moving.
  - kinetic energy T of this volume of fluid

$$T=\int_V \frac{1}{2}\rho(\bar{u}\cdot\bar{u})dV=\frac{1}{2}\rho\int_V \nabla\phi\cdot\nabla\phi dV$$

$$=\frac{1}{2}\rho\int_{\Sigma}\phi\frac{\partial\phi}{\partial n}dS \qquad \text{Green's theorem}$$

$$=\frac{1}{2}\rho\int_{S_0}\phi\frac{\partial\phi}{\partial n}dS-\frac{1}{2}\rho\int_S\phi\frac{\partial\phi}{\partial n}dS$$

surface that encloses V (S and S_0)



Kinetic energy of a moving fluid

- From the continuity equation

$$\int_V \nabla\cdot\bar{u}dV=0\rightarrow\int_{S_0}\bar{u}\cdot\bar{n}dS-\int_S\bar{u}\cdot\bar{n}dS=0 \qquad \text{Gauss' Theorem}$$

$$\int_{S_0}\frac{\partial\phi}{\partial n}dS-\int_S V\cdot\bar{n}dS=0\rightarrow\int_{S_0}\frac{\partial\phi}{\partial n}dS=0$$

for any constant C,

$$\int_{S_0}C\frac{\partial\phi}{\partial n}dS=0$$

- From previous equation,

$$T=\frac{1}{2}\rho\int_{S_0}(\phi-C)\frac{\partial\phi}{\partial n}dS-\frac{1}{2}\rho\int_S\phi\frac{\partial\phi}{\partial n}dS$$

  - Fluid velocity far from the body is zero → ϕ can be a constant.
  - By considering the surface S_0 to be large and by choosing C to be the value of ϕ far from the body, $T=-\frac{1}{2}\rho\int_S\phi\frac{\partial\phi}{\partial n}dS$

Apparent Mass

- When a body moves through a quiescent fluid, a certain mass of the fluid is induced to move.
- Then, what equivalent mass of fluid, if it moved with the same velocity as the body, would exhibit the same kinetic energy as the actual case?

- If the fluid may be considered as being ideal, the mass of fluid referred to above is found to depend on the body shape only, and this mass of fluid is called the apparent (added) mass (M').

$$\frac{1}{2}M'U^2 = -\frac{1}{2}\rho\int_S \phi\frac{\partial\phi}{\partial n}dS$$

$$\therefore M' = -\frac{\rho}{U^2}\int_S \phi\frac{\partial\phi}{\partial n}dS$$

Apparent Mass

- Example: apparent mass for the sphere

$$\phi(r,\theta) = U\left(r + \frac{1}{2}\frac{a^3}{r^2}\right)\cos\theta$$

stationary sphere of radius *a* with a uniform flow of magnitude U approaching it

$$\phi(r,\theta) = U\left(r + \frac{1}{2}\frac{a^3}{r^2}\right)\cos\theta - Ur\cos\theta = \frac{1}{2}U\frac{a^3}{r^2}\cos\theta$$

$$\therefore \frac{\partial\phi}{\partial n}(r,\theta) = \frac{\partial\phi}{\partial r}(r,\theta) = -U\frac{a^3}{r^3}\cos\theta$$

  - On the sphere surface S, (r = a)

$$\phi\frac{\partial\phi}{\partial n} = -\frac{1}{2}U^2 a\cos^2\theta \rightarrow M' = -\frac{\rho}{U^2}\int_0^{\pi}\left(-\frac{1}{2}U^2 a\cos^2\theta\right)a^2\sin\theta d\theta$$

$$M' = \frac{2}{3}\pi a^3\rho$$

- This may be added to the actual mass of the sphere, and the total mass may be used.
- The existence of the fluid may be ignored if the apparent mass of fluid is added to the actual mass of the body.

Surface Waves

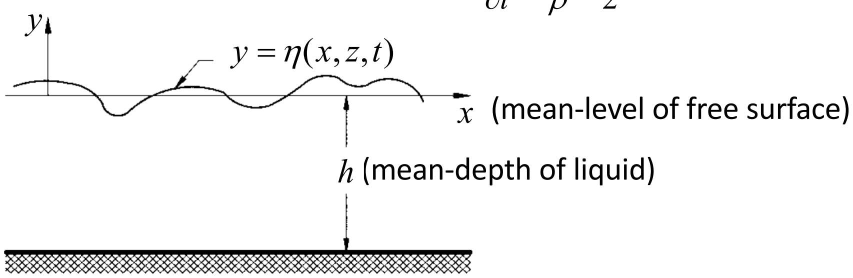
- The effect of gravity on liquid surfaces
- Flows associated with surface waves to be assumed "potential", which is a valid approximation for many free-surface phenomena
- Focus on two-dimensional flow

General Surface-Wave Problem

- In most cases, the motion of liquid induced by surface waves can be considered to be irrotational → Velocity vector is expressed as a gradient of a velocity potential, which must satisfy the **Laplace equation**. $\nabla^2\phi = 0$
  - Boundary condition @ $y = \eta$
    - Kinematic condition: $\dfrac{D}{Dt}(y-\eta) = \dfrac{\partial}{\partial t}(y-\eta) + \bar{u}\cdot\nabla(y-\eta) = 0$

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial z}\frac{\partial \eta}{\partial z} = \frac{\partial \phi}{\partial y}$$

    - Dynamic condition: $\dfrac{\partial \phi}{\partial t} + \dfrac{P}{\rho} + \dfrac{1}{2}\nabla\phi\cdot\nabla\phi + g\eta = F(t)$ Unsteady, irrotational



General Surface-Wave Problem

- Boundary condition @ *y* = -*h*? $\left.\frac{\partial \phi}{\partial y}\right)_{y=-h} = \frac{\partial \phi}{\partial y}(x,-h,t) = 0$

- The difficulty in solving surface-wave problems may be seen to be in the boundary conditions rather than the differential equation.
- However, for many interesting features of surface-wave flows, the difficulties may be avoided by linearizing the problem.

Small-amplitude Plane Waves

- Plane waves: two-dimensional $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$
- Small-amplitude: wave amplitude is small compared to other characteristic length scales

$$\eta \ll 1, \ \frac{\partial \eta}{\partial x} \ll 1, \ \frac{\partial \phi}{\partial x} \ll 1$$

- Kinematic boundary condition becomes

$$\frac{\partial \eta}{\partial t} + \cancel{\frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x}} + \cancel{\frac{\partial \phi}{\partial z}\frac{\partial \eta}{\partial z}} = \frac{\partial \phi}{\partial y} \rightarrow \frac{\partial \eta}{\partial t}(x,t) = \frac{\partial \phi}{\partial y}(x,\eta,t)$$

$$\frac{\partial \phi}{\partial y}(x,\eta,t) = \frac{\partial \phi}{\partial y}(x,0,t) + \eta \frac{\partial^2 \phi}{\partial y^2}(x,0,t) + O(\eta^2)$$

$$\therefore \frac{\partial \eta}{\partial t}(x,t) = \frac{\partial \phi}{\partial y}(x,0,t)$$

- Dynamic condition becomes $\frac{\partial \phi}{\partial t}(x,\eta,t) + \frac{P(x,t)}{\rho} + g\eta(x,t) = F(t)$

$$\therefore \frac{\partial^2 \phi}{\partial t^2}(x,0,t) + \frac{1}{\rho}\frac{\partial P(x,t)}{\partial t} + g\frac{\partial \phi}{\partial y}(x,0,t) = 0$$